de Sitter Extremal Surfaces and Entanglement in Ghost Systems

K. Narayan
Chennai Mathematical Institute

- de Sitter space, dS/CFT and extremal surfaces
- Entanglement entropy in some $c = -2$ ghost CFTs
- Entangled "ghost-spins" and spins

Based on arXiv:1501.03019, 1504.07430, 1602.06505 (KN),
1608.08351 (Dileep Jatkar, KN), and in progress.

Entanglement Entropy, Holography

Entanglement entropy: entropy of reduced density matrix of subsystem.

EE for spatial subsystem A, $S_A = -tr\rho_A \log \rho_A$, with partial trace $\rho_A = tr_B \rho$.

Quantum Field Theory: in general difficult to compute EE.

Corrl'ns strongest near interface $\rightarrow$ leading scaling, d-dim area law $\mathcal{N}_{dof} \frac{V_{d-2}}{\epsilon^{d-2}}$.

[$\epsilon =$ UV cutoff] (Bombelli, Koul, Lee, Sorkin; Srednicki) [exceptions: 2d CFT, Fermi surfaces]

2-dim conformal field theory (single interval): $S_A = \frac{c}{3} \log \frac{l}{\epsilon}$ ($c =$ central charge)

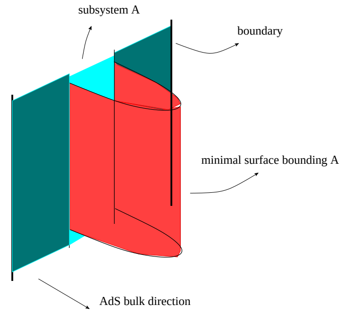
(Holzhey,Larsen,Wilczek) ["replica": $tr\rho_A^n = \frac{Z_n}{(Z_1)^n}$, $S_A^{EE} = -\lim_{n\rightarrow 1} \partial_n tr\rho_A^n$ (Calabrese,Cardy)]

Ryu-Takayanagi: $EE = \frac{A_{min.surf.}}{4G}$ [motivated by black hole entropy]

Codim-2 minimal surface in gravity dual.

Substantial evidence by now (see recent Lewkowycz, Maldacena).

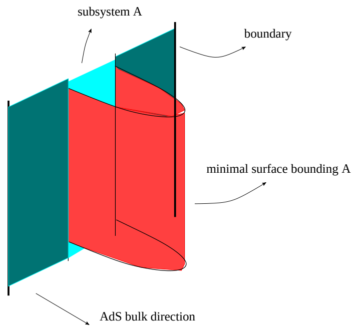
Non-static situations: extremal surfaces.
(Hubeny, Rangamani, Takayanagi)



EE a bulk surface probe [akin to correlation fns (geodesics), Wilson loops (bulk strings), . . .]

Holographic Entanglement Entropy

Ryu-Takayanagi: $EE = \frac{A_{min.surf.}}{4G}$



(i) Define boundary spatial subsystem on const time slice ,

(ii) corresponding const time slice in bulk, surface bounding subsystem,

(iii) extremize codim-2 surface area functional $\rightarrow$ minimal area.

Example: CFT ground state = empty AdS_{d+1}, $ds^2 = \frac{R^2}{r^2}(dr^2 - dt^2 + dx_i^2)$.

Strip, width $\Delta x = l$, infinitely long. Bulk surface $x(r)$. Turning point r_*.

$S_A \sim \frac{R^{d-1}}{G_{d+1}}\left(\frac{V_{d-2}}{\epsilon^{d-2}} - c_d \frac{V_{d-2}}{l^{d-2}}\right)$, $\frac{R^3}{G_5} \sim N^2$ [4d], $\frac{R^2}{G_4} \sim N^{3/2}$ [3d].

$S_A = \frac{R}{2G_3} \log \frac{l}{\epsilon}$, $\frac{3R}{2G_3} = c$ [2d].

CFT thermal state (AdS black brane): minimal surface wraps horizon. $S^{fin} \sim N^2 T^3 V_{d-2} l$

Spherical extremal surfaces: subleading log-div. $\rightarrow$ anomaly. Casini, Huerta, Myers derive EE.

$$[S_A = \frac{1}{4G_{d+1}} \int_{-\infty}^{\infty} \prod_{i=1}^{d-2} \frac{R dy_i}{r} \int \frac{R\sqrt{dr^2+dx^2}}{r} = \frac{V_{d-2} R^{d-1}}{4G_{d+1}} \int \frac{dr}{r^{d-1}} \sqrt{1+(\partial_r x)^2} \quad \rightarrow$$

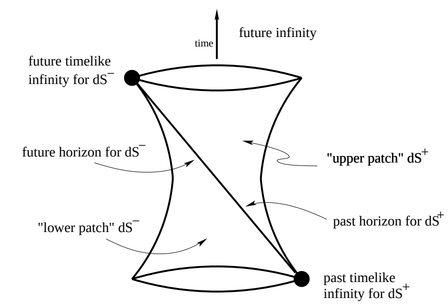
$$\frac{l}{2} = \int_0^{r_*} \frac{dr\ (r/r_*)^{d-1}}{\sqrt{1-(r/r_*)^{2d-2}}}, \qquad S = \frac{V_{d-2} R^{d-1}}{4G_{d+1}} \int_\epsilon^{r_*} \frac{dr}{r^{d-1}} \frac{2}{\sqrt{1-(r/r_*)^{2d-2}}}. \ ]$$

de Sitter space and dS/CFT

de Sitter space $ds^2 = \frac{R_{dS}^2}{\tau^2}(-d\tau^2 + dw^2 + dx_i^2)$.

Fascinating for various reasons.

dS/CFT: fluctuations about dS encoded in dual Euclidean non-unitary CFT on boundary at future timelike infinity $\mathcal{I}^+$ (Strominger; Witten).



(Maldacena '02) analytic continuation $r \to -i\tau,\ R_{AdS} \to -iR_{dS}$ from Eucl AdS $\to$ Hartle-Hawking wavefunction of the universe $\Psi[\varphi] = Z_{CFT}$.

Energy-momentum tensor $\langle TT \rangle$ 2-pt fn $\to$ dual CFT central charge

$\mathcal{C}_d \sim i^{1-d} \frac{R_{dS}^{d-1}}{G_{d+1}}$, negative or imaginary. $\mathcal{C}_3 \sim -\frac{R_{dS}^2}{G_4}$ for dS_4.

Anninos,Hartman,Strominger: Higher-spin dS_4 dual to $Sp(N)$ ghost $CFT_3, \ldots$

[Bulk EAdS regularity conditions, deep interior $\to$ Bunch-Davies initial conditions in deSitter, $\varphi_k(\tau) \sim e^{ik\tau}$, for large $|\tau|$. $Z_{CFT} = \Psi[\varphi] \sim e^{iS_{cl}[\varphi]}$ (semiclassical).

[Dual CFT: $\langle O_k O_{k'} \rangle \sim \frac{\delta^2 Z}{\delta\varphi_k^0 \varphi_{k'}^0}$] [Bulk expectation values $\langle f_1 f_2' \rangle \sim \int D\varphi f_1 f_2' |\Psi|^2$.]

Wavefunction $\Psi[\varphi]$ not pure phase $\to$ complex saddle points contribute to observables.]

de Sitter extremal surfaces

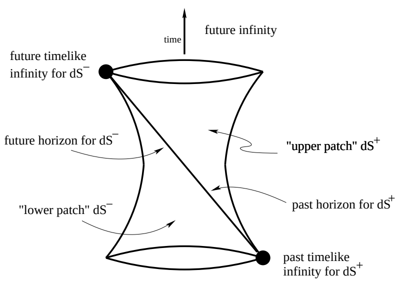
A speculative generalization of Ryu-Takayanagi to de Sitter space $\rightarrow$

dS (Poincare): $ds^2 = \frac{R_{dS}^2}{\tau^2}(-d\tau^2 + dw^2 + dx_i^2)$

Eucl time slice $w = const$,

subregion at future timelike infinity $\rightarrow$

codim-2 extremal surfaces in de Sitter space.



$\longrightarrow$ bulk analog of setting up entanglement entropy in dual CFT: consider boundary Euclidean time slice, construct spatial subsystem, trace over complement.

- Exploring this $\rightarrow$ complex extremal surfaces, negative area, dS_4.
  Recall dS/CFT via $Z_{CFT} = \Psi_{dS}$: $dS_4 \ \rightarrow \ c < 0$.
- Replica in 2-dim toy ghost CFTs gives negative EE.
- "Ghost-spins", toy models for systems with negative norm states.
  Reduced density matrix after tracing over some ghost-spins has some negative eigenvalues $\rightarrow Re(EE) < 0$.

de Sitter extremal surfaces

de Sitter extremal surfaces

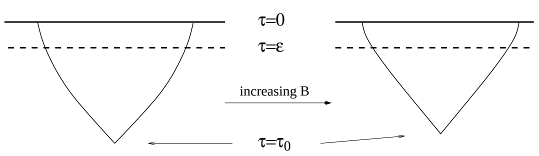
de Sitter (Poincare): $ds^2_{d+1} = \frac{R^2_{dS}}{\tau^2}(-d\tau^2 + dw^2 + dx_i^2) \quad \rightarrow$ EE in dual Eucl CFT $\rightarrow$

bulk: Eucl time slice $w = const$, subregion at future timelike infnty $\rightarrow$ codim-2 extremal surface.

Expectations based on dual CFT central charge being negative or pure imaginary suggest real surfaces will not work $\longrightarrow$

[strip] $S_{dS} = \frac{1}{4G_{d+1}} \int \prod_{i=1}^{d-2} \frac{R_{dS} dy_i}{\tau} \frac{R_{dS}}{\tau} \sqrt{dx^2 - d\tau^2}$

$\longrightarrow \qquad S_{dS} \propto \frac{R^{d-1}_{dS} V_{d-2}}{4G_{d+1}} \int \frac{d\tau}{\tau^{d-1}} \sqrt{1 - \dot{x}^2}.$



$Extremize \rightarrow \dot{x}^2 = \frac{B^2 \tau^{2d-2}}{1+B^2\tau^{2d-2}}, \quad B^2 = const$, conserved quantity.

• Sign difference from $AdS \Rightarrow$ no real "turning point". $x(\tau)$ hyperboloid.

