Cosmologies, Singularities & Quantum Extremal Surfaces

K. Narayan

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- AdS Kasner cosmologies; classical RT/HRT surfaces
- 2d dilaton-gravity-matter; quantum extremal surfaces
- QES: null Kasner singularities
- de Sitter extremal surfaces

arXiv:2111.14906, Goswami, KN, Saini; arXiv:2012.07351, Manu, KN, Paul;

2010.12955, 2002.11950, KN

[recent collaborations: Ritabrata Bhattacharya, Kaberi Goswami, A. Manu, Partha Paul, Hitesh Saini]

Holography, singularities, time . . .

~ 25 yrs since AdS/CFT '97 Maldacena; '98 Gubser,Klebanov,Polyakov; Witten.

Understand time-dependent phenomena in string theory and holography.

Big-Bang/Crunch singularities: curvatures, tidal forces divergent.

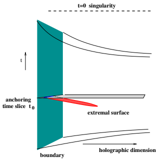
General relativity, notions of spacetime geometry break down.

Hope: holographic (or "stringy") description $\rightarrow$ UV completion.

Entanglement: insightful holographic probe.

Classical RT/HRT & quantum extremal surface probes?

Ryu,Takayanagi; Hubeny,Rangamani,Takayanagi



Prelude: singularities and string theory

Long history of studying singularities and their resolution in string theory.

Orbifolds (conical singularities): singular for point particles.
Strings → extra light winding modes → resolution modes of singularity.

Null singularities: *e.g.* singular plane waves, null Kasner singularities.
Strings become highly excited in the vicinity of singularity.

Spacelike Big-Bang/Crunch singularities: unclear.
Toy models of holographic cosmologies with Big-Bang/Crunch singularities: bulk gravitational description breaks down. Dual field theory?

Resolvable singularities in string theory and holography?

AdS cosmologies & duals; classical extremal surfaces

AdS Big-Crunch singularities, duals

Das, KN, Trivedi, Awad, Michelson, Nampuri, '06-'08

e.g.

isotropic AdS_5 Kasner: $ds^2 = \frac{R_{AdS}^2}{r^2}(-dt^2 + t^{2/3}dx_i^2 + dr^2), \quad e^\Psi = t^{2/\sqrt{3}}$.

$t = 0$, spacelike Big-Crunch singularity: $g_{YM}^2 = e^{\Psi(t)} \to 0$, weakly coupled CFT?

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Dual theory lives on isotropically Crunching space, with time-dep coupling.

Weak coupling analysis: interactions important. Gauge theory response $\to$ singular.

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Hindsight: Generic severe time-dependent deformations on vacuum state expected to thermalize on long timescales (*i.e.* black hole formation in the bulk).

Suggests non-generic initial conditions for Big-Crunch: dual CFT state nontrivial.

Further interesting insights by Engelhardt, Hertog, Horowitz '14-'16 and other groups.

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($\tilde{A}_\mu$) [Possible caveats: subtle renormalization effects (with near singularity cutoff).]

Large family of such cosmologies: AdS-Kasner (anisotropic), -FRW, -BKL, null singularities etc

AdS/CFT, time-dep deformations $\to ds^2 = \frac{R^2}{r^2}(\tilde{g}_{\mu\nu}dx^\mu dx^\nu + dr^2), \quad \Psi = \Psi(t).$

[Solutions to sugra if: $\tilde{R}_{\mu\nu} = \frac{1}{2}\partial_\mu\Psi\partial_\nu\Psi, \quad \tilde{\Box}\Psi = 0$: deformations constrained.]

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Entanglement: useful holographic probe. Explore behaviour of higher dim extremal surfaces with regard to Big-Crunch singularity *e.g.* AdS Kasner. ▸ RT

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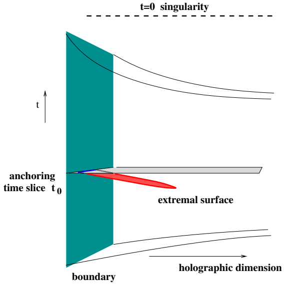
Strip shaped subsystem: codim-2 classical RT/HRT extremal surface $(t(r), x(r))$. Time dependence $\Rightarrow$ surface dips into time direction also (besides radial).

(Manu,KN,Paul)



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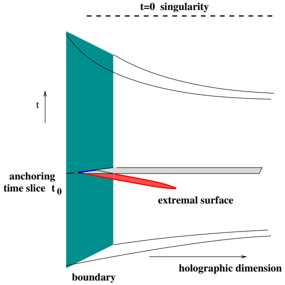

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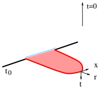

Geometry near turning point (t_*, r_*) & maximin
$\rightarrow$ $[t_* = t(r_*)$; t.p.: $(\partial_r x)^2 \rightarrow \infty$ and $\frac{dt}{dx} = 0]$

Extremization in *semiclassical* region far from singularity $(t = 0)$ is reliable $\rightarrow$

