Hyperscaling violation and entanglement entropy in gauge/string theory

K. Narayan
Chennai Mathematical Institute

• Introduction, summary
• Lightcone SYM, string theory, $AdS$ plane waves
• $AdS$ plane waves, hyperscaling violation, entanglement entropy

Introduction

Interesting to explore holography with reduced symmetries.
Generalizations of AdS/CFT to nonrelativistic systems
  $\rightarrow$ holographic condensed matter, . . .
[Son; Balasubramanian, McGreevy; Adams et al; Herzog et al; Maldacena et al; . . .]
  $\rightarrow$ Phases of gauge/string theory with non-relativistic symmetries.
Introduction

Generalizations of AdS/CFT to nonrelativistic systems

→ holographic condensed matter, . . .

[Son; Balasubramanian, McGreevy; Adams et al; Herzog et al; Maldacena et al; . . .]

→ Phases of gauge/string theory with non-relativistic symmetries.

Sometimes qualitative features of some of these phases from the gravity/string descriptions can be distilled to give interesting, perhaps unexpected insights for gauge theories.
Introduction

Nonrelativistic AdS/CFT → holographic condensed matter . . .

→ Phases of gauge/string theory with non-relativistic symmetries.
Sometimes qualitative features of some of these phases from the gravity/string descriptions can be distilled to give interesting, perhaps unexpected insights for gauge theories.

In the present context: entanglement entropy in various 3- and 4-dim quantum field theories exhibits a leading divergence called the area law.

Ryu-Takayanagi bulk prescription for EE using gravity dual.

• Gravity systems with hyperscaling violation exhibiting area law deviations can be realized in string theory in terms of strongly coupled CFTs with uniform lightcone momentum density $T_{++}$.

• Lightcone super Yang-Mills conformal field theories with $T_{++} \rightarrow$ logarithmic violation of area law for holographic entanglement entropy.

• Stronger area law deviations in some strongly coupled 3-dim CFTs.
Non-relativistic AdS/CFT

4-dim $\mathcal{N}=4$ superconformal Yang-Mills theory
dual to IIB string theory on $AdS_5 \times S^5$.

$$[ds^2 = \frac{R^2}{r^2}(-2 dx^+ dx^- + dx_i^2 + dr^2) + R^2 d\Omega_5^2, \quad R^4 \sim g_{YM}^2 N \alpha'\alpha'^2]$$

Large $N$ limit $\rightarrow$ classical gravity description useful.

A fairly traditional way that gives non-relativistic quantum field theory is to use lightcone variables ($x^- \equiv$ time), and study the system with nonzero lightcone momentum $[e.g. \quad p_\mu p^\mu = 0 \rightarrow p_- = \frac{1}{2p_+}p_i^2].$

For instance, $x^+$-DLCQ of relativistic $\mathcal{N}=4$ SYM $\rightarrow$

$z = 2$ nonrelativistic (Galilean) 2+1-dim system.

DLCQ $x^+$ of $AdS_5$ in lightcone coordinates — nonrelativistic, Schrodinger (Galilean) symmetries $[\text{Goldberger, Barbon et al, Maldacena et al}].$
Holography and Lifshitz scaling

**Lifshitz:** \( t, x_i \)-translations, \( x_i \)-rotations, scaling \( t \rightarrow \lambda^z t, x_i \rightarrow \lambda x_i \)

\([z: \text{dynamical exponent}]. \text{ (smaller than Galilean symmetries: e.g. boosts broken)}\)

Landau-Ginzburg action (free \( z = 2 \) Lifshitz): \( S = \int d^3 x ((\partial_t \varphi)^2 - \kappa (\nabla^2 \varphi)^2) \).