Join two half-extremal-surfaces with cusp $\rightarrow$ minimize area $\rightarrow$ null surface. Area vanishes.

Real codim-2 surfaces: featureless, no apparent relation to EE.

["outward bending" surfaces $\rightarrow$ null, $S_{dS} = 0$] [surfaces $x(\tau) = const$: $B = 0$, max area]

[Codim-1 surfaces: similar structure.]

de Sitter extremal surfaces

de Sitter (Poincare): $ds^2_{d+1} = \frac{R^2_{dS}}{\tau^2}(-d\tau^2 + dw^2 + dx_i^2) \quad \rightarrow \quad$ EE in dual Eucl CFT $\rightarrow$

bulk: Eucl time slice $w = const$, subregion at future timelike infnty $\rightarrow$ codim-2 extremal surface.

$$[\text{strip}] \quad S_{dS} = \frac{R_{dS}^{d-1} V_{d-2}}{4G_{d+1}} \int \frac{1}{\tau^{d-1}} \sqrt{dx^2 - d\tau^2} = \frac{R_{dS}^{d-1} V_{d-2}}{4G_{d+1}} \int \frac{d\tau}{\tau^{d-1}} \sqrt{\dot{x}^2 - 1}$$

$Extremize \quad \rightarrow \quad (\partial_\tau x)^2 = \frac{-A^2\tau^{2d-2}}{1-A^2\tau^{2d-2}}$. $\qquad$ [$A^2 < 0$ is earlier real-τ solution]

dS_4/CFT_3: consider $A^2 > 0$. Near $\tau \rightarrow 0$: $\quad \dot{x}^2 \sim -A^2\tau^4 \quad i.e.$

$x(\tau) \sim \pm iA\tau^3 + x(0)$. $\quad$ This is spatial direction in Eucl CFT $\Rightarrow$

$x(\tau)$ real-valued $\Rightarrow \quad \tau = iT$ $\qquad$ [can show width Δx also real]

$x(\tau) \rightarrow$ complex extremal surface, τ along imaginary path $\tau = iT$.

$(\frac{dx}{dT})^2 = \frac{A^2T^4}{1-A^2T^4}$. $\quad$ Note turning point: $T_* = \frac{1}{\sqrt{A}} \quad$ (where $|\dot{x}|^2 \rightarrow \infty$).

Can now smoothly join half-extremal-surfaces at turning point.

Overall sign: match with dS_4/CFT_3 central charge, and conformal anomaly (sphere).

Can generalize to dS_{d+1} extremal surfaces.

de Sitter extremal surfaces, dS/CFT

Complex extremal surfaces: compare dS_{d+1}/CFT_d central charges.

[Strip width l real (CFT spatial direction) $\Rightarrow$ path $\tau = iT \;\rightarrow$ extremal surface with turning point]

dS_4: area $\quad S_{dS_4} = -\frac{R_{dS}^2}{4G_4} V_1 \int_\epsilon^l \frac{dT/T^2}{\sqrt{1-T^4/T_*^4}} \sim -\frac{R_{dS}^2}{G_4} V_1 (\frac{1}{\epsilon} - c\frac{1}{l})$

dS_{d+1}, even d: area $\quad S_{dS} = i^{1-d} \frac{R_{dS}^{d-1}}{2G_{d+1}} V_{d-2} \int_\epsilon^{T_*} \frac{dT/T^{d-1}}{\sqrt{1+(-1)^{d-1}(T/T_*)^{2d-2}}}$

$$\sim i^{1-d} \frac{R_{dS}^{d-1}}{2G_{d+1}} V_{d-2} (\frac{1}{\epsilon^{d-2}} - c_d \frac{1}{l^{d-2}})$$

$\equiv$ analytic continuation $r \rightarrow -i\tau, R \rightarrow -iR_{dS}$ from AdS Ryu-Takayanagi.

$S_{AdS}[R, x(r), r] = \frac{R^{d-1}}{4G_{d+1}} V_{d-2} \int \frac{dr}{r^{d-1}} \sqrt{1 + (\frac{dx}{dr})^2}\,, \qquad (x')^2 = \frac{A^2 r^{2d-2}}{1-A^2 r^{2d-2}} \;\rightarrow$

$\dot{x}^2 = \frac{-(-1)^{d-1} A^2 \tau^{2d-2}}{1-(-1)^{d-1} A^2 \tau^{2d-2}}, \qquad S_{dS} = -i \frac{R_{dS}^{d-1}}{4G_{d+1}} V_{d-2} \int \frac{d\tau}{\tau^{d-1}} \frac{1}{\sqrt{1-(-1)^{d-1} A^2 \tau^{2d-2}}}.$

- leading "area law" divergence $\;\mathcal{C}_d \frac{V_{d-2}}{\epsilon^{d-2}} \;\rightarrow$

  central charges $\mathcal{C}_d = i^{1-d} \frac{R_{dS}^{d-1}}{G_{d+1}}$ match dS/CFT using $Z_{CFT} = \Psi$.

- finite cutoff-independent parts $\sim\; i^{1-d} \frac{R_{dS}^{d-1}}{G_{d+1}} \frac{V_{d-2}}{l^{d-2}}$.
- Spherical extremal surfaces: subleading log-div. Anomaly coeff exactly matches Ψ log-coeff.
- dS_4 black brane, CFT_3 at uniform energy density: S_{dS}^{fin} resembles extensive thermal entropy.

dS_4 surfaces, negative EE

dS_4 [strip]: $S_A \sim -\frac{R_{dS}^2}{G_4}(\frac{V_1}{\epsilon} - \frac{V_1}{l}) < 0$ $\quad\rightarrow\quad$ various odd features:

- Mutual information $I[A,B] = S[A] + S[B] - S[A \cup B]$ negative definite

  for disjoint strip subregions A, B sufficiently nearby (vanishes beyond critical separation).

- Bigger subregion more ordered than smaller one.

  Consider two strip subregions, width l_2 and $l_1 > l_2$ $(l_1, l_2 \ll V_1)$.

  Then $S(l_1) - S(l_2) = -\frac{R_{dS}^2}{G_4}(\frac{V_1}{l_2} - \frac{V_1}{l_1}) < 0\,,\ \ i.e.\ \ S(l_1) < S(l_2)$

  [conventional unitary CFT: $S(l_1) > S(l_2)$, $i.e.$ bigger subregion more disordered]

- Entropic c-function $c(l)$ increases: degrees of freedom *integrated in*?

  $c(l) = \frac{l^{d-1}}{V_{d-2}}\frac{dS_A}{dl}$. $\ c(l) \equiv \frac{l^2}{V_1}\frac{dS_A}{dl} = -\frac{R_{dS}^2}{G_4} < 0$ $\ i.e.$ as l increases,

$S_A(l)$ decreases. Asymptotically dS_4 spaces, $S_A < 0 \Rightarrow c'(l) > 0$, $i.e.$ $c(l)$ increases with l.

$|\tau_*| \sim l$: increasing size $l \rightarrow$ going to larger $|\tau_*|$ (earlier times in past).

dS complex extremal surfaces area resembles EE in dual CFT:

from $Z_{CFT} = \Psi$, note CFT is non-unitary ($c < 0$ for dS_4).

Bulk EE uses bulk reduced density matrix (via $\Psi^*\Psi$) $\neq$ EE$_{dS/CFT}$.

(Maldacena, Pimentel)

Negative EE, 2-dim ghost CFTs, replica

Negative entanglement, 2-dim CFT

These complex dS extremal surfaces with dS/CFT in mind suggest negative EE in dual CFT$_3$ for dS_4 (negative central charge).

Can $EE < 0$ at all arise from a (i) CFT or (ii) QM calculation?

2-dim ghost CFTs ($c < 0$) — toy models for studying EE ($e.g.$ replica).

Stress tensor $T(w) = (\partial_w z)^2 T(z) + \frac{c}{12}\{z, w\}$ under conformal transfmn

$w \to z$, with Schwarzian derivative $\{z, w\} = \frac{2\partial_w^3 z \partial_w z - 3(\partial_w^2 z)^2}{2(\partial_w z)^2}$.

($w = x + it_E$ with t_E Euclidean time)

Subsystem A — single interval (betw $x = u, v$ on slice $t_E = const$).

Replica w-space $\to$ z-plane under conf transfmn $z = (\frac{w-u}{w-v})^{1/n}$.

z-plane: $\langle T(z)\rangle_{\mathbb{C}} = 0$ $\to$ z-plane maps to $SL(2, Z)$ inv vacuum.

$\langle T(w)\rangle_{\mathcal{R}_n} = \frac{c}{12}\{z, w\} = \frac{\langle T(w)\Phi_n(u)\Phi_{-n}(v)\rangle}{\langle \Phi_n(u)\Phi_{-n}(v)\rangle} = \frac{\int D\varphi T(w) e^{-S}}{\int D\varphi e^{-S}}$ $\Rightarrow$ Twist ops at $w = u, v$.

Then $tr\rho_A^n \equiv \frac{Z_n}{Z_1^n}$ transforms as twist operator 2-pt function $\to$

$S_A = -\lim_{n\to 1} \partial_n \, tr\rho_A^n \ \to \ \frac{c}{3}\log\frac{l}{\epsilon}$ $\longrightarrow$ $c < 0$ suggests $S_A < 0 \ldots ?$

bc-ghost CFTs

- $SL(2)$ vacuum $|0\rangle \neq$ ghost ground state $|\downarrow\rangle$ in general.

$S \sim \int d^2z\, b\bar{\partial}c$, $(h_b, h_c) = (\lambda, 1-\lambda)$, $c = 1 - 3Q^2 < 0$, Background Charge $Q = 1 - 2\lambda$

$b(z) = \sum \frac{b_m}{z^{m+\lambda}}$, $c(z) = \sum \frac{c_m}{z^{m+1-\lambda}}$; $L_0 = \sum_{n>0} n(b_{-n}c_n + c_{-n}b_n) + \frac{\lambda(1-\lambda)}{2}$.

SL(2,Z) invariant vacuum $|0\rangle$: $T(z)|0\rangle = \sum_m \frac{L_m}{z^{m+2}}|0\rangle = regular$

$\Rightarrow$ $L_{m\geq -1}|0\rangle = 0$, $b_{m\geq 1-\lambda}|0\rangle = 0$, $c_{m\geq\lambda}|0\rangle = 0$ whereas $b_0|\downarrow\rangle = 0$

- $j_0^\dagger = -(j_0 + Q)$ Charge asymmetry.

