Aspects of Nonrelativistic Holography: Entanglement, Viscosity and AdS_2

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- Holography, nonrelativistic theories, string realizations
- Entanglement entropy and shear viscosity
- AdS_2 holography from redux, renormalization group flows and c-functions

Based on work in part with Koushik Balasubramanian, Kedar Kolekar, Debangshu Mukherjee, Tadashi Takayanagi, Sandip Trivedi

AdS/CFT and holography

21yrs since AdS/CFT '97 Maldacena; '98 Gubser,Klebanov,Polyakov; Witten.

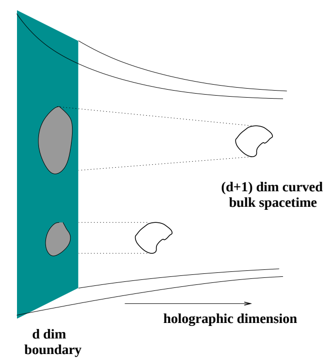
Holography: quantum gravity in $\mathcal{M} \leftrightarrow$ dual without gravity on $\partial\mathcal{M}$ ('t Hooft, Susskind).

N D3-branes, near horizon $\rightarrow$ IIB strings on $AdS_5 \times S^5$

$\longleftrightarrow$ 4-dim max susy Yang-Mills (CFT) (Maldacena)

$\rightarrow$ best known AdS_{d+1}/CFT_d example.

Large $N \rightarrow$ gravity approximation



Holographic handle on strongly coupled gauge theory (CFT) physics.

Over the years, many explorations and generalizations of AdS/CFT

$\rightarrow$ geometric handle on physical observables in field theory.

[AdS = anti de Sitter space] [CFT conformal symmetry = bulk AdS isometry group.]

[Dictionary: $Z_{CFT} = Z_{string} \simeq e^{-S_{sugra}}$ (Gubser,Klebanov,Polyakov; Witten)]

[AdS black brane $\xrightarrow{\text{dual to}}$ CFT thermal state]

Nonrelativistic Holography

Holography with reduced symmetries. Nonrelativistic (condmat-like) phases of gauge theories $\leftrightarrow$ new phases of black holes/branes.

Lifshitz spacetime: $ds^2 = -\frac{dt^2}{r^{2z}} + \frac{dx_i^2+dr^2}{r^2}$. (Kachru,Liu,Mulligan; Taylor)

scaling $t \to \lambda^z t,\ x_i \to \lambda x_i$ [dynamical exponent z $(z \geq 1)$] t, x_i-translations, x_i-rotations

[smaller than Schrodinger symm $e.g.$ ~~Galilean boosts~~] [gravity, $\Lambda < 0$, massive gauge field]

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More general gravity phases: $ds^2 = r^{2\theta/d_i}\big(-\frac{dt^2}{r^{2z}} + \frac{dx_i^2+dr^2}{r^2}\big)$.

θ = hyperscaling violation exponent; d_i = boundary spatial dim (x_i).

- Conformally Lifshitz. Effective Einstein-Maxwell-Dilaton theories (Trivedi et al; Kiritsis et al, ...)

  $S \sim T^{(d_i-\theta)/z}$. Thermodynamics $\sim$ space dim $d_{eff} = d_i - \theta$: actual space is d_i-dim.

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- $\theta = d_i - 1$: entanglement entropy $\sim$ $\log l$, logarithmic behaviour.

  Gravity duals of Fermi surfaces? (Ogawa,Takayanagi,Ugajin; Huijse,Sachdev,Swingle)

- $d_i - 1 \leq \theta < d_i$: EE area law violations. (Dong,Harrison,Kachru,Torroba,Wang)

  [Energy conditions: $(d_i - \theta)(d_i(z-1) - \theta) \geq 0$, $(z-1)(d_i + z - \theta) \geq 0$.]

Lif/hvLif, gauge/string realizations

UV completion. Narrow gravity parameter space. Identify recognizable CFT deformations/regimes.

- $z=2$ Lifshitz: AdS nonnormalizable defmns, x^+redn (Balasubramanian,KN; Donos,Gauntlett..)