**Lifshitz spacetime:** \( ds_4^2 = -\frac{dt^2}{r^{2z}} + \frac{dx_i^2 + dr^2}{r^2} \). Kachru, Liu, Mulligan

\([z = 1: \text{AdS}]\)

Solution to 4-dim gravity with \( \Lambda < 0 \) and massive gauge field \( A \sim \frac{dt}{r^z} \)

(or alternatively gauge field + 2-form: dualize to get \( A \)-mass ) Taylor

[More general Lifshitz-attractor solutions in Einstein-Maxwell-scalar theories Trivedi et al]

[Focus on zero temperature solutions] String realizations?

Hyperscaling violation and entanglement entropy in gauge/string theory, K. Narayan, CMI – p.6/24
Lifshitz scaling in string theory

Lifshitz: $t, x_i$-translations, $x_i$-rotations, scaling $t \to \lambda^z t$, $x_i \to \lambda x_i$

$[z$: dynamical exponent]. (smaller than Galilean symmetries: e.g. boosts broken)

Lifshitz spacetime: $ds_4^2 = -\frac{dt^2}{r^{2z}} + \frac{dx_i^2 + dr^2}{r^2}$. Kachru, Liu, Mulligan

String construction for $z = 2$ Lifshitz [Balasubramanian, KN]:

relativistic $\mathcal{N}=4$ SYM $\xrightarrow{x^+ - \text{DLCK}} z = 2$ non-rel (Galilean) 2+1-dim system.

Gauge coupling $g_{YM}^2(x^+) = e^{\Phi(x^+)}$ varying in lightlike $x^+$-direction $\to$ breaks $x^+$-shift reducing to 2+1-dim Lifshitz symmetries.

**Bulk:** $AdS + g_{++} \sim r^0 \xrightarrow{x^+-\text{dim.redn.}} z = 2$ Lifshitz.

Kaluza-Klein $x^+$-reduction of non-normalizable null deformations of $AdS_5 \times S^5$: $g_{++}$ sourced by lightlike scalar (dilaton $\Phi(x^+)$) $[x^- \equiv \text{time}].$

$[ds_5^2 = \frac{R^2}{r^2}[-2dx^+ dx^- + dx_i^2 + dr^2] + \# r^0 (dx^+)^2 \to ds_4^2 = -\frac{(dx^-)^2}{r^4} + \frac{dx_i^2 + dr^2}{r^2}]$

More general: $g_{++}$ sourced by axion-dilaton, fluxes [Donos, Gauntlett; et al].
**SYM, $T_{++} \neq 0$: AdS plane waves**

[Relativistic $\mathcal{N}=4$ SYM $\xrightarrow{x^+\text{DLCQ}} z = 2$ non-rel (Galilean) 2+1-dim system. Lightcone $AdS_5 \xrightarrow{x^+\text{DLCQ}} z = 2$ non-rel Schrodinger (Galilean) symmetries].

Consider $\mathcal{N}=4$ SYM with uniform lightcone momentum density $T_{++}$. At large $N$, we expect the gravity dual to this excited state to be some normalizable deformation of $AdS_5 \times S^5$, with $g_{++}$ nonzero.
SYM, $T_{++} \neq 0$: AdS plane waves

[Relativistic $\mathcal{N}=4$ SYM $\xrightarrow{x^+-\text{DLCQ}} z = 2$ non-rel (Galilean) 2+1-dim system. Lightcone $AdS_5 \xrightarrow{x^+-\text{DLCQ}} z = 2$ non-rel Schrodinger (Galilean) symmetries].

Consider $\mathcal{N}=4$ SYM with uniform lightcone momentum density $T_{++}$. At large $N$, we expect the gravity dual to this excited state to be some normalizable deformation of $AdS_5 \times S^5$, with $g_{++}$ nonzero.