$U(1)$ charge symmetry $\delta b = -i\epsilon b$, $\delta c = i\epsilon c$ $\rightarrow$ ghost current $j(z) = - : bc :$

$j(z) = \sum_m \frac{j_m}{z^{m+1}}$, $[L_m, j_n] = -nj_{m+n} + \frac{1}{2}Qm(m+1)\delta_{m,-n}$

$[j_0, O_p] = pO_p$, $j_0|q\rangle = q|q\rangle$ $\Rightarrow$ $p\langle q'|O_p|q\rangle = \langle q'|[j_0, O_p]|q\rangle = (-q' - Q - q)\langle q'|O_p|q\rangle$

Corrn fn $\neq 0$ only if Bgnd Charge cancelled $i.e.$ $p = -(q + q' + Q)$ $\Rightarrow$ $\langle -q - Q|q\rangle = 1.$

- $\lambda = 1$: $SL(2)$ vacuum $|0\rangle = |\downarrow\rangle$ ghost ground state

$\lambda = 1$ $\rightarrow$ $(h_b, h_c) = (1, 0)$, $c = -2$, $Q = -1.$

$b_{m\geq 0}|0\rangle = 0$, $c_{m\geq 1}|0\rangle = 0$, $\langle +1|0\rangle = \langle 0|c_0|0\rangle = 1$ $\leftarrow$ zero mode insertion

$\langle b(z)c(w)\rangle_0 \equiv \langle 0|c_0 \sum_{m,n} \frac{b_m}{z^{m+1}} \frac{c_n}{w^n}|0\rangle = \langle 0|c_0 \sum_{m=0}^{\infty} \frac{w^m}{z^{m+1}} b_m c_{-m}|0\rangle = \frac{1}{z-w}\langle 0|c_0|0\rangle$

whereas $\langle 0|b(z)c(w)|0\rangle = \frac{1}{z-w}\langle 0|0\rangle = 0.$ Plethora of negative norm states

bc-ghosts, $c=-2$: replica and EE

$T(w)=(\partial_w z)^2 T(z)+\frac{c}{12}\{z,w\}$, Schwarzian derivative $\{z,w\}=\frac{2\partial_w^3 z\partial_w z-3(\partial_w^2 z)^2}{2(\partial_w z)^2}$

($w=x+it_E$, Euclidean time t_E)

Subsystem A — single interval between $x=u$ and $x=v>u$ on fixed time slice $t_E=const$.

Replica w-space $\to$ z-plane under conformal transformation $z=(\frac{w-u}{w-v})^{1/n}$.

z-plane: $\langle T(z)\rangle_{\mathbb{C}}=0 \quad \to \quad$ z-plane corresponds to $SL(2)$ vacuum.

$$\langle T(w)\rangle_{\mathcal{R}_n}=\frac{c}{12}\{z,w\}=\frac{c(1-\frac{1}{n^2})}{24}\frac{(v-u)^2}{(w-u)^2(w-v)^2}=\frac{\langle T(w)\Phi_n(u)\Phi_{-n}(v)\rangle}{\langle\Phi_n(u)\Phi_{-n}(v)\rangle}=\frac{\int D\varphi T(w)e^{-S}}{\int D\varphi e^{-S}}$$

Twist operators at $w=u,v \ \to\ tr\rho_A^n\equiv\frac{Z_n}{Z_1^n}$ transforms as twist op 2-pt fn $\to\ S_A$.

- Replica argument is useful for the ghost ground state only if it is the $SL(2)$ vacuum: for $c=-2$, we have $|\downarrow\rangle=|0\rangle$ with $L_0=0$.
- Regularity condition $\langle T(z)\rangle_{\mathbb{C}}=0$ vacuous unless background charge incorporated, $i.e.$ we require $\langle -Q|\,T(z)\,|0\rangle=0$ (else trivially zero due to zero modes). $c=-2\to\lambda=1,\ Q=-1$.

Replica formulation formally applies now: $c<0\ \Rightarrow\ S_A<0$.

$\mathbb{Z}_N$ bc-orbifold CFTs (Saleur, Kausch, Flohr, ...'90s) confirm negative conf dims of twist ops [$l\equiv v-u$]

$$tr\rho_A^n=\prod_{k=1}^{n-1}\langle 0|\sigma^-_{k/N}(v)\sigma^+_{k/N}(u)|0\rangle=l^{\frac{1}{3}(n-1/n)}\to S_A=-\lim_{n\to 1}\partial_n\, tr\rho_A^n=-\frac{2}{3}\log\frac{l}{\epsilon}$$

bc-ghosts, $c = -2$: replica and EE

Consider n-sheeted replica boundary conditions (for b_k and likewise c_k)

$$b_k(e^{2\pi i}(w-u)) = b_{k+1}(w-u),\ \ b_k(e^{2\pi i}(w-v)) = b_{k-1}(w-v),\ \ k=1\ldots n.$$

$[(b,c)_k \to (b,c)_{k+1}$ under $w-u \to e^{2\pi i}(w-u)$ and $(b,c)_k \to (b,c)_{k-1}$ around $w=v]$

Diagonalize: $\tilde{b}_k = \frac{1}{n}\sum_{l=1}^{n} e^{2\pi i l k/n} b_k, \quad \tilde{c}_k = \frac{1}{n}\sum_{l=1}^{n} e^{-2\pi i l k/n} c_k.$

$$\tilde{b}_k(e^{2\pi i}(w-u)) = e^{-2\pi i k/n}\tilde{b}_k(w-u), \quad \tilde{c}_k(e^{2\pi i}(w-u)) = e^{2\pi i k/n}\tilde{c}_k(w-u).$$

$\mathbb{Z}_N$ orbifold of bc-ghost CFTs: (Saleur, Kausch, Flohr, ...'90s)

$$b_t(e^{2\pi i}z) = e^{-2\pi i k/N} b_t(z),\ \ c_t(e^{2\pi i}z) = e^{2\pi i k/N} c_t(z) \qquad (k = 1,\ldots,N-1)$$

Using twist fields $\sigma_{k/N} \equiv \sigma^+_{k/N}$: $b_t(e^{2\pi i}z)\sigma_{k/N}(0) = e^{-2\pi i k/N} b_t(z)\sigma_{k/N}(0)\ \ldots$

Anti-twist $\sigma^-_{k/N} \equiv \sigma^+_{1-k/N}$; Corrn fns are *e.g.* $\langle bc\rangle \equiv \langle 0|\sigma^-_{k/N} b_t(z) c_t(w) \sigma^+_{k/N}|0\rangle$

Regularizing (point-splitting) $\to$ conformal dimn and $U(1)$ charge

$$\langle T(z)\rangle_{k/N} = \lim_{z\to w}\Big(\langle -:b_t(z)\partial c_t(w):\rangle + \frac{1}{(z-w)^2}\Big) = -\frac{1}{2}\frac{k}{N}\Big(1-\frac{k}{N}\Big)\frac{1}{z^2} \equiv \frac{h_{\sigma_{k/N}}}{z^2}$$

$$\langle j(z)\rangle_{k/N} = \lim_{z\to w}\Big(\langle -:b^t(z)c^t(w):\rangle + \frac{1}{z-w}\Big) = \frac{k/N}{z}$$

$$b_t(z) = \sum_{m\in\mathbb{Z}} \frac{b_{m+k/N}}{z^{m+1+k/N}},\ \ c_t(z) = \sum_m \frac{c_{m-k/N}}{z^{m-k/N}}; \quad \{b_{m+k/N},\ c_{n-k/N}\} = \delta_{m+n,0}$$

$$\langle 0|b_t(z)\partial c_t(w)|0\rangle_{k/N} = \frac{1}{z}\Big(\frac{z}{w}\Big)^{1-k/N}\ \frac{\frac{k}{N}z + (1-\frac{k}{N})w}{(z-w)^2}$$

bc-ghosts, $c = -2$: replica and EE

Twist: $\sigma^+_{k/N}$ has dim $-\frac{1}{2}\frac{k}{N}(1-\frac{k}{N})$ $(h_{\sigma_{k/N}} < 0)$; $U(1)$ charge is $\frac{k}{N}$

Anti-twist: $\sigma^-_{k/N} \equiv \sigma^+_{1-k/N}$ has dim $-\frac{1}{2}\frac{k}{N}(1-\frac{k}{N})$; $U(1)$ charge is $1-\frac{k}{N}$

Nonvanishing twist field correlation functions also require total $U(1)$ charge to cancel background charge when calculated in untwisted $SL(2)$ vacuum: $\langle\sigma^+_{\lambda_1}\sigma^+_{\lambda_2}\ldots\rangle \equiv \langle 0|\sigma^+_{\lambda_1}\sigma^-_{1-\lambda_2}\ldots|0\rangle \neq 0 \quad \Rightarrow \quad \sum_i \lambda_i = 1$

Nonvanishing 2-point function has the form $\langle 0|\sigma^+_\lambda\sigma^-_\lambda|0\rangle \equiv \langle 0|\sigma^+_\lambda\sigma^+_{1-\lambda}|0\rangle \longrightarrow$ automatically contains an unpaired c-field cancelling background charge $Q = -1$.

$$\Rightarrow\ tr\rho_A^n = \prod_{k=1}^{n-1}\langle 0|\sigma^-_{k/N}(v)\sigma^+_{k/N}(u)|0\rangle \ = \ (v-u)^{-4\sum_{k=1}^{n-1}\ h_{\sigma_{k/N}}} = (v-u)^{\frac{1}{3}(n-1/n)}$$

$\rightarrow$ Entanglement entropy $\quad S_A = -\lim_{n\rightarrow 1}\partial_n\, tr\rho_A^n = -\frac{2}{3}\log\frac{l}{\epsilon} \quad [l \equiv v-u]$

Bosonized: $j(z) = i\partial\phi$ and $b(z) = e^{-\phi},\ c(z) = e^{\phi}$, in untwisted $c = -2$ theory.

In sector twisted by $\lambda = \frac{k}{N}$, twist fields are $\sigma_\lambda = e^{i\lambda\phi} = \sigma^+_\lambda$

Neighbourhood of each singularity does not contain zero modes (twisted sector): however $n \rightarrow 1$ limit is smooth requiring total $U(1)$ charge to cancel background charge (for finite interval).