  $g_{++}\sim r^0$ sourced by lightlike matter, *e.g.* $g_{++}\sim(\partial_+c_0)^2$, axion $c_0=Kx^+$: $(x^-\equiv t)$

  $$ds^2=\frac{R^2}{r^2}[-2dx^+dx^-+dx_i^2+dr^2]+K^2R^2(dx^+)^2\longrightarrow ds^2=-\frac{dt^2}{r^4}+\frac{\sum_{i=1}^{d_i}dx_i^2+dr^2}{r^2}$$

- Hyperscaling violation: nonconformal branes (Dong,Harrison,Kachru,Torroba,Wang)

  $$ds_{st}^2=\frac{r^{(7-p)/2}}{R_p^{(7-p)/2}}dx_{\|}^2+\frac{R_p^{(7-p)/2}}{r^{(7-p)/2}}(dr^2+r^2d\Omega_{8-p}^2),\qquad e^{\Phi}=g_s\left(\frac{R_p^{7-p}}{r^{7-p}}\right)^{\frac{3-p}{4}}.$$

  $$g_{YM}^2\sim g_s\alpha'^{(p-3)/2},\quad R_p^{7-p}\sim g_{YM}^2N\alpha'^{5-p}$$

  Dp-branes $\longrightarrow$ S^{8-p} redux $\longrightarrow$ $d_i=p,\ z=1,\ \theta=p-\frac{9-p}{5-p}$.

- Hyperscaling violation: AdS_{d+1} plane waves (KN), (Singh)

  [Normalizable g_{++}] $\quad ds^2=\frac{R^2}{r^2}[-2dx^+dx^-+dx_i^2+dr^2]+R^2Qr^{d-2}(dx^+)^2\quad\longrightarrow$

  $$ds^2=r^{\frac{2\theta}{d_i}}\left(-\frac{dt^2}{r^{2z}}+\frac{\sum_{i=1}^{d_i}dx_i^2+dr^2}{r^2}\right),\quad z=\frac{d-2}{2}+2,\quad \theta=\frac{d-2}{2},\quad d_i=d-2.$$

  Anisotropic CFT excited state, energy-momentum density $T_{++}=Q$.

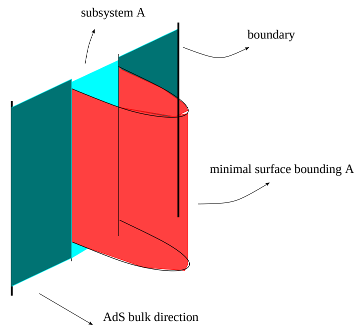
  AdS_5 plane wave: $d=4,\ d_i=2,\ \theta=1,\ z=3$. Logarithmic behaviour of EE.

Nonrelativistic gauge/gravity duality and entanglement entropy

Holographic Entanglement Entropy

Entanglement entropy: entropy of reduced density matrix of subsystem.

EE for spatial subsystem A, $S_A = -tr\rho_A \log \rho_A$, with partial trace $\rho_A = tr_B \rho$.



Ryu-Takayanagi: $EE = \frac{A_{min.surf.}}{4G}$

[$\sim$ black hole entropy] Area of codim-2 minimal surface in gravity dual.

Non-static situations: extremal surfaces (Hubeny, Rangamani, Takayanagi).

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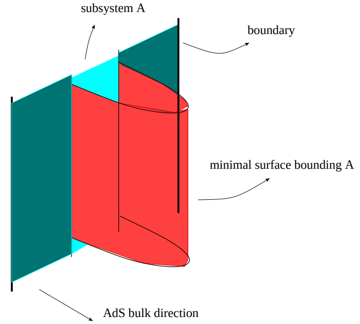

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Example: CFT$_d$ ground state = empty AdS_{d+1}, $ds^2 = \frac{R^2}{r^2}(dr^2 - dt^2 + dx_i^2)$.

Strip, width $\Delta x = l$, infinitely long. Bulk surface $x(r)$. Turning point r_*.

$$S_A = \frac{V_{d-2}R^{d-1}}{4G_{d+1}} \int \frac{dr}{r^{d-1}} \sqrt{1 + (\partial_r x)^2} \quad \rightarrow \quad (\partial_r x)^2 = \frac{(r/r_*)^{2d-2}}{1-(r/r_*)^{2d-2}} \,, \quad \frac{l}{2} = \int_0^{r_*} dr \partial_r x$$

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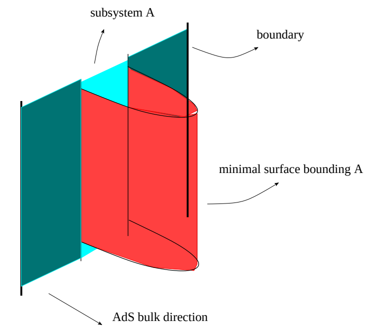