Can identify this precisely: $AdS_5$ plane wave $[AdS + g_{++}]$,

$$ ds^2 = \frac{R^2}{r^2} \left[ -2dx^+dx^- + dx_i^2 + dr^2 \right] + R^2 Qr^2(dx^+)^2 + R^2 d\Omega_5^2 $$

Holographic boundary stress tensor using AdS/CFT rules gives

$$ T_{++} = \frac{2Q}{8\pi G_5} \rightarrow \text{uniform lightcone momentum density.} $$
SYM, $T_{++} \neq 0$: AdS plane waves

AdS$_5$ plane wave: $ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + R^2 Qr^2(dx^+)^2 + R^2 d\Omega_5^2$

- AdS analogs of plane-waves: $AdS + g_{++}$. Supersymmetric. Likely $\alpha'$-exact string backgrounds due to lightlike nature.

- Nonzero (boundary) lightcone momentum density $T_{++}$: null energy condition $\Rightarrow T_{++} \geq 0$.

- Can be realized as “zero temperature”, highly boosted double scaling limit of boosted AdS Schwarzschild black branes (Singh).

  Simple excited pure states in $\mathcal{N}=4$ SYM CFT (on D3-branes):
  double scaling limit of SYM thermal state.

- Generalize to AdS$_D$ plane wave (e.g. states in M2-, M5- CFTs):
  $ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + R^2 Qr^{D-3}(dx^+)^2 + R^2 d\Omega_5^2$

  Bulk $x^+$-dimensional reduction?
**AdS plane waves: \( x^+ \)-dim. redn.**

\[
\begin{align*}
    ds^2 &= \frac{R^2}{r^2} \left[ -2dx^+ dx^- + dx_i^2 + dr^2 \right] + R^2 Q r^2 (dx^+)^2 + R^2 d\Omega_5^2 \\
    AdS \text{ plane wave: } Q &\sim \text{lightcone-momentum density } T_{++} \quad [R^4 \sim g^2_{YM} N \alpha'^2].
\end{align*}
\]

Dimensionally reduce 10-dim metric on \( S^5 \) and on \( x^+ \)-circle:

4-dim Einstein metric \((x^- \equiv t)\):

\[
    ds^2_E = \frac{R^3 \sqrt{Q}}{r} \left( -\frac{dt^2}{Q r^4} + dx_i^2 + dr^2 \right),
\]

Electric gauge field \( A = -\frac{dt}{Q r^4} \), scalar \( e^\phi \sim r \). Nontrivial IR scales \( R, Q \).
Hyperscaling violation

\[ ds^2 = \frac{R^2}{r^2} [-2dx^+dx^- + dx_i^2 + dr^2] + R^2Qr^2(dx^+)^2 + R^2d\Omega_5^2 \]

AdS plane wave: \( Q \sim \) lightcone-momentum density \( T_{++} \) \( [R^4 \sim g^2_{YM} N \alpha'^2] \).

Dimensionally reduce 10-dim metric on \( S^5 \) and on \( x^+ \)-circle:

4-dim Einstein metric \((x^- \equiv t)\): \( ds^2_E = \frac{R^3\sqrt{Q}}{r} \left( -\frac{dt^2}{Qr^4} + dx_i^2 + dr^2 \right) \),

Electric gauge field \( A = -\frac{dt}{Qr^4} \), scalar \( e^\phi \sim r \). Nontrivial IR scales \( R, Q \).

\[ ds^2 = r^{2\theta/d}( -\frac{dt^2}{r^{2z}} + \frac{dx_i^2+dr^2}{r^2}) \]. \( d = \) “boundary” spatial dim \((x_i)\).

\( \theta = \) hyperscaling violation exponent.

Conformal to Lifshitz (dynamical exponent \( z \)).

Arise in effective gravity+vector+scalar theories (Trivedi et al, Kiritsis et al)

Thermodynamics reflects effective space dim \((d - \theta)\) \( [e.g. S \sim T^{(d-\theta)/z}] \).

Above: \( d = 2, \ z = 3, \ \theta = 1 \) \( (d_{eff} = d - \theta = 1) \).
Hyperscaling violation

\[
ds^2 = \frac{R^2}{r^2} \left[ -2dx^+dx^- + dx_i^2 + dr^2 \right] + R^2 Q r^{D-3}(dx^+)^2
\]

*AdS* \(D\) plane wave, \(D = d + 3\), \(Q\) lightcone-momentum density, \((D - 1)\)-dim.