Surface lies almost on $t \sim const$ slice & bends away from singularity $(t_* > t_0)$

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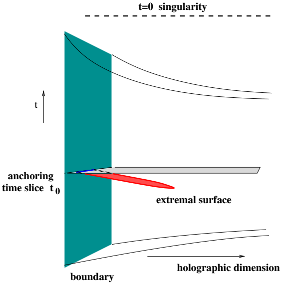

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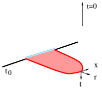

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Area extrmz'n, $x(r)$: $(\partial_r x)^2 = \phi_*^2 \left(\frac{1}{t^{2/d_i}}\right) \frac{1-(\partial_r t)^2}{\phi^2 - \phi_*^2}$; width $l \sim r_*$, $\phi = \frac{t}{r^{d_i}}$;

$t(r)$ extrmz'n eqn $\rightarrow$ $t(r) \sim t_0 + \sum c_n r^n$, $c_n \sim \frac{1}{t_0^{\#}}$ $\rightarrow t_* > t_0$ $(r_* \lesssim t_0)$

Extremization exhibits maximin structure (r-min and t-max). ▸ $t(r)$

2-dim dilaton gravity & cosmologies

Generic 2-dim dilaton gravity

Nearly AdS_2 holography: extremal black holes/branes $\xrightarrow{near\ horizon}$ $AdS_2 \times X$.

Compactify $X \xrightarrow{IR}$ JT gravity + matter: $S_{JT} = \frac{1}{16\pi G_2} \int d^2x \sqrt{-g}\ \phi(\mathcal{R} + 2)$

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Generic 2-dim dilaton gravity $\leftrightarrow$ 2-dim subsector of higher dim gravity

on $\mathcal{M}_2 \times X$: compactify X $\boxed{S = \frac{1}{16\pi G_2} \int d^2x \sqrt{-g}\, \big(\phi\mathcal{R} - U(\phi)\big)}$

dilaton $\phi =$ higher dim transverse area; $U(\phi)$ general dilaton potential.

[Dimensional reduction $ds_D^2 = g_{\mu\nu}^{(2)} dx^\mu dx^\nu + \phi^{2/d_i} d\sigma_{d_i}^2$; Weyl: $g_{\mu\nu} = \phi^{(d_i-1)/d_i} g_{\mu\nu}^{(2)}$]

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"Effective holography": subset of bulk observables & duals.

Generic 2-dim gravity akin to $D \geq 4$ gravity (not "near JT").

[*e.g.*(AdS_{d_i+2} reduction $\rightarrow U = 2\Lambda\phi^{1/d_i}$); observables reflect higher dim theory (holo EM tensor, corrn fns)]

$\boxed{\text{Use 2-dim theory as diagnostic of full higher dim theory.}}$

(adequate for some aspects incl entanglement). (UV-incomplete effective theory)

2dg

2-dim gravity: Big-Crunch singularities

2-dim dilaton gravity: interesting playground for Big-Crunch/Bang?

e.g. higher dim cosmologies $\rightarrow$ dim'nal reduction.

$$\boxed{S = \frac{1}{16\pi G_2} \int d^2x \sqrt{-g} \left(\phi \mathcal{R} - U(\phi, \Psi) - \frac{1}{2}\phi(\partial\Psi)^2 \right)}$$ (Bhattacharya,KN,Paul)

ϕ is 2d dilaton. Extra scalar $\Psi \rightarrow$ nontrivial dynamics.

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EOM: $g_{\mu\nu} \nabla^2\phi - \nabla_\mu \nabla_\nu \phi + \frac{g_{\mu\nu}}{2} \left(\frac{\phi}{2} (\partial\Psi)^2 + U \right) - \frac{\phi}{2} \partial_\mu \Psi \partial_\nu \Psi = 0$;

$\mathcal{R} - \frac{\partial U}{\partial \phi} - \frac{1}{2} (\partial\Psi)^2 = 0; \quad \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, \phi \partial^\mu \Psi) - \frac{\partial U}{\partial \Psi} = 0.$ → conformal gauge $e^f \eta_{\mu\nu}$

Near Big-Crunch singularity:

→ "universal" subsector: $\phi \sim t, \quad e^f \sim t^a, \quad e^\Psi \sim t^\alpha; \quad a = \frac{\alpha^2}{2}$

Rapid time variation → divergence. Time derivatives dominant ⇒ dilaton potential U disappears.