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$$S_A = \frac{V_{d-2}R^{d-1}}{4G_{d+1}} \int_\epsilon^{r_*} \frac{dr}{r^{d-1}} \frac{2}{\sqrt{1-(r/r_*)^{2d-2}}} \longrightarrow S_A = \frac{R}{2G_3} \log \frac{l}{\epsilon} , \quad \frac{3R}{2G_3} = c \quad [2d].$$

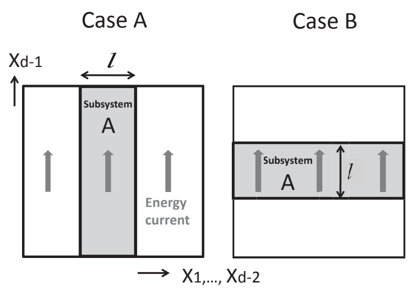
$$S_A \sim \frac{R^{d-1}}{G_{d+1}} \left(\frac{V_{d-2}}{\epsilon^{d-2}} - c_d \frac{V_{d-2}}{l^{d-2}} \right) , \qquad \frac{R^3}{G_5} \sim N^2 \quad [4d], \quad \frac{R^2}{G_4} \sim N^{3/2} \quad [3d].$$

CFT thermal state (AdS black brane): minimal surface wraps horizon. $S^{fin} \sim N^2 T^3 V_2 l$

Entanglement, AdS plane waves

$ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + R^2 Q r^{d-2}(dx^+)^2$, dual to CFT state, $T_{++} \sim Q$

EE, spacelike strips (width l, $\Delta x^+ > 0 > \Delta x^-$). (KN, Takayanagi, Trivedi)



Non-static spacetime → extremal surfaces.
Spacelike subsystem, UV cutoff ϵ:
leading divergence is area law $\sim \frac{V_{d-2}}{\epsilon^{d-2}}$

Case A: width direction x_i. Strip along energy flux.

Finite cutoff-independent part of EE: size-dependent measure of entanglement $S^{fin} \sim N^2\sqrt{Q}V_2 \log(lQ^{1/4})$ [d=4].

[ground st] $-N^2\frac{V_2}{l^2} < S^{fin} < N^2T^3V_2 l$ [thermal entr]

Case B: Strip ⊥ flux.

Phase transition (no connected surface if $\Delta x^+ > 0 > \Delta x^-$).

S_A saturated for $l \gtrsim Q^{-1/4}$.

Boosted black branes (Maldacena,Martelli,Tachikawa): large boost λ, low temperature r_0 limit (Singh)

$$ds^2 = \frac{R^2}{r^2}\Big(- 2dx^+dx^- + \frac{r_0^4 r^4}{2}(\lambda dx^+ + \lambda^{-1}dx^-)^2 + \sum_i dx_i^2\Big) + \frac{R^2 dr^2}{r^2(1-r_0^4r^4)} .$$

regulated AdS_5 plane wave $\big(Q = \frac{\lambda^2 r_0^4}{2}\big)$ CFT thermal state, highly boosted: anisotropic.

Mutual Information

MI (disjoint subsystems A & B): $I[A,B] = S[A] + S[B] - S[A \cup B]$.

$I[A,B] \geq 0$. Cutoff-dependent divergences cancel. Gives bound for correlation fns.

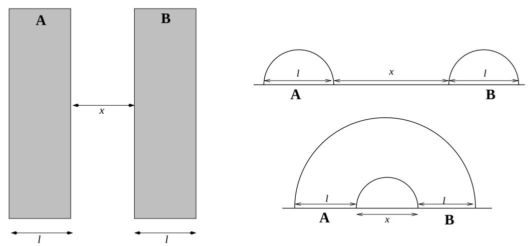
Holographic mutual information: find extremal surface for $A \cup B$.

Subsystems far, two disjoint minimal surfaces: $MI = 0$.

Subsystems nearby, connected surface has lower area.

Ryu-Takayanagi $\Rightarrow$ MI disentangling transition (Headrick).

[This is large N: expect softer subleading decay for MI.]



Similar disentanglement for thermal states (Fischler,Kundu,Kundu): $\frac{x_c}{l} \sim 0$ (for $x, l \gg \frac{1}{T}$).