Dim’nal redn: \(ds^2_E = \frac{R^2(R^2 Q)^{1/(D-3)}}{r^2} \left( - \frac{dt^2}{Q r^{D-1}} + dx_i^2 + dr^2 \right)\).

“boundary” spatial dimension \(d = D - 3\), \(z = \frac{d}{2} + 2\), \(\theta = \frac{d}{2}\).

In particular, for the conformal branes of M-theory:

* \(M2\)-brane stacks \(\rightarrow\) *AdS* \(4\) deformations, \(d = 1\), \(z = \frac{5}{2}\), \(\theta = \frac{1}{2}\).

* \(M5\)-brane stacks \(\rightarrow\) *AdS* \(7\) deformations, \(d = 4\), \(z = 4\), \(\theta = 2\).

[General dim’nal reduction: \(\int d^D x \sqrt{-g^{(D)}} (R^{(D)} - 2\Lambda) = \int dx^+ d^{D-1} x \sqrt{-g^{(D-1)}} (R^{(D-1)} - \# \Lambda e^{-2\phi/(D-3)} - \# (\partial \phi)^2 - \# e^{2(D-2)\phi/(D-3)} F_{\mu \nu}^2)\)

(recall effective gravity+vector+scalar theories, *e.g.* Trivedi et al, Kiritsis et al, Takayanagi et al)]
Hyperscaling violation, holog. EE

Entanglement entropy for subsystem $A$, $S_A = -tr\rho_A \log \rho_A$,
partial trace over $A$-complement, $\rho_A = tr_B \rho$.

• Ryu-Takayanagi bulk prescription:
find bulk surface bounding $A$ with minimal area.

Holographic entanglement entropy $S_A$ for subsystem $A$
$\propto$ area of bulk minimal surface bounding $A$.

E.g.: SYM CFT ground state, strip with width $l$, bndry area $L^2$:
ordinary time-$t$ slice in bulk $AdS_5$ $\rightarrow$ $S_A \sim \frac{R^3}{G_5} \left( \frac{L^2}{\epsilon^2} - \# \frac{L^2}{l^2} \right)$, 
with leading divergence reflecting area law (Bombelli et al; Srednicki, ...)
$[\epsilon = SYM$ UV cutoff$]$, and finite part.

• Hyperscaling violating bulk metrics $ds^2 = r^{2\theta/d} \left( -\frac{dt^2}{r^2 z} + \frac{dx_i^2 + dr^2}{r^2} \right)$
with $\theta = d - 1$ $\rightarrow$ logarithmic violation of area law.

Gravitational dual of hidden Fermi surfaces? Takayanagi et al, Sachdev et al
AdS plane waves, holog. EE

- Ryu-Takayanagi bulk prescription: holographic entanglement entropy $S_E$ for subsystem $A$ 
  \( \propto \text{area of bulk minimal surface bounding } A \).

- Hyperscaling violating bulk metrics
  \( ds^2 = r^{2\theta/d} \left( -\frac{dt^2}{r^2z} + \frac{dx_i^2 + dr^2}{r^2} \right) \)
  with \( \theta = d - 1 \) \( \rightarrow \) logarithmic violation of area law.