$[-\partial_t^2 \phi + \dot{f} \partial_t \phi - \frac{\phi}{2} (\dot{\Psi})^2 \sim 0, \quad -\partial_t^2 \phi \sim 0, \quad \ddot{f} + \frac{1}{2} (\dot{\Psi})^2 \sim 0, \quad -\partial_t (\phi \partial_t \Psi) \sim 0.]$

2-dim gravity: Big-Crunch singularities

Higher dim cosmology : $ds_D^2 = \frac{e^f}{\phi^{(d_i-1)/d_i}}(-dt^2 + dr^2) + \phi^{2/d_i} dx_i^2 \quad \leftrightarrow$

2-dim fields ϕ, $g_{\mu\nu} = e^f \eta_{\mu\nu}$, Ψ : $\quad \phi = t^k r^m, \quad e^f = t^a r^b, \quad e^\Psi = t^\alpha r^\beta$

[EOM $\rightarrow$ algebraic equations for exponents]

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$$\rightarrow \boxed{\phi = \frac{t\,R^{d_i}}{r^{d_i}}, \qquad ds^2 = \frac{t^{(d_i-1)/d_i}\,R^{d_i+1}}{r^{d_i+1}}(-dt^2 + dr^2), \qquad e^\Psi = t^{\sqrt{2(d_i-1)/d_i}}\,.}$$

$U = 2\Lambda\phi^{1/d_i}$, $\Lambda = -\frac{1}{2}\,d_i(d_i+1)$; $\quad R = AdS$ scale. Kasner scale t_K suppressed, *e.g.* $t^{2p} \rightarrow (t/t_K)^{2p}$.

Various other families of cosmological solutions also exist.

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Various other families of cosmological solutions also exist.

- Flat space: $U = 0 \ \ \leftrightarrow$ reduction of “mostly isotropic” Kasner.
- Hyperscaling violating cosmologies: here $\theta < 0$.
  $U(\phi,\Psi) = 2\Lambda\phi^{1/d_i}\,e^{\gamma\Psi}, \quad \Lambda = -\frac{1}{2}(d_i+1-\theta)(d_i-\theta), \quad \gamma = \frac{-2\theta}{\sqrt{2d_i(d_i-\theta)(-\theta)}}\ .$
- Hyperscaling violating Lifshitz cosmologies: nontrivial z exponent.
  More constrained, complicated.

2-dim dilaton-gravity, Big-Crunches & quantum extremal surfaces

Quantum extremal surfaces

Exciting recent developments in black hole information paradox

Penington; Almheiri,Engelhardt,Marolf,Maxfield; Almheiri,Mahajan,Maldacena,Zhao; ... .

New insights from generalized entropy and quantum extremal surfaces.

2-dim CFT techniques → adapt Calabrese,Cardy formula for subleading bulk matter contribution to entanglement entropy. $S_{gen} = \frac{\phi}{4G_2} + S_{bulk}$

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But perhaps studying simple toy models & quantum extremal surfaces as probes will give insight into cosmological singularities.

We will mostly study closed universes: no horizons (no appreciable entropy), bulk matter in ground state (reasonable far from singularity), no entanglement with "elsewhere" (auxiliary universes).

Quantum extremal surfaces

$S_{gen} = S_{cl} + S_{bulk} = \frac{\phi}{4G_2} + \frac{c}{12} \log \left(\Delta^2 \, e^f \big|_{(t,r)} \right) + \ \ldots$ $\quad (\Delta^2 = r^2 - (t - t_0)^2)$

$\rightarrow$ extremize $\rightarrow$ QES. (retaining only terms relevant for extremization)

[holographic boundary observer at $(t_0, 0)$]

- S_{gen} pertains to 2-dim theory from dim. red'n of higher dim space.
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- 2d CFT matter in ground state: reasonable far from singularity.
- $1 \ll c \ll \frac{1}{G}$: classical area term S_{cl} dominant but S_{bulk} appreciable.
- If S_{bulk} overpowers S_{cl}, Bekenstein bound violated $\rightarrow$ islands.

Hartman,Jiang,Shagoulian

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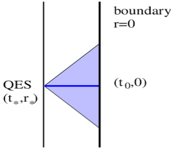
Time-independent examples: everything is on const time slice ($t = t_0$ above).

AdS_{d_i+2} reduction (earlier): $\phi = \frac{R^{d_i}}{r^{d_i}}, \quad ds^2 = \frac{R^{d_i+1}}{r^{d_i+1}}(-dt^2 + dr^2)$

$$S_{gen} = \frac{\phi_r}{4G}\frac{R^{d_i}}{r^{d_i}} + \frac{c}{12}\log\left(\frac{r^2/\epsilon_{UV}^2}{(r/R)^{d_i+1}}\right) \Rightarrow \partial_r S_{gen} = -\frac{\phi_r}{4G}\frac{d_i R^{d_i}}{r^{d_i+1}} - \frac{c}{6}\left(\frac{d_i-1}{2}\right)\frac{1}{r} = 0$$



Both terms negative: $c > 0$ and $d_i > 1$

$\Rightarrow \quad r_* \to \infty \ \to$ entire Poincare wedge

$\equiv$ usual AdS_{d_i+2} entanglement wedge. (maximin)

[$S^{o.s.} \sim 0$: AdS ground state]

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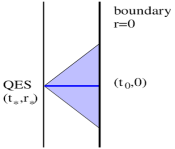

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Can be recast as $S_{gen} \sim \frac{\phi}{4G} + \frac{c}{12}\frac{d_i-1}{d_i}\log\phi$

$\Rightarrow\quad S_{bulk}$ is subleading to classical area [ϕ not too small]

$\to$ Bekenstein bound not violated $\to$ no islands.