(Mukherjee, KN) MI for AdS plane wave excited states $\rightarrow$

critical separation $\frac{x_c}{l}$ between subsystems smaller than in ground state.

Mutual information disentangling occurs faster.

Suggests energy density disorders system.

Nonrelativistic holography and shear viscosity

The viscosity bound

Strongly coupled theories with Einstein gravity duals obey the

viscosity bound, $\frac{\eta}{s} \geq \frac{1}{4\pi}$. (Kovtun,Son,Starinets) [RHIC, QGP $\rightarrow$ ideal fluid . . .]

Nonrelativistic theories?

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Nonrelativistic theories?

• Membrane paradigm: black brane near horizon $\sim$ fluctuating membrane; shear perturbations $h_{xy}, h_{ty}, a_y \rightarrow$ appropriate currents satisfying Fick's law $j^x = -\mathcal{D}\partial_x j^t$ & conservation $\partial_t j^t + \partial_x j^x = 0$; recast Einstein eqns for shear perturbations on this "stretched horizon" as diffusion eqn $\partial_t j^t = \mathcal{D}\partial_x^2 j^t$ via self-consistent approximations $\rightarrow$ shear diffusion constant $\mathcal{D}$. (Kolekar, Mukherjee, KN)

• Quasinormal modes: diffusive solutions to linearized perturbation equations, ingoing at horizon, vanishing at boundary $\rightarrow$ $\omega = -i\mathcal{D}q^2$, diffusion eqn in momentum space. (Mukherjee, KN)

Kubo formula: $\eta = \lim_{\omega\rightarrow 0} \frac{G_{xy,xy}(\omega,q=0)}{i\omega}$ with $G_{xy,xy} = \langle T^x_y T^x_y \rangle$.

Black brane horizon area $\rightarrow$ entropy density s.

The viscosity bound

Finite temperature 4-dim hvLif black brane: temperature $T = \frac{d_i+z-\theta}{4\pi} r_0^z$

$ds^2 = r^{2\theta/d_i} (-\frac{f(r)}{r^{2z}} dt^2 + \frac{dr^2}{f(r) r^2} + \sum_{d_i} \frac{dx_i^2}{r^2}), \quad f(r) = 1 - (r_0 r)^{d_i+z-\theta}$

$\underline{z < d_i + 2 - \theta}$: viscosity bound satisfied.

$$\mathcal{D} = \frac{r_0^{z-2}}{d_i+2-z-\theta} = \frac{1}{d_i+2-z-\theta} \cdot \left(\frac{4\pi}{d_i+z-\theta}\right)^{1-2/z} T^{\frac{z-2}{z}}$$

$$\frac{\eta}{s} = \frac{d_i+2-z-\theta}{4\pi} \left(\frac{4\pi}{d_i+z-\theta}\right)^{\frac{2-z}{z}} \mathcal{D} T^{\frac{2-z}{z}} = \frac{1}{4\pi}$$

Continuously connected to AdS, $z = 1, \theta = 0 \ \rightarrow \ \frac{\eta}{s} = \mathcal{D}T = \frac{1}{4\pi}$

Near horizon long-wavelength modes $\rightarrow \ \mathcal{D}$, hydrodynamics.

$\underline{z = d_i + 2 - \theta}$: viscosity bound satisfied, but some curious features.

Quasinormal mode analysis $\rightarrow \ \frac{\eta}{s} = \frac{1}{4\pi}$ from $q = 0$ Kubo limit

Shear diffusion const $\rightarrow$ logarithmic scaling $\mathcal{D} = \frac{1}{z} (\frac{2\pi}{z-1})^{1-2/z} \cdot T^{\frac{z-2}{z}} \log \frac{\Lambda}{T}$ (Λ UV cutoff).

Perhaps some transport coefficients nontrivial $\rightarrow$ these exponents arise from redux of AdS plane waves (highly boosted; lightlike limit?).

$\underline{z > d_i + 2 - \theta}$: universal near horizon low energy modes do not dictate hydrodynamics.