- As we have seen, AdS plane waves give microscopic (top-down) descriptions for some of these: e.g.
  \( ds^2 = \frac{R^2}{r^2} [-2dx^+ dx^- + dx_i^2 + dr^2] + R^2 Q r^2 (dx^+)^2 + R^2 d\Omega_5^2, \)

  \( \text{AdS}_5 \text{ plane wave: } \frac{x^+ - \text{dim.redn.}}{\frac{1}{2}} \rightarrow d = 2, \ z = 3, \ \theta = 1 \)

- Concrete gauge/string example: $\mathcal{N}=4$ SYM in pure excited state
  with uniform lightcone momentum density $T_{++} \rightarrow$
  logarithmic behaviour (rather than area law) of entanglement entropy holographically.
AdS plane waves, holog. EE

\( AdS_5 \) plane wave: \( ds^2 = \frac{R^2}{r^2} [-2dx^+dx^- + dx_i^2 + dr^2] + R^2 Q r^2 (dx^+)^2 + R^2 d\Omega_5^2 \).

- \( AdS_5 \) plane wave \( \xrightarrow{x^+\text{-dim. redn.}} \) hyperscaling violation \( ds^2 = r^{2\theta/d} \left( -\frac{dt^2}{r^{2z}} + \frac{dx_i^2 + dr^2}{r^2} \right) \)

with \( d = 2, \ z = 3, \ \theta = 1 \).

4-dim \( \mathcal{N}=4 \) SYM CFT in these pure excited states (\( T_{++} \sim Q \)) exhibits logarithmic behaviour of holog. entanglement entropy (not area law).

Holographic EE \( S_E = \text{area of bulk minimal surface bounding A} \).
Subsystem A = strip in \( x_i \)-plane, width \( l \) (possibly wrapping \( x^+ \)-direction), lying on const-\( x^- \) slice (\( \equiv \) constant time-\( t \) slice (4-dim)).

Logarithmic behaviour of EE: \( S_A = \frac{R^3 \sqrt{Q}}{2G_5} L + L \log \frac{l}{\epsilon} \).

[This uses only 5-d spacetime: also applies to various \( \mathcal{N}=1 \) SYM CFTs.]

Explore further . . . Weak coupling field theory?
**AdS plane waves, holo. EE**

$AdS_5$ plane wave:  \[ ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + R^2Qr^2(dx^+)^2 \]

[EE for subsystem A, $S_A = -tr \rho_A \log \rho_A$, partial trace over $A$-complement, $\rho_A = tr_B \rho$.]

Want to de-mystify entanglement entropy in null slicing:

consider spacelike subsystems ($\Delta x^+ > 0 > \Delta x^-$, strip, width $l$).

Non-static spacetime: use covariant HEE (Hubeny, Rangamani, Takayanagi).

HEE $\sim$ area of bulk extremal surface bounding $A$

(stationary point of area functional; if several surfaces exist, choose minimal area).

Two choices for subsystem depending on energy flux $T_{++}$ direction:

![Diagram showing two cases for subsystems A and B]

(spacelike subsystem: leading divergence is area law $\sim \frac{V_2}{\epsilon^2}$.).
\(AdS\) plane waves, holo. EE

\(AdS_5\) plane wave:
\[
ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + R^2 Qr^2 (dx^+)^2
\]

Simple excited pure states in \(\mathcal{N}=4\) SYM CFT: \(T_{++} = \text{const.}\)

Case A: width direction is \(x_i\).

Finite piece \(V_2 \sqrt{Q} \log(lQ^{1/4})\), grows with size.

Smaller than \(\sim l\) (thermal entropy),

larger than \(\sim l^{-2}\) (CFT ground state).

Strip along energy flux: (heuristically) size increases, entanglement increases.

Case B: width along \(x_3\) (and flux).

Phase transition: \(S_A \sim \text{constant}\) (saturated) beyond \(l_c \sim Q^{-1/4}\)

(no connected surface, \(\Delta x^+ > 0 > \Delta x^-\); disconnected surfaces for large \(l\)).

Strip orthogonal to energy flux: correlation length \(\sim Q^{-1/4}\).

Large \(l\): entanglement saturation.