$\sigma_{k/N}$ dim negative means long distance divergence in $\langle\sigma\sigma\rangle$ corrn fn $\rightarrow$ suggests replica theory has some instability (vanishes in $n \rightarrow 1$ limit).

Logarithmic ghost CFTs, $c = -2$

bc-ghost CFTs are like the nonlogarithmic subsector of more general logarithmic CFTs. (Gurarie, Flohr, . . . , '90s)

L_0 not diagonalizable — generic conformal fields have logarithmic partners (comprising ghost zero mode composites).

Example: anticommuting scalars $\chi, \bar{\chi} \rightarrow$ complex ghost $S = \int d^2 z \partial\chi\bar{\partial}\bar{\chi}$ CFT, $c = -2$

(motivated by Anninos,Hartman,Strominger, higher-spin dS_4/CFT_3, Sp(N) symplectic fermions)

Zero modes in $\chi \;\rightarrow\;$ log-partner of identity op $(\xi_0\bar{\xi}_0)$.

With single insertion: derivative ops $e.g.\ \partial\chi$ have no logs in corrn fns.

Also no logarithms in twist op 2-pt fn. So single interval entanglement entropy has no subleading logarithms.

Multiple intervals: logs arise in twist op corrn fns.

Mutual information?

“Ghost-spins”

"Ghost-spins"

To abstract away from the various technical issues of ghost CFTs and subtleties of the replica formulation there: cook up simple quantum mechanical toy model of ghost-like systems with negative norm states $\rightarrow$ reduced density matrix (RDM) after partial trace $\rightarrow$ EE.

Recall ordinary spin: $\langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 1, \qquad \langle\uparrow|\downarrow\rangle = \langle\downarrow|\uparrow\rangle = 0$

$$|\psi\rangle = c_1|\uparrow\rangle + c_2|\downarrow\rangle \;\;\rightarrow\;\; \langle\psi|\psi\rangle = |c_1|^2 + |c_2|^2 > 0$$

"Ghost-spin" $\rightarrow$ 2-state spin variable with indefinite norm.

$$\langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 0, \qquad \langle\uparrow|\downarrow\rangle = \langle\downarrow|\uparrow\rangle = 1$$

$|\psi\rangle = c_1|\uparrow\rangle + c_2|\downarrow\rangle \;\;\rightarrow\;\; \langle\psi|\psi\rangle = c_1 c_2^* + c_2 c_1^* \not> 0.$ $\;e.g.\; |\uparrow\rangle - |\downarrow\rangle$ has norm -2.

$|\pm\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle); \quad \langle +|+\rangle = \gamma_{++} = 1, \;\; \langle -|-\rangle = \gamma_{--} = -1, \;\; \langle +|-\rangle = \langle -|+\rangle = 0$

Two ghost-spins: $|\psi\rangle = \sum \psi^{ij}|ij\rangle$, adjoint: $\langle\psi| = \sum \langle ij|\psi^{ij\,*}$,

$$\langle\psi|\psi\rangle \;=\langle k|i\rangle\langle l|j\rangle\psi^{ij}\psi^{kl\,*} \;\equiv\; \gamma_{ik}\gamma_{jl}\psi^{ij}\psi^{kl\,*} = \gamma_{ii}\gamma_{jj}|\psi^{ij}|^2.$$

$\rho = |\psi\rangle\langle\psi| \;\rightarrow\;$ trace over one ghost-spin $\;\rightarrow\;$ reduced density matrix for remaining ghost-spin $\;\rightarrow\;$ von Neumann entropy.

Even number of ghost-spins $\rightarrow$ calculations, interpretation sensible.

Two ghost-spins

$$|\psi\rangle = \sum \psi^{ij}|ij\rangle = \psi^{++}|++\rangle\ + \psi^{+-}|+-\rangle\ + \psi^{-+}|-+\rangle\ + \psi^{--}|--\rangle\ ,$$

$$\langle\psi|\psi\rangle = \gamma_{ik}\gamma_{jl}\psi^{ij}\psi^{kl*} = |\psi^{++}|^2 + |\psi^{--}|^2 - |\psi^{+-}|^2 - |\psi^{-+}|^2 = \pm 1 \quad [\gamma_{\pm\pm} = \pm 1]$$

Trace over one ghost-spin $\rightarrow$ ρ_A for remaining ghost-spin $\rightarrow$ von Neumann entropy S_A.

RDM: $\rho_A = tr_B\rho \equiv (\rho_A)^{ik}|i\rangle\langle k|\,, \quad (\rho_A)^{ik} = \gamma_{jl}\psi^{ij}\psi^{kl*} = \gamma_{jj}\psi^{ij}\psi^{kj*}$

$$(\rho_A)^{++} = |\psi^{++}|^2 - |\psi^{+-}|^2, \qquad (\rho_A)^{+-} = \psi^{++}\psi^{-+*} - \psi^{+-}\psi^{--*},$$
$$(\rho_A)^{-+} = \psi^{-+}\psi^{++*} - \psi^{--}\psi^{+-*}, \quad (\rho_A)^{--} = |\psi^{-+}|^2 - |\psi^{--}|^2.$$

EE: $S_A = -\gamma_{ij}(\rho_A \log\rho_A)^{ij} = -\gamma_{++}(\rho_A\log\rho_A)^{++} - \gamma_{--}(\rho_A\log\rho_A)^{--}$

Define $\log\rho_A$ using expansion, using mixed-index RDM $(\rho_A)_i{}^k = \gamma_{ij}(\rho_A)^{jk}$.

Simple subfamily, diagonal ρ_A: $\rho_A^{+-} = 0 \rightarrow \psi^{-+*} = \psi^{+-}\psi^{--*}/\psi^{++} \rightarrow$

$$(|\psi^{++}|^2 - |\psi^{+-}|^2)\big(1 + \frac{|\psi^{--}|^2}{|\psi^{++}|^2}\big) = \pm 1, \qquad (\rho_A)^{ik}|i\rangle\langle k| = \pm x|+\rangle\langle +| \mp (1-x)|-\rangle\langle -|.$$

$$(\rho_A)_+^+ = \pm x, \quad (\rho_A)_-^- = \pm(1-x), \qquad x = \frac{|\psi^{++}|^2}{|\psi^{++}|^2 + |\psi^{--}|^2}, \quad (0 < x < 1)$$

$(\log\rho_A)_+^+ = \log(\pm x)$ etc, and $S_A = -(\rho_A)_+^+(\log\rho_A)_+^+ - (\rho_A)_-^-(\log\rho_A)_-^- \longrightarrow$

$\langle\psi|\psi\rangle > 0$: $S_A = -x\log x - (1-x)\log(1-x) > 0$ **$+ve$ norm $\Rightarrow$ $+ve$ EE.**

$\langle\psi|\psi\rangle < 0$: $S_A = x\log(-x) + (1-x)\log(-(1-x)) = x\log x + (1-x)\log(1-x) + i\pi$

$-ve$ norm $\Rightarrow$ some ρ^A eigenvalues $-ve$ $\Rightarrow$ $-ve$ Re(EE), const Im(EE).

Spins and ghost-spins, disentangled

Spin metric $+ve$ definite: $g_{ij} = \delta_{ij}$; Ghost-spins: $\gamma_{++} = 1,\ \gamma_{--} = -1$.

Observable of spin variables alone: $\langle\psi|O_s|\psi\rangle = tr_s(O_s\rho^s) \ \longrightarrow\ \rho^s = tr_{gs}\rho$

Disentangled spins and ghost-spins $\Rightarrow$ product states $\longrightarrow$

$$|\psi\rangle = |\psi_s\rangle\, |\psi_{gs}\rangle, \quad \langle\psi|\psi\rangle = \langle\psi_s|\psi_s\rangle\, \langle\psi_{gs}|\psi_{gs}\rangle$$

$$\langle\psi_s|\psi_s\rangle = g_{i_1j_1}\ldots g_{i_nj_n}(\psi_s)^{i_1i_2\cdots}(\psi_s)^{j_1j_2\cdots *} > 0,$$

$$\langle\psi_{gs}|\psi_{gs}\rangle = \gamma_{i_1j_1}\ldots \gamma_{i_nj_n}(\psi_{gs})^{i_1i_2\cdots}(\psi_{gs})^{j_1j_2\cdots *}$$

RDM after tracing over all ghost-spins: $\rho_A^s = tr_{gs}\big(|\psi_s\rangle\, |\psi_{gs}\rangle\langle\psi_s|\, \langle\psi_{gs}|\big)$

$$(\rho_A^s)^{i_1\cdots,k_1\cdots} = \langle\psi_{gs}|\psi_{gs}\rangle\, (\psi_s)^{i_1\cdots}(\psi_s)^{k_1\cdots *}$$

Normalize positive/negative norm states $\rightarrow$ norm ± 1 respectively:

$$\langle\psi_{gs}|\psi_{gs}\rangle \gtrless 0 \quad \Rightarrow \quad \langle\psi|\psi\rangle = \langle\psi_s|\psi_s\rangle\, \langle\psi_{gs}|\psi_{gs}\rangle = \pm 1 \qquad [\langle\psi_s|\psi_s\rangle > 0]$$

$$(\rho_A^s)^{i_1\cdots,k_1\cdots} = \pm\frac{1}{\langle\psi_s|\psi_s\rangle}(\psi_s)^{i_1\cdots}(\psi_s)^{k_1\cdots *} \quad \Rightarrow \quad tr\rho_A^s = \pm 1 \quad \big(\langle\psi|\psi\rangle \gtrless 0\big)$$

$+ve$ norm: ρ_A^s +ve definite, eigenvalues $0 < \lambda_i < 1$ with $\sum_i \lambda_i = 1 \quad \Rightarrow$

$$S_A = -\mathrm{tr}_s\, \rho_A^s \log\rho_A^s = -\sum_i \lambda_i \log\lambda_i > 0$$

$-ve$ norm: ρ_A^s negative definite, eigenvalues $-\lambda_i \quad \Rightarrow$

$$S_A = -\mathrm{tr}_s\, \rho_A^s \log\rho_A^s = -\sum_i (-\lambda_i)\log(-\lambda_i) = \sum_i \lambda_i \log\lambda_i + i\pi$$

Entangled ghost-spins & spins

Trace over ghost-spins $\rightarrow$ RDM for spins $\rightarrow$ in general, new EE patterns.