AdS_2 holography from redux, renormalization group flows and c-functions (Kolekar, KN)

Extremal black branes and AdS_2

4-dim general relativity: charged black holes $(G = c = 1)$

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + \frac{dr^2}{1-2m/r+Q^2/r^2} + r^2 d\Omega_2^2$$

Extremality $Q^2 = m^2 \rightarrow$ zero Hawking temperature (not due to susy)

Near horizon $r_0 = Q$: $ds^2 \sim -\frac{(r-r_0)^2}{r_0^2}dt^2 + \frac{r_0^2 dr^2}{(r-r_0)^2} + r_0^2 d\Omega_2^2 \rightarrow \boxed{AdS_2 \times S^2}$

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AdS_2: universal near horizon geometry, extremal black holes/branes.

- Einstein-Maxwell theory, $\Lambda < 0$: extremal relativistic black branes.
- Einstein-Maxwell-scalar $+ U(1)$: extremal hvLif black branes.

$$S = \int \sqrt{-g}\Big(R - \frac{1}{2}(\partial\Psi)^2 + |\Lambda| e^{\gamma\Psi} - \frac{e^{\lambda_1\Psi}}{4} {F^{(1)}_{\mu\nu}}^2 - \frac{e^{\lambda_2\Psi}}{4} {F^{(2)}_{\mu\nu}}^2\Big) \rightarrow \boxed{AdS_2 \times R^2}$$

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Effective 2-dim dynamics: $ds^2 = g^{(2)}_{\mu\nu}dx^\mu dx^\nu + \Phi^2 d\vec{x}^2$, Φ dilaton.

Leading backreaction $\rightarrow$ breaking of AdS_2 isometries $\equiv$ breaking of local boundary time reparametrizations (modulo global $SL(2)$ symm)

$\rightarrow$ universally captured by Schwarzian derivative action.

Almheiri,Polchinski; Jensen; Maldacena,Stanford,Zhang; Engelsoy,Mertens,Verlinde; ...

Schwarzian: breaking of emergent conformality in SYK (Sachdev,Ye,Kitaev)

Extremal branes, 2d dilaton-gravity

Einstein-Maxwell: $S = \int \sqrt{-g}[\frac{1}{16\pi G_4}(\mathcal{R} - 2\Lambda) - \frac{1}{4} F_{MN}F^{MN}], \quad \Lambda = -\frac{3}{R^2}$

$ds^2 = -\frac{r^2 f(r)}{R^2} dt^2 + \frac{R^2}{r^2 f(r)} dr^2 + \frac{r^2}{R^2} dx_i^2 \,, \quad f(r) = 1 - (\frac{r_0}{r})^3 + \frac{Q_e^2}{r^4}(1 - \frac{r}{r_0}) \,, \ Q_e^2 = 3r_0^4$

[Dim.reduction: $ds^2 = g_{\mu\nu}^{(2)} dx^\mu dx^\nu + \Phi^2 dx_i^2$]

Solve for $F_{\mu\nu}$, put back in EOM/action $\rightarrow$ dim.redux $\rightarrow$ equivalent dilaton-gravity + potential

$\longrightarrow \quad S = \frac{1}{16\pi G_2} \int d^2x \sqrt{-g} \left(\Phi^2 \mathcal{R} - U(\Phi)\right), \qquad U(\Phi) = 2\Lambda\Phi + \frac{2Q_e^2}{R^6 \Phi^3}$

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Einstein-Maxwell-scalar $+U(1)$: ($\gamma,\lambda_1,\lambda_2$ fns of z,θ)

$$S=\int\sqrt{-g}(R-\frac{1}{2}(\partial\Psi)^2+V_0e^{\gamma\Psi}-\frac{e^{\lambda_1\Psi}}{4}{F^{(1)}_{\mu\nu}}^2-\frac{e^{\lambda_2\Psi}}{4}{F^{(2)}_{\mu\nu}}^2)$$

$$ds^2=(\frac{r}{r_{hv}})^{-\theta}[-\frac{r^{2z}f(r)}{R^{2z}}dt^2+\frac{R^2}{r^2f(r)}dr^2+\frac{r^2}{R^2}dx_i^2],\qquad Q^2\sim r_0^{2(1+z-\theta)}$$

[Also $\Psi,\ F^{(1)}_{tr},\ F^{(2)}_{tr}$ fns of r] [Energy condns: $z\geq 1,\ 2z-2-\theta\geq 0,\ 2-\theta\geq 0$]

Equivalent 2-dim dilaton-gravity-scalar theory + potential:

$$S=\frac{1}{16\pi G_2}\int d^2x\sqrt{-g}\,\big(\Phi^2\mathcal{R}-\frac{\Phi^2}{2}(\partial\Psi)^2-U(\Phi,\Psi)\big),$$

$$U(\Phi,\Psi)=-V_0e^{\gamma\Psi}\Phi\ +\ \frac{1}{\Phi^3}\,\big(\frac{V_1}{e^{\lambda_1\Psi}}+\frac{V_2Q^2}{e^{\lambda_2\Psi}}\big)\qquad [V_0,V_1,V_2 \text{ fns of } z,\theta]$$

Is the near-AdS_2 theory Jackiw-Teitelboim?