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AdS plane waves, holo. EE

\( \text{AdS}_5 \) plane wave: \( ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + R^2Qr^2(dx^+)^2 \)

To understand phase transition better, consider regulated \( \text{AdS}_5 \) plane wave: boosted black D3-brane (Singh).

\[
ds^2 = \frac{1}{r^2} \left[ -2dx^+dx^- + Qr^4 \left( dx^+ + \frac{r_0^4}{2Q} dx^- \right)^2 + dx_i^2 \right] + \frac{dr^2}{r^2(1-r_0^4 r^4)}.
\]

Boost \( \lambda^2 = \frac{2Q}{r_0^4} \): as \( r_0 \to 0 \), \( Q = \text{fixed} \), recover \( \text{AdS}_5 \) plane wave.

For small \( r_0 \), the scale \( Q \) dominates, so phase transition persists (no connected extremal surface for large size with spacelike subsystem).

Connected extremal surface for Case B can be found by scaling towards horizon, and towards double zero of surface equation (which gives large size) [becomes disconnected surface in \( \text{AdS} \) plane wave limit].
Hyperscaling violation cont’d

\[ ds^2 = \frac{R^2}{r^2} [-2dx^+ dx^- + dx_i^2 + dr^2] + R^2 Q r^{D-3} (dx^+)^2 \]

AdS\(_D\) plane wave, \( D = d + 3 \), \( Q \) lightcone-momentum density, \((D-1)\)-dim.

Dim’nal redn:
\[ ds^2_E = \frac{R^2 (R^2 Q)^{1/(D-3)}}{r} (- \frac{dt^2}{Q r^{D-1}} + dx_i^2 + dr^2) . \]

“boundary” spatial dimension \( d = D - 3 \), \( z = \frac{d}{2} + 2 \), \( \theta = \frac{d}{2} \).

In particular, for the conformal branes of M-theory:

\( M2 \)-brane stacks \( \rightarrow \) AdS\(_4\) deformations, \( d = 1 \), \( z = \frac{5}{2} \), \( \theta = \frac{1}{2} \).

\( M5 \)-brane stacks \( \rightarrow \) AdS\(_7\) deformations, \( d = 4 \), \( z = 4 \), \( \theta = 2 \).

[General dim’nal reduction:
\[ \int d^D x \sqrt{-g^{(D)}} (R^{(D)} - 2\Lambda) = \]
\[ \int dx^+ d^{D-1} x \sqrt{-g^{(D-1)}} (R^{(D-1)} - \# \Lambda e^{-2\phi/(D-3)} - \# (\partial \phi)^2 - \# e^{2(D-2)\phi/(D-3)} F_{\mu\nu}^2) \]
(_recall effective gravity+vector+scalar theories, \text{e.g. Trivedi et al, Kiritsis et al, Takayanagi et al)\]

\textbf{M2-brane CFT: super 3-dim Chern-Simons theory with matter.}

\textbf{Deviations from area law stronger than logarithmic.}

\textbf{Entanglement (finite piece) grows with subsystem size as} \( L_+ \sqrt{Ql} \).
Hyperscaling violation

(Recall Dp-brane phases, Itzhaki, Maldacena, Sonnenschein, Yankielowicz)

Conformal M-theory brane stack $ds^2 = ds^2_{AdS \times S} + g_{++} (dx^+)^2$,
for $g_{++}$ (non-)normalizable deformations, gives rise in appropriate
IIA string/sugra phase to hyperscaling violation and new $z, \theta$ with

$$ds^2_E = r^{2/\theta/d} (\frac{dt^2}{r^{2z}} + \frac{dx_i^2 + dr^2}{r^2})$$

$$\left[ ds^2_{M-th} \rightarrow 10-d ds^2_{st} + \text{dilaton } \Phi \rightarrow ds^2_E \rightarrow \text{dim.redn.}^{\text{sphere}S} \rightarrow \right.$$  
null-deformed solution, non-compact $x^+ \rightarrow \text{dim.redn.}^{x^+} \longrightarrow z, \theta$.

Note: 10-d solutions can be checked independently in IIA supergravity.