Quantum extremal surfaces: AdS Kasner

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$\boxed{t_* \sim t_0\ \textit{i.e.}\ \Delta t \text{ small},\quad r_* \to \infty\ ,\ \ t_* \to \infty.}$

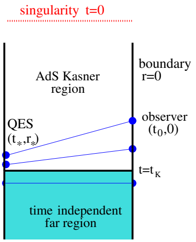
Quantum extremal surfaces: AdS Kasner

$[S_{gen} = \frac{\phi}{4G} + \frac{c}{12}\log(\Delta^2\, e^f|_{(t,r)}), \quad \Delta^2 = r^2 - (\Delta t)^2]$

AdS_{d_i+2} Kasner reduction: $\phi = \frac{t/t_K}{(r/R)^{d_i}}, \quad ds^2 = \frac{(t/t_K)^{(d_i-1)/d_i}}{(r/R)^{d_i+1}}(-dt^2 + dr^2)$

Observer in semiclassical region far from singularity:
mild time-dependence $\Rightarrow$ not too different from AdS.
matter in ground state: reasonable far from singularity.

$\boxed{t_* \sim t_0 \text{ i.e. } \Delta t \text{ small}, \; r_* \to \infty \,, \; t_* \to \infty.}$



$$\frac{3\phi_r}{Gc}\,\frac{d_i\, t/t_K}{r^{d_i+1}/R^{d_i}} + \frac{d_i+1}{r}\left(\frac{\frac{d_i-1}{d_i+1}r^2 - (\Delta t)^2}{r^2 - (\Delta t)^2}\right) = 0\,;$$

$$\frac{c}{6}\,\frac{t-t_0}{\Delta^2} = \frac{\phi_r}{4G}\,\frac{1/t_K}{r^{d_i}/R^{d_i}} + \frac{c}{12}\,\frac{d_i-1}{d_i\, t}\,.$$

QES driven to semiclassical region:
entanglement wedge excludes near singularity region.

[maximin] [2d cosmology reasonable approx'n to higher dim]

Quantum extremal surfaces: AdS Kasner

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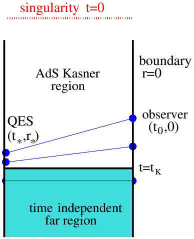

Regulate $r_*\to R_c$ large, finite: (semicl. $t\sim t_0$, $\Delta^2\sim R_c^2$)

Quantum extremal surfaces: AdS Kasner

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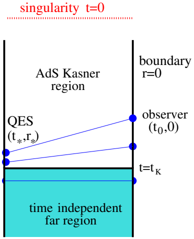

QES spacelike-separated $\Rightarrow \Delta t > 0$ $(\Delta^2 > 0 \to t_* > t_0)$.

QES lags behind observer (direction away from singularity).

$$\frac{\Delta t}{R_c^2} \sim \frac{1}{2K_c} + \frac{d_i-1}{2d_i\, t_0} \qquad \Big[\frac{1}{K_c} = \frac{3\phi_r}{Gc}\frac{1/t_K}{R_c^{d_i}/R^{d_i}}\Big]\,.$$

QES lag Δt increases ($t_0 \downarrow$ as observer evolves in time)

Append time-independent region far from singularity: QES evolution confirms above picture.

Islands?

AdS Kasner: searching for islands

QES $\rightarrow$ semiclassical region: entanglement wedge excludes near singularity region.

$$\frac{3\phi_r}{Gc}\,\frac{d_i\,t/t_K}{r^{d_i+1}/R^{d_i}} + \frac{d_i+1}{r}\left(\frac{\frac{d_i-1}{d_i+1}r^2-(\Delta t)^2}{r^2-(\Delta t)^2}\right) = 0\,; \qquad \frac{c}{6}\,\frac{t-t_0}{\Delta^2} = \frac{\phi_r}{4G}\,\frac{1/t_K}{r^{d_i}/R^{d_i}} + \frac{c}{12}\,\frac{d_i-1}{d_i\,t}\,.$$

Spacelike-separated island-like region for $\sqrt{\frac{d_i-1}{d_i+1}} \lesssim \frac{\Delta t}{r} < 1$? (large finite r QES solution)

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Analysing in detail expanding near this island boundary shows inconsistency:
no island-like solution emerging continuously from QES in semiclassical region.

Consistent with previous studies of closed universes with no horizons,
no entanglement with “elsewhere”, no flat non-gravitating bath regions.

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More general 2-dim holographic cosmologies (incl hyperscaling violating Lifshitz asymptotics):

$$\phi = tr^m\,, \quad e^f = t^a r^b\,, \quad a>0, \quad m<0, \quad b<0.$$

Qualitatively similar:
QES $\rightarrow$ semiclassical region: entanglement wedge excludes near singularity region.
Potential island-like QES solution shows inconsistency generically.