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Einstein-Maxwell: $S = \int \sqrt{-g}[\frac{1}{16\pi G_4}(\mathcal{R} - 2\Lambda) - \frac{1}{4}F_{MN}F^{MN}], \quad \Lambda = -\frac{3}{R^2}$

$ds^2 = -\frac{r^2 f(r)}{R^2}dt^2 + \frac{R^2}{r^2 f(r)}dr^2 + \frac{r^2}{R^2}dx_i^2 \,, \quad f(r) = 1 - (\frac{r_0}{r})^3 + \frac{Q_e^2}{r^4}(1 - \frac{r}{r_0}) \,, \; Q_e^2 = 3r_0^4$

[Dim.reduction: $ds^2 = g^{(2)}_{\mu\nu}dx^\mu dx^\nu + \Phi^2 dx_i^2$]

Solve for $F_{\mu\nu}$, put back in EOM/action $\rightarrow$ dim.redux $\rightarrow$ equivalent dilaton-gravity + potential

$\rightarrow \quad S = \frac{1}{16\pi G_2}\int d^2x\sqrt{-g}\,\big(\Phi^2\mathcal{R} - U(\Phi)\big), \qquad U(\Phi) = 2\Lambda\Phi + \frac{2Q_e^2}{R^6\Phi^3}$

Einstein-Maxwell-scalar $+\, U(1)$: ($\gamma, \lambda_1, \lambda_2$ fns of z, θ)

$S = \int \sqrt{-g}(R - \frac{1}{2}(\partial\Psi)^2 + V_0 e^{\gamma\Psi} - \frac{e^{\lambda_1\Psi}}{4}{F^{(1)}_{\mu\nu}}^2 - \frac{e^{\lambda_2\Psi}}{4}{F^{(2)}_{\mu\nu}}^2)$

$ds^2 = (\frac{r}{r_{hv}})^{-\theta}[-\frac{r^{2z}f(r)}{R^{2z}}dt^2 + \frac{R^2}{r^2 f(r)}dr^2 + \frac{r^2}{R^2}dx_i^2], \qquad Q^2 \sim r_0^{2(1+z-\theta)}$

[Also $\Psi,\ F^{(1)}_{tr},\ F^{(2)}_{tr}$ fns of r] [Energy condns: $z \geq 1,\ 2z - 2 - \theta \geq 0,\ 2 - \theta \geq 0$]

Equivalent 2-dim dilaton-gravity-scalar theory + potential:

$S = \frac{1}{16\pi G_2}\int d^2x\sqrt{-g}\,\big(\Phi^2\mathcal{R} - \frac{\Phi^2}{2}(\partial\Psi)^2 - U(\Phi,\Psi)\big),$

$U(\Phi,\Psi) = -V_0 e^{\gamma\Psi}\Phi \;+\; \frac{1}{\Phi^3}\,\big(\frac{V_1}{e^{\lambda_1\Psi}} + \frac{V_2 Q^2}{e^{\lambda_2\Psi}}\big)$ [V_0, V_1, V_2 fns of z, θ]

Is the near-AdS_2 theory Jackiw-Teitelboim? Yes.

Linearized fluctuations $\rightarrow$ Ψ-scalar massive, propagates on AdS_2. Quadratic corrections, ...

Generic 2-dim dilaton-gravity-scalar

D-dim gravity-scalar theory (after incorporating $(F^I_{\mu\nu})^2$ contributions)

$$S = \frac{1}{16\pi G_D}\int d^Dx\sqrt{-g^{(D)}}\left(\mathcal{R}^{(D)} - \frac{h_{IJ}}{2}\partial_M\Psi^I\partial^M\Psi^J - V(\Psi^I, g)\right)$$

Dim.redux: $ds^2 = g^{(2)}_{\mu\nu}dx^\mu dx^\nu + \Phi^{\frac{4}{D-2}}\sum_{i=1}^{D-2}dx_i^2, \qquad g^{(D)}_{xx} \equiv \Phi^{\frac{4}{D-2}}$