- One spin, two ghost-spins $\psi^{i,\alpha\beta}|i\rangle|\alpha\beta\rangle$: $\quad (\rho_A)^{ik} = \gamma_{\alpha\sigma}\gamma_{\beta\rho}\psi^{i,\alpha\beta}(\psi^*)^{k,\sigma\rho}$

$$|\psi\rangle = \psi^{+,++}|+\rangle|++\rangle + \psi^{+,--}|+\rangle|--\rangle + \psi^{-,++}|-\rangle|++\rangle + \psi^{-,--}|-\rangle|--\rangle$$

Correlated ghost-spins: $+ve$ norm $\Rightarrow$ $+ve$ EE. Subspace always exists for even ghost-spins.

Also for Hilbert space component continuously connected to this subsector, but not in general.

Physical requirement: $+ve$ norm $\Rightarrow +ve$ EE $\quad\longrightarrow\quad$ Even number of ghost-spins: sensible.

- One spin, one ghost-spin $\psi^{i,\alpha}|i\rangle|\alpha\rangle$: $\quad (\rho_A)^{ik} = \gamma_{\alpha\beta}\psi^{i,\alpha}(\psi^*)^{k,\beta}$

Simple entangled state: $\quad |\psi\rangle = \psi^{+,+}|+\rangle|+\rangle + \psi^{-,-}|-\rangle|-\rangle \;\Rightarrow$

$$(\rho_A)^{++} = |\psi^{+,+}|^2, \quad (\rho_A)^{--} = -|\psi^{-,-}|^2$$

$$|\psi^{+,+}|^2 - |\psi^{-,-}|^2 = \pm 1\,, \quad (\log\rho_A)^+_+ = \log(|\psi^{+,+}|^2), \quad (\log\rho_A)^-_- = \log(-|\psi^{-,-}|^2)$$

EE: $\quad S_A = -|\psi^{+,+}|^2 \log\left(|\psi^{+,+}|^2\right) + |\psi^{--}|^2 \log\left(|\psi^{-,-}|^2\right) + |\psi^{-,-}|^2 (i\pi)$

$\longrightarrow$ $+ve$ norm does not give $+ve$ EE.

- Multiple ghost-spins: n odd as above $\rightarrow$ $+ve$ norm does not give $+ve$ EE.

$$|\psi\rangle = \psi^{++\cdots}|++\ldots\rangle + \psi^{--\cdots}|--\ldots\rangle, \quad \langle\psi|\psi\rangle = |\psi^{++\cdots}|^2 + (-1)^n|\psi^{--\cdots}|^2$$

$$(\rho_A)^+_+ = (\rho_A)^{++} = |\psi^{++\cdots}|^2, \quad (\rho_A)^-_- = -(\rho_A)^{--} = (-1)^n|\psi^{--\cdots}|^2$$

Conclusions, questions

- Complex codim-2 extremal surfaces in de Sitter space (Poincare) $\rightarrow$ analytic continuation from Ryu-Takayanagi in AdS. Resembles EE for dual CFT via dS/CFT with $Z_{CFT} = \Psi_{dS}$. dS_4: negative area $\rightarrow$ $EE < 0$ $\leftarrow$ negative central charge.
- Replica in some 2-dim $c = -2$ ghost CFTs gives $EE < 0$. Ghost-spins: toy QM models, negative norm states, $Re(EE) < 0$. Entangled ghost-spins & spins. Subsectors: $+ve$ norm $\rightarrow$ $+ve$ EE.

??

- Bulk EE (via $\Psi^*\Psi$) vs $\text{EE}_{dS/CFT}$ above? EE as probe of dS/CFT?
- Lattice discretization of ghost-CFT ...
  - Ghost-spin chains, ghost-spin glasses?
  - Toy models for $Sp(N)$ and dS_4: hints at emergence of time?
- Conceptual issues with Bell pairs of spins and ghost-spins etc ...?

Details

de Sitter extremal surfaces

de Sitter (Poincare): $ds^2_{d+1} = \frac{R^2_{dS}}{\tau^2}(-d\tau^2 + dw^2 + dx_i^2) \quad \rightarrow \quad$ EE in dual Eucl CFT $\rightarrow$

bulk: Eucl time slice $w = const$, subregion at future timelike infnty $\rightarrow$ codim-2 extremal surface.

[strip] $S_{dS} = \frac{R^{d-1}_{dS} V_{d-2}}{4G_{d+1}} \int \frac{1}{\tau^{d-1}} \sqrt{dx^2 - d\tau^2} = \frac{R^{d-1}_{dS} V_{d-2}}{4G_{d+1}} \int \frac{d\tau}{\tau^{d-1}} \sqrt{\dot{x}^2 - 1}$

Extremize $\rightarrow \quad (\partial_\tau x)^2 = \frac{-A^2\tau^{2d-2}}{1-A^2\tau^{2d-2}}\ .$ $\qquad$ [$A^2 < 0$ is earlier real-τ solution]

dS_4/CFT_3: consider $A^2 > 0$. Near $\tau \rightarrow 0$: $\quad \dot{x}^2 \sim -A^2\tau^4 \quad i.e.$

$x(\tau) \sim \pm iA\tau^3 + x(0)$. $\quad$ This is spatial direction in Eucl CFT $\Rightarrow$

$x(\tau)$ real-valued $\Rightarrow \quad \tau = iT$ $\qquad$ [can show width Δx also real]

$x(\tau) \rightarrow$ complex extremal surface, τ along imaginary path $\tau = iT$.

$(\frac{dx}{dT})^2 = \frac{A^2T^4}{1-A^2T^4}\ .$ $\quad$ Note turning point: $T_* = \frac{1}{\sqrt{A}}$ $\quad$ (where $|\dot{x}|^2 \rightarrow \infty$).

Can now smoothly join half-extremal-surfaces at turning point. $\qquad$ [$\tau_{UV} = i\epsilon,\ \ \tau_* \sim il$]

$$\frac{\Delta x}{2} = \frac{l}{2} = \int_0^{\tau_*} d\tau \frac{iA\tau^2}{\sqrt{1-A^2\tau^4}} = \int_0^{T_*} \frac{(T^2/T_*^2)\ dT}{\sqrt{1-(T^4/T_*^4)}} \sim T_*$$

$$S_{dS_4} = -i\frac{R^2_{dS}}{4G_4} V_1 \int_{\tau_{UV}}^{\tau_*} \frac{d\tau}{\tau^2} \frac{1}{\sqrt{1-\tau^4/\tau_*^4}} = -\frac{R^2_{dS}}{4G_4} V_1 \int_\epsilon^l \frac{dT/T^2}{\sqrt{1-T^4/T_*^4}} \sim -\frac{R^2_{dS}}{G_4} V_1 (\frac{1}{\epsilon} - c\frac{1}{l})$$

Overall sign $\rightarrow$ match with dS_4/CFT_3 central charge, and conformal anomaly (sphere).

de Sitter extremal surfaces

de Sitter (Poincare): $ds^2_{d+1} = \frac{R^2_{dS}}{\tau^2}(-d\tau^2 + dw^2 + dx_i^2) \quad \rightarrow \quad$ EE in dual Eucl CFT $\rightarrow$

bulk: Eucl time slice $w = const$, subregion at future timelike infnty $\rightarrow$ codim-2 extremal surface.

[strip] $S_{dS} = \frac{R^{d-1}_{dS} V_{d-2}}{4G_{d+1}} \int \frac{1}{\tau^{d-1}} \sqrt{dx^2 - d\tau^2} = \frac{R^{d-1}_{dS} V_{d-2}}{4G_{d+1}} \int \frac{d\tau}{\tau^{d-1}} \sqrt{\dot{x}^2 - 1}$

$Extremize \quad \rightarrow \quad (\partial_\tau x)^2 = \frac{-A^2\tau^{2d-2}}{1-A^2\tau^{2d-2}}$. [$A^2 < 0$ is the earlier real solution]

$\underline{dS_{d+1}/CFT_d}$ (d even): near $\tau \rightarrow 0$, $\dot{x} \sim \pm\sqrt{-A^2}\,\tau^{d-1}$ $i.e.$

$x(\tau) \sim \pm\sqrt{-A^2}\,\tau^d + x(0)$. This is spatial direction in Eucl CFT

$\Rightarrow$ $\boxed{x(\tau) \text{ real-valued } \Rightarrow A^2 < 0,\ \tau = iT}$ [can show width Δx also real]

$x(\tau) \rightarrow$ complex extremal surface, τ along imaginary path $\tau = iT$.

$(\frac{dx}{dT})^2 = \frac{A^2 T^{2d-2}}{1+(-1)^{d-1}A^2T^{2d-2}}$. Note turning point: $T_*^{2d-2}A^2 = 1$.

$$S_{dS} = -i\frac{R^{d-1}_{dS}}{4G_{d+1}} V_{d-2} \int_{\tau_{UV}}^{\tau_*} \frac{d\tau}{\tau^{d-1}} \frac{2}{\sqrt{1+A^2\tau^{2d-2}}}$$

$$= i^{1-d}\frac{R^{d-1}_{dS}}{2G_{d+1}} V_{d-2} \int_\epsilon^{T_*} \frac{dT/T^{d-1}}{\sqrt{1+(-1)^{d-1}A^2T^{2d-2}}} \sim i^{1-d}\frac{R^{d-1}_{dS}}{2G_{d+1}} V_{d-2}\left(\frac{1}{\epsilon^{d-2}} - c_d\frac{1}{l^{d-2}}\right)$$

Spherical extremal surfaces, dS/CFT

(KN)

$ds^2 = \frac{R_{dS}^2}{\tau^2}(-d\tau^2 + dw^2 + dr^2 + r^2 d\Omega_{d-2}^2) \rightarrow w = const$, sphere subregion. $0 \le r \le l$

$$S_{dS} = \frac{R_{dS}^{d-1}\Omega_{d-2}}{4G_{d+1}} \int \frac{d\tau}{\tau^{d-1}} r^{d-2}\sqrt{(\frac{dr}{d\tau})^2 - 1}\,, \quad \text{extremize: } r(\tau) = \sqrt{l^2+\tau^2}\,, \quad \dot{r} = \frac{\tau}{\sqrt{l^2+\tau^2}}$$

Real τ: outward-bending, $r(\tau) \ge l$. Timelike: $\dot{r} \le 1$. No "end" at finite τ.