Quantum extremal surfaces: Null Kasner

Null Kasner singularities: $ds^2 = -dx^+ dx^-, \quad \phi = \phi(x^+), \quad \Psi = \Psi(x^+)$ $[x^\pm = t \pm r]$

Higher dim null cosmology $ds^2 = -\phi^{-(d_i-1)/d_i} dx^+ dx^- + \phi^{2/d_i} dy_i^2$ [no holography here]

Only x^+-dependence $\Rightarrow$ *Holomorphic structure*

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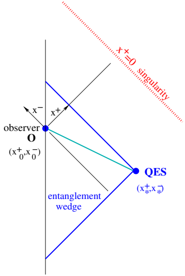

EOM $-\partial_+^2\phi - \frac{\phi}{2}(\partial_+\Psi)^2 = 0 \;\Rightarrow$

$$\boxed{0 < k \le 1\,, \quad \phi = (-x^+)^k\,, \quad e^\Psi = (-x^+)^{\pm\sqrt{2k(1-k)}}} \qquad (x^+ < 0)$$

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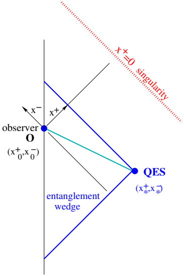

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Spacelike-separated QES: $\Delta^2 = -\Delta x^+ \Delta x^- > 0.$

$$\boxed{\begin{array}{ll} \Delta x^+ > 0, & x_*^+ = x_0^+ + \frac{2Gc}{3k\phi_r}(-x_*^+)^{1-k} > x_0^+ \\ \Delta x^- < 0, & x_*^- \to X_c^- \sim -\infty \quad \text{(spatial regulator)} \end{array}} \qquad \text{(null-time max)}$$

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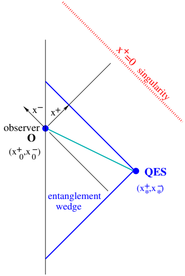

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$x_0^+,\ x_*^+$ not driven to semiclassical region. QES lies towards singularity.

Qualitatively different from AdS Kasner etc: $x^+ = 0$ is a solution when $x_0^+ = 0$
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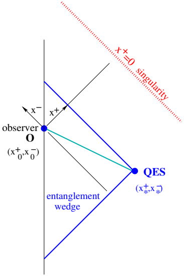

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However on-shell generalized entropy appears singular generically: thus near singularity region best regarded as unreliable.

Strings known to become highly excited in the vicinity of null singularities: perhaps consistent with above. Explore?

Conclusions, questions

- *AdS* Kasner: classical RT/HRT surfaces anchored in reliable semiclassical region bend away from singularity. Quantum extremal surfaces $\rightarrow$ semiclassical region.
- Null Kasner singularities: QES can reach singularity but on-shell generalized entropy generically singular near singularity.

Appropriate non-ground state models for bulk matter entropy ("stringy entanglement"), initial conditions in semiclassical region far from singularity: dual CFT state?

Which Big-Crunch singularities are accessible via entanglement?

Black hole interior singularity? could be qualitatively different from Big-Bang/Crunch ...

- dS: timelike-separated QES generically (spacelike-separated QES with regulator: not late times)

Classically: dS future-past extremal surfaces – way to organize bulk entanglement.

Maybe de Sitter suggests new object, "temporal entanglement"?

Quantum extremal surfaces: de Sitter (Poincare)

dS_{d_i+2}: $ds^2 = \frac{R^2}{\tau^2}(-d\tau^2 + dx^2 + dy_i^2) \quad \rightarrow \quad \phi = \frac{R^{d_i}}{(-\tau)^{d_i}}\,, \quad ds^2 = \frac{R^{d_i+1}}{(-\tau)^{d_i+1}}(-d\tau^2 + dx^2)$

Generalized entropy: $S_{gen} = \frac{\phi_r}{4G}\,\frac{R^{d_i}}{(-\tau)^{d_i}} + \frac{c}{6}\log\Big(\Delta^2\,\frac{R^{(d_i+1)/2}}{(-\tau)^{(d_i+1)/2}}\Big)$, $\quad \Delta^2 = (\Delta x)^2 - (\tau-\tau_0)^2$

Extremization: $\boxed{\frac{c}{3}\,\frac{\Delta x}{\Delta^2} = 0\,, \quad \frac{d_i\phi_r}{4G}\,\frac{R^{d_i}}{(-\tau)^{d_i+1}} + \frac{c}{12}\,\frac{d_i+1}{(-\tau)} - \frac{c}{3}\,\frac{\tau-\tau_0}{\Delta^2} = 0}$

- <u>Timelike-separated QES:</u> ($d_i = 1 \leftrightarrow dS_2$, Chen,Gorbenko,Maldacena)

$$\Delta x = 0\,, \quad \Delta^2 = -(\tau-\tau_0)^2\,; \qquad \frac{d_i\phi_r}{4G}\,\frac{R^{d_i}}{(-\tau)^{d_i+1}} + \frac{c}{12}\,\frac{d_i+1}{(-\tau)} + \frac{c}{3}\,\frac{1}{\tau-\tau_0} = 0$$

Late-time observer $\tau_0 \sim 0$: $\quad \Delta x = 0\,, \quad \tau_* = -R\big(\frac{d_i}{3-d_i}\,\frac{3\phi_r}{Gc}\big)^{1/d_i}$

Timelike-separated $\Rightarrow \Delta^2 < 0 \rightarrow$ generalized entropy acquires imaginary part.