D$p$-branes $\rightarrow d = p, \ z = 1, \ \theta \neq 0$. (Dong, Harrison, Kachru, Torroba, Wang)

M2 $\rightarrow$ D2 null non-normalizable: $d = 1, \ z = \frac{7}{3}, \ \theta = 0$.

M2 $\rightarrow$ D2 null normalizable: $d = 1, \ z = 3, \ \theta = \frac{2}{3}$.

M5 $\rightarrow$ D4 null non-normalizable: $d = 3, \ z = 2, \ \theta = -1$.

M5 $\rightarrow$ D4 null normalizable: $d = 3, \ z = 4, \ \theta = \frac{1}{3}$.

Note: some of these (with $d = 1$) have $d - 1 \leq \theta \leq d$ $\rightarrow$
Violation of area law of entanglement entropy. Field theory duals nontrivial: explore?
Hyperscaling violation

(Following D<sub>p</sub>-brane phase structure of Itzhaki, Maldacena, Sonnenschein, Yankielowicz): These solutions are of the form of null-deformed D<sub>p</sub>-brane systems, with rich phase structure. \( z, \theta \)-values flow.

**Null-deformed D2-brane phases:** flow from DLCQ<sub>x+</sub> of 2+1-dim perturbative SYM (UV) regime \( \rightarrow \) IIA supergravity region with null-deformed D2-solution (valid in some intermediate regime where \( x^+ \)-circle large relative to 11-th circle) \( \rightarrow \) 11-dim \( AdS_4 \) null-deformed M2-brane IR phase (dual to DLCQ of null-defmn of Chern-Simons ABJM-like theory).

**Null-deformed D4-brane phases:** flow from \( AdS_7 \) null-deformed M5-brane UV phase (dual to null-deformation of (2, 0) M5-theory) \( \rightarrow \) intermediate IIA supergravity region with null-deformed D4-solution \( \rightarrow \) DLCQ<sub>x+</sub> of 4+1-dim perturbative SYM IR phase.
**AdS** null defmns + inhomogeneities

Most general family of (static) AdS plane waves:

\[ ds^2 = \frac{1}{r^2} (-2dx^+dx^- + dx_i^2) + g_{++}[r, x_i](dx^+)^2 + \frac{dr^2}{r^2} + d\Omega_5^2 \]

- AdS analogs of plane-waves: AdS + g_{++}. Supersymmetric. Likely \(\alpha'\)-exact string backgrounds due to lightlike nature.

- Restricting to normalizable modes (near boundary), these are general lightcone states in SYM with inhomogeneities: nonzero lightcone momentum density \(T_{++}\) varying spatially (frozen in lightcone time).

- Static normalizable backgrounds: generically, \(g_{++}\) vanishes at specific locii in the interior, even if positive definite near boundary (i.e. \(T_{++} \geq 0\)). Effectively a horizon: time-like Killing vector \(\partial_- \rightarrow \text{null}\).

- On \(x^+\)-dimensional reduction, this means string modes winding around \(x^+\)-circle become light in the vicinity of \(g_{++} = 0\) locii. New stringy physics beyond the gravity approximation: new operators in SYM with low anomalous dimensions.

Similar inhomogenous solutions exist for asymptotically Lifshitz solutions too.
Conclusions, open questions

- $AdS$ plane waves $\rightarrow$ dim’nal redux
  $\rightarrow$ Lifshitz scaling, hyperscaling violation.
Dual to 4-d SYM CFT (or more exotic 3d CFT) excited state with $T_{++}$.
- Some of these lead to logarithmic (or stronger) deviations of entanglement entropy holographically relative to area law.
- $AdS$ plane waves: simple excited pure states. Entanglement entropy can be studied explicitly for strip-shaped subsystems, results depend on strip direction w.r.t. energy flux. For strip orthogonal to flux, phase transition.
More general $AdS$ null deformations with inhomogeneities.

Entanglement entropy area law deviations from field theory?
What are these materials?

… …