$\rightarrow \quad \epsilon < |\tau| < \infty \quad \rightarrow S_{dS}$ real, no finite cutoff-indep parts.

$\tau = iT$: now $0 \le r(\tau) < l$ and $\Delta r = l$. Turning point $\tau_* = il$.

$$S_{dS} = \frac{R_{dS}^{d-1}\Omega_{d-2}}{4G_{d+1}} \int_{i\epsilon}^{\tau_*} \frac{d\tau}{\tau^{d-1}}(-il)(l^2+\tau^2)^{(d-3)/2} \quad \rightarrow \quad S_{dS_4} = -\frac{\pi R_{dS}^2}{2G_4}(\frac{l}{\epsilon} - 1)$$

d even: $\log\frac{l}{\epsilon}$ divergence. Coeff $\rightarrow -i\frac{R_{dS}}{2G_3}$ $[dS_3]$; $-i\frac{\pi R_{dS}^3}{2G_5}$ $[dS_5]$, ...

Free energy of CFT_d on sphere: log-div, related to conformal anomaly.

Casini,Huerta,Myers: $-F_{CFT} = \log Z_{CFT} = a\log\epsilon + \ldots$, integ. trace anomaly $a = \int\langle T^k{}_k\rangle$.

$Z_{CFT} = e^{-F} = \Psi \sim e^{iS_{cl}}$ for auxiliary global dS. $T_{ij} \sim \frac{2}{\sqrt{h}}\frac{\delta(-F_{CFT})}{\delta h^{ij}} \sim i\frac{2}{\sqrt{h}}\frac{\delta S}{\delta h^{ij}}$

$\rightarrow$ log-div coeff matches $\rightarrow$ equivalent to analytic continuation from AdS.

$[S_{CFT}^{EE} = -\lim_{n\rightarrow 1}\partial_n \frac{Z_n}{(Z_1)^n}$; scale change $l\frac{\partial}{\partial l}S_{CFT}^{EE} \sim \int\langle T_\mu{}^\mu\rangle$; here $S_{CFT}^{EE} = S_{dS}]$

$S_{cl} = \frac{2d\ \Omega_d R_{dS}^{d-1}}{16\pi G_{d+1}} \int \frac{dt}{R_{dS}}\ (\cosh\frac{t}{R_{dS}})^d \rightarrow$ log-div $[ds^2 = -dt^2 + R_{dS}^2(\cosh\frac{t}{R_{dS}})^2 d\Omega_d^2]$

bc-ghost CFTs

$S = \frac{1}{2\pi}\int d^2z\, b\bar{\partial}c, \quad (h_b, h_c) = (\lambda, 1-\lambda) \quad \longrightarrow \quad b(z)c(w) \sim \frac{1}{z-w}$

$T(z) = :(\partial b)c: - \lambda\partial(:bc:) \;=\; \frac{1}{2}\big(:(\partial b)c: - :b\partial c: \big) + \frac{1}{2}Q\partial(:bc:) \quad \longrightarrow$

Central charge $c = 1 - 3(2\lambda-1)^2 = 1 - 3Q^2$;

Background charge $Q = 1 - 2\lambda$.

$$b(z) = \sum_{m\in\mathbb{Z}} \frac{b_m}{z^{m+\lambda}}, \quad c(z) = \sum_{m\in\mathbb{Z}} \frac{c_m}{z^{m+1-\lambda}}.$$

$\{b_m, c_n\} = \delta_{m+n,0}$, $\{b_m, b_n\} = 0$, $\{c_m, c_n\} = 0$ and Virasoro algebra with $L_m = \sum_{n=-\infty}^{\infty}(m\lambda - n)b_n c_{m-n} \quad [m\neq 0]; \quad L_0 = \sum_{n>0} n(b_{-n}c_n + c_{-n}b_n) + \frac{\lambda(1-\lambda)}{2}$.

$$b_0|\downarrow\rangle = 0, \quad c_0|\downarrow\rangle = |\uparrow\rangle, \qquad b_0|\uparrow\rangle = |\downarrow\rangle, \quad c_0|\uparrow\rangle = 0.$$

Conventional to take $|\downarrow\rangle$ as ghost ground state (b_0 is annihilation op).

SL(2,Z) invariant vacuum $|0\rangle$: $\quad T(z)|0\rangle = \sum_m \frac{L_m}{z^{m+2}}|0\rangle = regular$

$\Rightarrow \quad L_m|0\rangle = 0\,, \quad m \geq -1, \qquad b_{m\geq 1-\lambda}|0\rangle = 0, \quad c_{m\geq\lambda}|0\rangle = 0.$

In general, $SL(2)$ vacuum $|0\rangle \neq$ ghost ground state $|\downarrow\rangle$.

$(L_0|\downarrow\rangle = \frac{\lambda(1-\lambda)}{2}|\downarrow\rangle)$

$|0\rangle = b_{-1}b_{-2}\ldots b_{1-\lambda}|\downarrow\rangle = \prod_{1-\lambda\leq m<0} b_m|\downarrow\rangle \qquad [\lambda\neq 1]$

bc-ghost CFTs

$U(1)$ charge symmetry $\delta b = -i\epsilon b,\ \delta c = i\epsilon c\ \rightarrow$ ghost current $j(z) = -:bc:$

$$j(z)b(w) \sim -\frac{1}{z-w}b(w),\ \ j(z)c(w) \sim \frac{1}{z-w}c(w) \qquad [(\partial_z w)j'(w) = j(z) - \frac{Q}{2}\frac{\partial_z^2 w}{\partial_z w}]$$

$$j(z)j(w) \sim \frac{1}{(z-w)^2}\ , \quad T(z)j(w) \sim \frac{Q}{(z-w)^3} + \frac{1}{(z-w)^2}j(w) + \frac{1}{z-w}\partial j(w)$$

$$N_g = \int_0^{2\pi} \frac{dw}{2\pi i}\, j^{cyl}(w) = \sum_{n=1}^{\infty}(c_{-n}b_n - b_{-n}c_n) + c_0 b_0 - \frac{1}{2}. \quad [w = \log z]$$

$$N_g^z = \oint \frac{dz}{2\pi i} j(z) = N_g - \frac{Q}{2}; \qquad [N_g, b_m] = -b_m,\ [N_g, c_m] = c_m$$

$$N_g|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle,\ N_g|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle \ \Rightarrow\ N_g^z|\downarrow\rangle = -\frac{Q+1}{2}|\downarrow\rangle = (\lambda-1)|\downarrow\rangle.$$

$$|0\rangle = \prod_{1-\lambda\le m<0} b_m|\downarrow\rangle \quad [\lambda \neq 1] \qquad N_g|0\rangle = \big(-\tfrac{1}{2} - (\lambda-1)\big)|0\rangle = \frac{Q}{2}|0\rangle\ ;\ N_g^z|0\rangle = 0.$$

$$j(z) = \sum_m \frac{j_m}{z^{m+1}}\ , \quad j_n^\dagger = -j_n \text{ for } n \neq 0 \ \text{(with } j_n = -\sum_m b_m c_{n-m}) \longrightarrow$$

$$[L_m, j_n] = -nj_{m+n} + \frac{1}{2}Qm(m+1)\delta_{m,-n}$$

$$[L_1, j_{-1}] = j_0 + Q, \quad [L_1, j_{-1}]^\dagger = [L_{-1}, j_1] = -j_0 \quad \Rightarrow$$

$$j_0^\dagger = -(j_0 + Q) \qquad \text{Charge asymmetry.}$$

With $[j_0, O_p] = pO_p$, and $j_0|q\rangle = q|q\rangle \ \Rightarrow$

$$p\langle q'|O_p|q\rangle = \langle q'|[j_0, O_p]|q\rangle = (-q' - Q - q)\langle q'|O_p|q\rangle$$

$i.e.$ correlation function non-vanishing only if $p = -(q + q' + Q)$.

For $p = 0$, this gives $q' = -q - Q \quad \Rightarrow \qquad \langle -q - Q|q\rangle = 1.$

bc-ghost CFTs

$\lambda = 1: \quad (h_b, h_c) = (1, 0), \quad c = -2, \; Q = -1.$ $\; b(z) = \sum_m \frac{b_m}{z^{m+1}}, \;\; c(z) = \sum_m \frac{c_m}{z^m}.$

$b_{m\geq 0}|0\rangle = 0, \;\; c_{m\geq 1}|0\rangle = 0 \Rightarrow$ $SL(2)$ vacuum $|0\rangle = |\downarrow\rangle$ ghost ground state

$L_{m\neq 0} = \sum_{n=-\infty}^{\infty}(m-n)b_n c_{m-n}; \quad L_0 = \sum_{n>0} n(b_{-n}c_n + c_{-n}b_n), \quad [L_0|\downarrow\rangle = 0]$
$N_g^z|\downarrow\rangle = N_g^z|0\rangle = 0, \quad c_0|0\rangle = |\uparrow\rangle \quad \longrightarrow \quad L_0 = 0, \;\; N_g^z c_0|0\rangle = c_0|0\rangle.$

Smallest nonvanishing corrn fn $\langle 0|c(z)|0\rangle = \langle 0|\sum_m \frac{c_m}{z^m}|0\rangle = \langle 0|c_0|0\rangle \equiv \langle +1|0\rangle;$

Adjoint of $|0\rangle$: $\quad \langle +1| = \langle 0|c_0, \; i.e. \; \langle 0|^\dagger = \langle 0|c_0.$

$$\langle b(z)c(w)\rangle_0 \equiv \langle 0|c_0 \sum_{m,n} \frac{b_m}{z^{m+1}} \frac{c_n}{w^n}|0\rangle = \langle 0|c_0 \sum_{m=0}^{\infty} \frac{w^m}{z^{m+1}} b_m c_{-m}|0\rangle$$

$$= \frac{1}{z}\frac{1}{1-\frac{w}{z}}\langle 0|c_0|0\rangle = \frac{1}{z-w} \quad \text{whereas} \quad \langle 0|b(z)c(w)|0\rangle = \frac{1}{z-w}\langle 0|0\rangle = 0$$

Plethora of negative norm states, with L_0 non-negative

(also zero norm states): $\quad e.g. \; (b_{-1} - c_{-1})|0\rangle$ with norm

$$\langle 0|c_0(b_1 - c_1)(b_{-1} - c_{-1})|0\rangle = -\langle 0|c_0\big(\{b_1, c_{-1}\} + \{c_1, b_{-1}\}\big)|0\rangle = -2\langle 0|c_0|0\rangle = -2$$

bc-ghosts, $c = -2$: replica and EE

Twist operator correlation functions in detail? Negative conf dims? Zero modes as $n \to 1$?