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  Timelike-separated $\Rightarrow \Delta^2 < 0 \rightarrow$ generalized entropy acquires imaginary part.

- Spacelike-separated QES: exist in certain regimes with regulator.

  $$\Delta^2 \sim R_c^2\,, \quad \frac{d_i\phi_r}{4G}\frac{R^{d_i}}{(-\tau)^{d_i+1}} + \frac{c}{12}\frac{d_i+1}{(-\tau)} \sim \frac{c}{3}\frac{\tau-\tau_0}{R_c^2}.$$

  $*$ $R_c \rightarrow \infty \ \Rightarrow \tau \rightarrow -\infty$. $\quad *$ Late-times $\rightarrow$ no real solution.

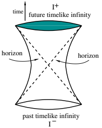
FRW, scalar source $p = w\rho$: $ds^2 = -dt^2 + a(t)^2 dx_i^2 \ \rightarrow\ \phi = a^{d_i}\,, \quad ds^2 = a^{d_i+1}(-d\tau^2 + dx^2)$

de Sitter: classical extremal surfaces

dS/CFT: dual CFT on boundary at future/past timelike infinity $\mathcal{I}^{\pm}$

'01 Strominger; Witten. $\boxed{\Psi_{dS}^{HH} = Z_{CFT}}$ (Maldacena '02)

analytic continuation $r \to -i\tau,\ R_{AdS} \to -iR_{dS}$ from Eucl AdS.

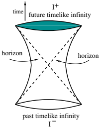


Bulk expectation values $\langle\varphi_k\varphi_{k'}\rangle \sim \int D\varphi\ \varphi_k\varphi_{k'}|\Psi|^2 \to$ dual $\equiv$ two CFT copies.

de Sitter entropy = area of cosmological horizon. (Gibbons,Hawking)

de Sitter entropy as some sort of entanglement entropy?

de Sitter: classical extremal surfaces



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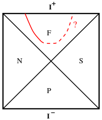


One possible generalization of Ryu-Takayanagi to de Sitter space $\equiv$ bulk analog of setting up entanglement entropy in dual CFT $\rightarrow$ restrict to some boundary Eucl time slice $\rightarrow$ codim-2 RT/HRT surfaces anchored at I^+, dipping into holographic (time) direction.

No real $I^+ \rightarrow I^+$ turning point, surfaces do not return to I^+: end at I^-?

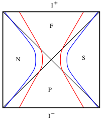
de Sitter: classical extremal surfaces

No real $I^+ \to I^+$ turning point, surfaces do not return to I^+: end at I^-?

Hartman-Maldacena rotated $\boxed{\text{Future-past surfaces stretching from } I^+ \text{ to } I^-}$

Limiting surface, subregion $\to$ all $I^\pm$ ($\Delta w \to \infty$): $S^{div} \sim \frac{\pi l^2}{G_4} \frac{l}{\epsilon_c}$, $S^{fin} \sim \frac{\pi l^2}{G_4} \Delta w$

Features: vanishing mutual information, "entanglement wedge", subregion duality, ...



Suggest TFD-like entangled dual of two CFT copies at future boundary.

With ordinary spatial (AdS-like) boundary, a spacelike RT/HRT surface gives real area. Then as the holographic boundary is rotated to timelike infinity, we acquire a relative minus sign, suggesting complex areas for timelike extremal surfaces. In this sense, future-past surfaces have overall i which we are removing.
(a bit like calling the length of a timelike geodesic as time, rather than i·space ?)

So perhaps this is a new object, "temporal entanglement"?

[Also complex extremal surfaces: AdS RT analytic cont'n ('15, KN; Sato; Miyaji,Takayanagi)]

Generic 2-dim dilaton gravity: dim. red'n

$$\boxed{S = \frac{1}{16\pi G_2}\int d^2x\sqrt{-g}\,(\phi\mathcal{R} - U(\phi))}$$

Equations of motion:
$$g_{\mu\nu}\nabla^2\phi - \nabla_\mu\nabla_\nu\phi + \frac{g_{\mu\nu}}{2}U = 0, \quad \mathcal{R} - \frac{\partial U}{\partial\phi} = 0.$$

- curvature divergence: $U \sim \phi^n \to \mathcal{R} \sim \phi^{n-1} \xrightarrow{n<1}$ IR singularity as $\phi \to 0$. Conformally AdS_2.
- on-shell action divergence: UV divergence at large ϕ — $S^{o.s.} = \frac{1}{16\pi G_2}\int d^2x\sqrt{-g}\,(\phi\partial_\phi U - U)$.

[Distinct from JT gravity ($U = -2\phi$): const curvature ($R = -2$); vanishing on-shell action.]

AdS/CFT experience: fixable - •introduce black hole •counterterms, holographic RG (stress tensor, correlators).

Suggests generic 2d gravity akin to $D \geq 4$ gravity $\to$ not "near JT".

UV-incomplete effective theory $\equiv$ thermodynamic ensemble. Extra scalar: similar features.

---

e.g. AdS_{d_i+2} reduction: bulk action & Gibbons-Hawking boundary term $\to$ renormalized with counterterm

$$S_{ren} = \frac{1}{16\pi G_2}\Big[\int d^2x\sqrt{-g}\,(\phi\mathcal{R} - U(\phi)) - 2\int dt\sqrt{-\gamma}\,\phi K - 2\int dt\sqrt{-\gamma}\,\phi^{\frac{d_i+1}{2d_i}}\,d_i\Big]; \qquad (U = 2\Lambda\phi^{\frac{1}{d_i}})$$

Varying renormalized action w.r.t. boundary metric $\hat\gamma \to$ holographic energy-momentum tensor

$$\boxed{T^{ren}_{tt} = -\frac{2}{\sqrt{-\hat\gamma}}\frac{\delta S_{ren}}{\delta\hat\gamma^{tt}} = \frac{\epsilon^{(d_i+1)/2}}{8\pi G_2}\Big(-\sqrt{g^{rr}}\partial_r\phi - d_i\,\phi^{\frac{d_i+1}{2d_i}}\Big)\gamma_{tt}}$$

vacuum: $T^{ren}_{tt} = 0$. Black hole - $T^{ren}_{tt} = \frac{d_i}{16\pi G_2\, r_0^{d_i+1}}$

T^{ren}_{tt} is ϵ-expansion: counterterms cancel divergences, pick finite pieces $\lim_{\epsilon\to 0}$.

Matches familiar higher dim δS^{grav} calculation, after reconciling redux+Weyl:
$$-\frac{1}{16\pi G_D}\int d^{d_i+1}x\sqrt{-h}\,(K_{\mu\nu} - Kh_{\mu\nu})\delta h^{\mu\nu} \to \frac{1}{16\pi G_2}\int dt\sqrt{-\gamma}\,\sqrt{g^{rr}}\partial_r\phi\,\gamma_{tt}\,\delta\gamma^{tt} + \ldots$$

Scalar probes with dilaton coupling: $-\int d^2x\sqrt{g}\,\phi_B(\partial\psi)^2 \;\to\; \psi = e^{-i\omega\tau}\psi_\omega(r) \xrightarrow{d_i\to 2}$

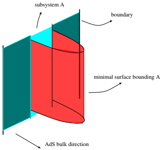
$-S \sim \int d\omega\,\frac{1}{\epsilon^{d_i}}\,\psi_{-\omega}\partial_r\psi_\omega\Big|_\epsilon \;\to\; \int d\omega\,\psi^0_{-\omega}\psi^0_\omega\left(\frac{-\omega^2}{\epsilon} + \omega^3 + \ldots\right)$. Remove divergence via counterterm $\to$

nonlocal term $\to \langle O(\tau)O(\tau')\rangle \sim \frac{1}{(\Delta\tau)^4}$. Similar to higher dim calc'n: 0+1 dim FT $\to O$ conf.dim. $1+\frac{d_i}{2}$.

Holographic entanglement entropy

Entanglement entropy: entropy of reduced density matrix of subsystem.

EE for spatial subsystem A, $S_A = -tr\rho_A \log \rho_A$, with partial trace $\rho_A = tr_B\,\rho$.



Ryu-Takayanagi: $EE = \frac{A_{min.surf.}}{4G}$

[$\sim$ black hole entropy] Area of codim-2 minimal surface in gravity dual.

Non-static situations: extremal surfaces (Hubeny, Rangamani, Takayanagi).

Operationally: const time slice, boundary subsystem $\rightarrow$ bulk slice, codim-2 extremal surface

Example: CFT$_d$ ground state = empty AdS_{d+1}, $ds^2 = \frac{R^2}{r^2}(dr^2 - dt^2 + dx_i^2)$.