Consider n-sheeted replica boundary conditions

$$b_k(e^{2\pi i}(w-u)) = b_{k+1}(w-u), \;\; c_k(e^{2\pi i}(w-u)) = c_{k+1}(w-u),$$

$$b_k(e^{2\pi i}(w-v)) = b_{k-1}(w-v), \;\; c_k(e^{2\pi i}(w-v)) = c_{k-1}(w-v), \quad k = 1, \ldots, n.$$

[$(b,c)_k \to (b,c)_{k+1}$ under $w-u \to e^{2\pi i}(w-u)$ and $(b,c)_k \to (b,c)_{k-1}$ around $w=v$]

Diagonalize: $\tilde{b}_k = \frac{1}{n}\sum_{l=1}^{n} e^{2\pi i l k/n} b_k, \quad \tilde{c}_k = \frac{1}{n}\sum_{l=1}^{n} e^{-2\pi i l k/n} c_k.$

Twist around $w=u \to$

$$\tilde{b}_k(e^{2\pi i}(w-u)) = e^{-2\pi i k/n}\tilde{b}_k(w-u), \quad \tilde{c}_k(e^{2\pi i}(w-u)) = e^{2\pi i k/n}\tilde{c}_k(w-u).$$

(likewise anti-twist around $w=v$) $\longrightarrow$ standard orbifold boundary conditions.

$\mathbb{Z}_N$ orbifold of bc-ghost:

$$b_t(e^{2\pi i}z) = e^{-2\pi i k/N} b_t(z), \;\; c_t(e^{2\pi i}z) = e^{2\pi i k/N} c_t(z) \quad (k = 1, \ldots, N-1)$$

$$b_t(z) = \sum_{m\in\mathbb{Z}} \frac{b_{m+k/N}}{z^{m+1+k/N}}, \quad c_t(z) = \sum_{m\in\mathbb{Z}} \frac{c_{m-k/N}}{z^{m-k/N}}; \; \{b_{m+k/N},\, c_{n-k/N}\} = \delta_{m+n,0}$$

Twist ground state $\quad b_{m+k/N}|0\rangle_{k/N} = 0, \; c_{m-k/N}|0\rangle_{k/N} = 0, \;\; m > 0$

Twisted sector: no zero modes. Untwisted: zero modes exist.

bc-ghosts, $c=-2$: replica and EE

Twist operator correlation functions in detail? Negative conf dims? Zero modes as $n\to 1$?

$\mathbb{Z}_N$ orbifold: $b_t(e^{2\pi i}z)=e^{-2\pi ik/N}b_t(z),\ \ c_t(e^{2\pi i}z)=e^{2\pi ik/N}c_t(z)\quad (k=1\ldots N-1)$

$$\langle b_t(z)c_t(w)\rangle_{k/N}=\sum_{m,n}\frac{\langle 0|b_{m+k/N}\ c_{n-k/N}|0\rangle_{k/N}}{z^{m+1+k/N}w^{n-k/N}}=\sum_{m=0}^{\infty}\frac{1}{z}\ \frac{w^{k/N}}{z^{k/N}}\ \left(\frac{w}{z}\right)^m=\left(\frac{z}{w}\right)^{-k/N}\frac{1}{z-w}$$

$$\langle b_t(z)\partial c_t(w)\rangle_{k/N}=\sum_{m,n}\frac{\langle b_{m+k/N}\ c_{n-k/N}\rangle}{z^{m+1+k/N}w^{n+1-k/N}}=\frac{1}{z}\ \left(\frac{z}{w}\right)^{1-k/N}\ \frac{\frac{k}{N}z+(1-\frac{k}{N})w}{(z-w)^2}$$

This can be recast in terms of twist fields $\sigma_{k/N}$.

$b_t(e^{2\pi i}z)\sigma_{k/N}(0)=e^{-2\pi ik/N}b_t(z)\sigma_{k/N}(0),$

$c_t(e^{2\pi i}z)\sigma_{k/N}(0)=e^{2\pi ik/N}c_t(z)\sigma_{k/N}(0)\quad (k=1,\ldots,N-1)$ equiv to OPEs

$b_t(z)\sigma_{k/N}(0)\sim z^{-k/N}\tau_{k/N},\quad \partial c_t(z)\sigma_{k/N}(0)\sim z^{-1+k/N}\tau'_{k/N}.$

Twist field $\sigma_{k/N}\equiv\sigma^+_{k/N}\ \longrightarrow$ anti-twist field $\sigma^-_{k/N}\equiv\sigma^+_{1-k/N}$

(OPEs have $\frac{k}{N}\to 1-\frac{k}{N}$) $\quad\to\quad$ nonvanishing corrn fns.

Then correlation functions are $e.g.$ $\langle bc\rangle\equiv\langle 0|\sigma^-_{k/N}b_t(z)c_t(w)\sigma^+_{k/N}|0\rangle$

Regularizing (point-splitting) $\longrightarrow$

$$\langle T(z)\rangle_{k/N}=\lim_{z\to w}\left(\langle -:b_t(z)\partial c_t(w):\rangle+\frac{1}{(z-w)^2}\right)=-\frac{1}{2}\frac{k}{N}\left(1-\frac{k}{N}\right)\frac{1}{z^2}\ \equiv\ \frac{h_{\sigma_{k/N}}}{z^2}$$

$U(1)$ charge: $\langle j(z)\rangle_{k/N}=\lim_{z\to w}\left(\langle -:b^t(z)c^t(w):\rangle+\frac{1}{z-w}\right)=\frac{k/N}{z}$

bc-ghosts, $c = -2$: replica and EE

Twist: $\sigma^+_{k/N}$ has dim $-\frac{1}{2}\frac{k}{N}(1-\frac{k}{N})$ $(h_{\sigma_{k/N}} < 0)$; $U(1)$ charge is $\frac{k}{N}$

Anti-twist: $\sigma^-_{k/N} \equiv \sigma^+_{1-k/N}$ has dim $-\frac{1}{2}\frac{k}{N}(1-\frac{k}{N})$; $U(1)$ charge is $1-\frac{k}{N}$

Nonvanishing twist field correlation functions also require total $U(1)$ charge to cancel background charge when calculated in untwisted $SL(2)$ vacuum: $\langle\sigma^+_{\lambda_1}\sigma^+_{\lambda_2}\ldots\rangle \equiv \langle 0|\sigma^+_{\lambda_1}\sigma^-_{1-\lambda_2}\ldots|0\rangle \neq 0 \quad \Rightarrow \quad \sum_i \lambda_i = 1$

Bosonized: $j(z) = i\partial\phi$ and $b(z) = e^{-\phi},\ c(z) = e^{\phi}$, in untwisted $c = -2$ theory.
In sector twisted by $\lambda = \frac{k}{N}$, twist fields are $\sigma_\lambda = e^{i\lambda\phi} = \sigma^+_\lambda$

Thus nonvanishing 2-point function has the form $\langle 0|\sigma^+_\lambda\sigma^-_\lambda|0\rangle \equiv \langle 0|\sigma^+_\lambda\sigma^+_{1-\lambda}|0\rangle \longrightarrow$ automatically contains an unpaired c-field e^ϕ cancelling background charge $Q = -1$.

$$\Rightarrow\ tr\rho_A^n = \prod_{k=1}^{n-1}\langle 0|\sigma^-_{k/N}(v)\sigma^+_{k/N}(u)|0\rangle \ =\ (v-u)^{-4\sum_{k=1}^{n-1}\ h_{\sigma_{k/N}}} = (v-u)^{\frac{1}{3}(n-1/n)}$$

$\rightarrow$ Entanglement entropy $\quad S_A = -\lim_{n\to 1}\partial_n\, tr\rho_A^n \ =\ -\frac{2}{3}\log\frac{l}{\epsilon}$ $\quad$ (with $l \equiv v-u$).

Neighbourhood of each singularity does not contain zero modes: however $n \to 1$ limit is smooth requiring total $U(1)$ charge to cancel background charge (for finite interval).
$\sigma_{k/N}$ dim negative means long distance divergence in $\langle\sigma\sigma\rangle$ corrn fn $\rightarrow$ suggests replica theory has some instability (vanishes in $n \to 1$ limit).

One spin, two ghost-spins, entangled

$$|\psi\rangle = \psi^{i,\alpha\beta}|i\rangle|\alpha\beta\rangle; \quad \langle\psi|\psi\rangle = g_{ij}\gamma_{\alpha\sigma}\gamma_{\beta\rho}\psi^{i,\alpha\beta}(\psi^*)^{j,\sigma\rho} = |\psi^{+,++}|^2 - |\psi^{+,+-}|^2$$
$$-|\psi^{+,-+}|^2 + |\psi^{+,--}|^2 + |\psi^{-,++}|^2 - |\psi^{-,+-}|^2 - |\psi^{-,-+}|^2 + |\psi^{-,--}|^2$$

RDM: $(\rho_A)^{ik} = \gamma_{\alpha\sigma}\gamma_{\beta\rho}\psi^{i,\alpha\beta}(\psi^*)^{k,\sigma\rho} = \gamma_{\alpha\alpha}\gamma_{\beta\beta}\psi^{i,\alpha\beta}(\psi^*)^{k,\alpha\beta}$

Physical requirement: after tracing over ghost-spins, RDM is for spins alone $\longrightarrow$ **Demand that $+ve$ norm states $\Rightarrow$ $+ve$ EE.**

- $|\psi\rangle = \psi^{+,++}|+\rangle|++\rangle + \psi^{+,--}|+\rangle|--\rangle + \psi^{-,++}|-\rangle|++\rangle + \psi^{-,--}|-\rangle|--\rangle$

  Correlated ghost-spins: positive norm so manifestly positive EE.

One spin, two ghost-spins, entangled

- Subfamily of states

$|\psi\rangle = \psi^{+,++}|+\rangle|++\rangle + \psi^{+,+-}|+\rangle|+-\rangle + \psi^{-,-+}|-\rangle|-+\rangle + \psi^{-,--}|-\rangle|--\rangle$

RDM: $(\rho_A)^{++} = |\psi^{+,++}|^2 - |\psi^{+,+-}|^2, \quad (\rho_A)^{--} = -|\psi^{-,-+}|^2 + |\psi^{-,--}|^2$

Remaining spin $\rightarrow +ve$ definite metric $\Rightarrow$

$S_A = -g_{ij}(\rho_A \log \rho_A)^{ij} = -(\rho_A \log \rho_A)^{++} - (\rho_A \log \rho_A)^{--}$

$\pm$ norm: $\langle\psi|\psi\rangle = |\psi^{+,++}|^2 - |\psi^{+,+-}|^2 - |\psi^{-,-+}|^2 + |\psi^{-,--}|^2 = \pm 1$

$|\psi^{+,++}|^2 - |\psi^{+,+-}|^2 \equiv x, \quad \langle\psi|\psi\rangle = x + (\pm 1 - x);$

$(\rho_A)^{+,+} = x, \ (\rho_A)^{-,-} = \pm 1 - x, \qquad S_A = -x\log x - (\pm 1 - x)\log(\pm 1 - x)$

Positive norm: $x > 0 \ \Rightarrow \ 1 - x > 0$ so $S_A = -x\log x - (1-x)\log(1-x) > 0$.

$x < 0 \Rightarrow \ (\rho_A)^{++} < 0, \ (\rho_A)^{--} > 0, \ S_A = |x|\log|x| - (1+|x|)\log(1+|x|) + i\pi|x|$.

$x > 0$ is component of Hilbert space that is continuously connected to the correlated ghost-spin sector. Small deformations don't change positivity.

Negative norm: ρ_A negative definite if $x < 0$ and $(-1-x) < 0$, so that $0 < |x| < 1 \Rightarrow$

$$S_A = |x|\log|x| + (1-|x|)\log(1-|x|) + i\pi$$

$x < -1 \Rightarrow \ (\rho_A)^{++} < 0, \ (\rho_A)^{--} > 0, \ S_A = |x|\log|x| - (|x|-1)\log(|x|-1) + i\pi|x|$

$x > 0 \Rightarrow \ (\rho_A)^{++} > 0, \ (\rho_A)^{--} < 0, \ S_A = -x\log x + (1+x)\log(1+x) + i\pi(1+x)$

$(\rho_A)^{++} > 0, \ (\rho_A)^{--} > 0, \ x > 0, \ -1 - x > 0$: not possible.

One spin, two ghost-spins, entangled

$|\psi\rangle = \psi^{i,\alpha\beta}|i\rangle|\alpha\beta\rangle$; RDM: $(\rho_A)^{ik} = \gamma_{\alpha\sigma}\gamma_{\beta\rho}\psi^{i,\alpha\beta}(\psi^*)^{k,\sigma\rho} = \gamma_{\alpha\alpha}\gamma_{\beta\beta}\psi^{i,\alpha\beta}(\psi^*)^{k,\alpha\beta}$

- $|\psi\rangle = \psi^{+,++}|+\rangle|++\rangle + \psi^{+,--}|+\rangle|--\rangle + \psi^{-,++}|-\rangle|++\rangle + \psi^{-,--}|-\rangle|--\rangle$

Correlated ghost-spins: $+ve$ norm $\Rightarrow$ $+ve$ EE. Subspace always exists for even ghost-spins.

- In general, new entanglement patterns. After tracing over ghost-spins, RDM is for spins alone

$\longrightarrow$ **Physical requirement: $+ve$ norm states $\Rightarrow$ $+ve$ EE.**

e.g. $|\psi\rangle = \psi^{+,++}|+\rangle|++\rangle + \psi^{+,+-}|+\rangle|+-\rangle + \psi^{-,-+}|-\rangle|-+\rangle + \psi^{-,--}|-\rangle|--\rangle$

RDM: $(\rho_A)^{++} = |\psi^{+,++}|^2 - |\psi^{+,+-}|^2$, $(\rho_A)^{--} = -|\psi^{-,-+}|^2 + |\psi^{-,--}|^2$

$\pm$ norm: $\langle\psi|\psi\rangle = |\psi^{+,++}|^2 - |\psi^{+,+-}|^2 - |\psi^{-,-+}|^2 + |\psi^{-,--}|^2 = \pm 1$

$$|\psi^{+,++}|^2 - |\psi^{+,+-}|^2 \equiv x, \quad \langle\psi|\psi\rangle = x + (\pm 1 - x);$$

$$(\rho_A)^{+,+} = x, \;\; (\rho_A)^{-,-} = \pm 1 - x, \qquad S_A = -x\log x - (\pm 1 - x)\log(\pm 1 - x)$$

Positive norm: $0 < x < 1 \;\;\Rightarrow\;\; S_A = -x\log x - (1-x)\log(1-x) > 0$.

$x > 0$ is component of Hilbert space that is continuously connected to the correlated ghost-spin sector. Small deformations don't change positivity.

Negative norm: ρ_A negative definite if $x < 0$, $(-1-x) < 0$, *i.e.* $0 < |x| < 1 \Rightarrow$

$$S_A = |x|\log|x| + (1-|x|)\log(1-|x|) + i\pi$$

[Other ranges for x: $+ve$ norm is not $+ve$ EE and $-ve$ norm is not $-ve$ Re(EE)]

One spin, one ghost-spin, entangled

$|\psi\rangle = \psi^{i,\alpha}|i\rangle|\alpha\rangle, \quad \langle\psi|\psi\rangle = g_{ij}\gamma_{\alpha\beta}\psi^{i,\alpha}(\psi^*)^{j,\beta} = \sum_{i,\alpha}\gamma_{\alpha\alpha}\psi^{i,\alpha}(\psi^*)^{i,\alpha}$

RDM: $(\rho_A)^{ik} = \gamma_{\alpha\beta}\psi^{i,\alpha}(\psi^*)^{k,\beta} = \gamma_{\alpha\alpha}\psi^{i,\alpha}(\psi^*)^{k,\alpha}$

Simple entangled state: $|\psi\rangle = \psi^{+,+}|+\rangle|+\rangle + \psi^{-,-}|-\rangle|-\rangle \Rightarrow$
$(\rho_A)^{++} = |\psi^{+,+}|^2, \quad (\rho_A)^{--} = -|\psi^{-,-}|^2$

$|\psi^{+,+}|^2 - |\psi^{-,-}|^2 = \pm 1\,, \ (\log\rho_A)^+_+ = \log(|\psi^{+,+}|^2), \ (\log\rho_A)^-_- = \log(-|\psi^{-,-}|^2)$

EE: $S_A = -|\psi^{+,+}|^2\log\left(|\psi^{+,+}|^2\right) + |\psi^{--}|^2\log\left(|\psi^{-,-}|^2\right) + |\psi^{-,-}|^2(i\pi)$

Positive norm: $|\psi^{-,-}|^2 = |\psi^{+,+}|^2 - 1$

$S_A = -x\log x + (x-1)\log(x-1) + (x-1)(i\pi)\,, \quad x = |\psi^{+,+}|^2, \ 1 \le x < \infty$

Real part negative, altho $+ve$ norm state.

Negative norm: $|\psi^{-,-}|^2 = |\psi^{+,+}|^2 + 1$

$S_A = -x\log x + (x+1)\log(x+1) + (x+1)(i\pi)\,, \quad x = |\psi^{+,+}|^2, \ 0 \le x < \infty$

Real part positive, altho $-ve$ norm state.

Also true for general entangled states.
Other contraction schemes for ρ_A can be studied: however this bug does not go away.

ρ_A acquires negative eigenvalues also for multiple spins entangled with single ghost-spin $\longrightarrow$ $+ve$ norm does not give $+ve$ EE.

Multiple ghost-spins, entangled

3 ghost-spins: possible to find subsectors where $+ve$ norm states have $+ve$ EE.
Structure has similarities to one spin and two ghost-spins entangled.

However there are weird entangled states: consider n ghost-spins —

$|\psi\rangle = \psi^{++\cdots}|++\ldots\rangle + \psi^{--\cdots}|--\ldots\rangle, \quad \langle\psi|\psi\rangle = |\psi^{++\cdots}|^2 + (-1)^n|\psi^{--\cdots}|^2$

$(\rho_A)^+_+ = (\rho_A)^{++} = |\psi^{++\cdots}|^2, \ (\rho_A)^-_- = -(\rho_A)^{--} = (-1)^n|\psi^{--\cdots}|^2\,,$

$S_A = -(\rho_A)^+_+(\log\rho_A)^+_+ - (\rho_A)^-_-(\log\rho_A)^-_-$

n even: $+ve$ norm so $+ve$ EE.

n odd: $|\psi^{++\cdots}|^2 - |\psi^{--\cdots}|^2 = \pm 1$ [$\pm$ norm]

$S_A = -|\psi^{++\cdots}|^2\log\left(|\psi^{++\cdots}|^2\right) + |\psi^{--\cdots}|^2\log\left(|\psi^{--\cdots}|^2\right) + |\psi^{--\cdots}|^2(i\pi)$

Similar to one spin and one ghost-spin entangled $\rightarrow$ thus $+ve$ norm has $-ve$ Re(EE) and Im. part, while $-ve$ norm gives $Re(S_A) > 0$.

One spin entangled with n ghost-spins:

$|\psi\rangle_{(1,n)} = \psi^{+,++\cdots}|+,++\ldots\rangle + \psi^{-,--\cdots}|-,--\ldots\rangle,$

${}_{(1,n)}\langle\psi|\psi\rangle_{(1,n)} = |\psi^{+,++\cdots}|^2 + (-1)^n|\psi^{-,--\cdots}|^2$

$(\rho_A)^+_+ = (\rho_A)^{++} = |\psi^{+,++\cdots}|^2, \ (\rho_A)^-_- = (\rho_A)^{--} = (-1)^n|\psi^{-,--\cdots}|^2$

$S_A = -(\rho_A)^+_+(\log\rho_A)^+_+ - (\rho_A)^-_-(\log\rho_A)^-_-$

n odd: $+ve$ norm does not give $+ve$ EE.

Even number of ghost-spins: sensible calculations and interpretation.