Strip, width $\Delta x = l$, infinitely long. Bulk surface $x(r)$. Turning point r_*.

$$S_A = \frac{V_{d-2}R^{d-1}}{4G_{d+1}} \int \frac{dr}{r^{d-1}}\sqrt{1+(\partial_r x)^2} \quad \rightarrow \quad (\partial_r x)^2 = \frac{(r/r_*)^{2d-2}}{1-(r/r_*)^{2d-2}}\,, \qquad \frac{l}{2} = \int_0^{r_*} dr\partial_r x.$$

$$S_A = \frac{V_{d-2}R^{d-1}}{4G_{d+1}} \int_\epsilon^{r_*} \frac{dr}{r^{d-1}} \frac{2}{\sqrt{1-(r/r_*)^{2d-2}}} \quad \longrightarrow \quad S_A = \frac{R}{2G_3}\log\frac{l}{\epsilon}\,, \qquad \frac{3R}{2G_3} = c \quad [2d].$$

$$S_A \sim \frac{R^{d-1}}{G_{d+1}}\left(\frac{V_{d-2}}{\epsilon^{d-2}} - c_d\frac{V_{d-2}}{l^{d-2}}\right), \qquad \frac{R^3}{G_5} \sim N^2 \quad [4d], \quad \frac{R^2}{G_4} \sim N^{3/2} \quad [3d].$$

CFT thermal state (AdS black brane): minimal surface wraps horizon. $S^{fin} \sim N^2T^3V_2 l$

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Big-Crunches, extremal surfaces

Surface $(t(r), x(r))$. Trng pt (t_*, r_*), $t_* = t(r_*)$. $S = \frac{V_{d_i-1}}{4G_{d_i+2}} \int dr\, \phi \sqrt{\frac{e^f}{\phi^{(d_i+1)/d_i}}(1-(\partial_r t)^2) + (\partial_r x)^2}$

$$\longrightarrow \quad (\partial_r x)^2 = A^2 \frac{\frac{e^f}{\phi^{(d_i+1)/d_i}}(1-(\partial_r t)^2)}{\phi^2 - A^2}, \qquad S = \frac{V_{d_i-1}}{4G_{d_i+2}} \int dr\, \frac{e^{f/2}\, \phi^{(3-1/d_i)/2}}{\sqrt{\phi^2 - A^2}} \sqrt{1-(\partial_r t)^2}$$

$$\boxed{\text{Time extrmz'n} \rightarrow \ (1-t'^2)\left(d_i^2 t' + \frac{r(t^2 - A^2 r^{2d_i})}{t^3} - \frac{d_i r}{t}\right) - \frac{(t^2 - A^2 r^{2d_i}) d_i r t''}{t^3} = 0}$$

$$\phi_* = A = \frac{t_*}{r_*^{d_i}} \ \rightarrow \ t'_* = \frac{r_*}{d_i\, t_*} > 0 \ \rightarrow \ \frac{dr}{dx}\Big|_* = 0, \quad \frac{dt}{dx}\Big|_* = \frac{t'_*}{(\partial_r x)_*} = 0 \ \Rightarrow \ t\text{-max (maximin)}$$

$$t' \ll 1 \ \rightarrow \ t\text{-eqn} \xrightarrow{approx} d_i^2 t^3 t' + r(t^2 - A^2 r^{2d_i}) - d_i r t^2 - (t^2 - A^2 r^{2d_i}) d_i r t'' = 0$$

Numerically solve $\longrightarrow$ AdS_5 Kasner: $A \sim \frac{t_0}{r_*^3} \gtrsim \frac{1}{t_0^2}$

$$t(r) = t_0 + \frac{1}{12 t_0} r^2 - \frac{1}{432 t_0^3} r^4 + \frac{1}{7776 t_0^5} r^6 + \frac{A^2}{160 t_0^3} r^8 - \frac{103 A^2}{160\cdot 540 t_0^5} r^{10}$$
$$+ \frac{3943 A^2}{160\cdot 540\cdot 252 t_0^7} r^{12} + \frac{A^4}{160\cdot 4 t_0^5} r^{14} - \frac{7A^4}{160\cdot 72 t_0^7} r^{16} + \frac{15011 A^4}{160\cdot 540\cdot 1120 t_0^9} r^{18}$$
$$+ \frac{91 A^6}{160\cdot 880\, t_0^7} r^{20} + \frac{8453 A^6}{22302720\, t_0^9} r^{22} + \frac{493338049 A^6}{3653185536000\, t_0^{11}} r^{24} + \frac{19 A^8}{56320 t_0^9} r^{26} + \ldots$$

$$\rightarrow \ t_* = t_0 + \frac{r_*^2}{t_0}\left(\frac{1}{12} + \frac{1}{160} + \frac{1}{160\cdot 4} + \frac{91}{160\cdot 880} + \frac{19}{160\cdot 352} + \ldots\right) \ \rightarrow \ t_* > t_0.$$

Surface lies almost on $t \sim const$ slice & bends away from singularity $(t_* \gtrsim t_0)$

IR limit $r_* \rightarrow \infty$ more delicate: $A \rightarrow 0 \quad (A \lesssim 1/t_0^2)$

$$t(r) = t_0 + \frac{1}{12\, t_0} r^2 - \frac{1}{432\, t_0^3} r^4 + \frac{1}{7776\, t_0^5} r^6 - \frac{17}{1866240\, t_0^7} r^8 + \frac{247}{335923200\, t_0^9} r^{10} + \ldots$$

$$r_* \rightarrow \infty, \quad t_0 \rightarrow \infty, \quad \frac{t_0}{r_*} \lesssim 1 \quad \Rightarrow \quad t_* > t_0$$

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