2-dim dilaton-gravity-matter theory

(Weyl transfm $g_{\mu\nu} = \Phi^{\frac{2(D-3)}{(D-2)}}g^{(2)}_{\mu\nu}$ to absorb Φ-kinetic term)

$$S = \frac{1}{16\pi G_2}\int d^2x\sqrt{-g}\left(\Phi^2\mathcal{R} - \frac{\Phi^2}{2}h_{IJ}\partial_\mu\Psi^I\partial^\mu\Psi^J - U(\Phi,\Psi^I)\right), \quad U(\Phi,\Psi^I) = V\Phi^{\frac{2}{D-2}}$$

EOM: $g_{\mu\nu}\nabla^2\Phi^2 - \nabla_\mu\nabla_\nu\Phi^2 + \frac{g_{\mu\nu}}{2}\left(\frac{\Phi^2}{2}h_{IJ}\partial_\mu\Psi^I\partial^\mu\Psi^J + U\right) - \frac{\Phi^2}{2}h_{IJ}\partial_\mu\Psi^I\partial_\nu\Psi^J = 0\,,$

$$\mathcal{R} - \frac{h_{IJ}}{2}\partial_\mu\Psi^I\partial^\mu\Psi^J - \frac{\partial U}{\partial(\Phi^2)} = 0, \qquad \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\Phi^2 h_{IJ}\partial^\mu\Psi^J) - \frac{\partial U}{\partial\Psi^I} = 0$$

<u>AdS_2 critical point:</u> $\quad U_b = 0\,, \quad \frac{\partial U}{\partial(\Phi^2)}\Big|_b = \frac{-2}{L^2}\,, \quad \frac{\partial U}{\partial\Psi^I}\Big|_b = 0$

Static bgnd, away from AdS_2: nontrivial radial flow.

Interpret as holographic renormalization group flow? c-function?

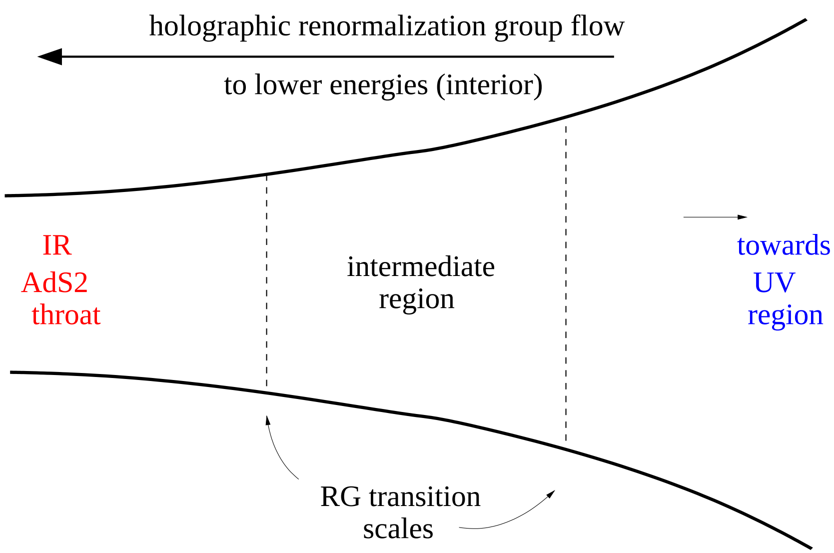
2d holographic RG, NECs, c-function

$$ds^2 = -B^2dt^2 + \frac{du^2}{B^2} + \Phi^{\frac{4}{D-2}} \sum_{i=1}^{D-2} dx_i^2 \qquad [\zeta^t, \zeta^r \neq 0, \quad \xi^t, \xi^i \neq 0]$$

Null energy conditions: $R_{MN}\zeta^M\zeta^N = -2B^2\Big[\frac{\Phi''}{\Phi} - \frac{(D-4)}{(D-2)}\frac{(\Phi')^2}{\Phi^2}\Big] \geq 0\,,$

$$R_{MN}\xi^M\xi^N = \frac{B^2}{2}\Big[\frac{(B^2)''}{B^2} - \frac{2}{(D-2)}\frac{(\Phi^2)''}{\Phi^2} + \frac{2(D-4)}{(D-2)}\frac{(B^2)'\Phi'}{B^2\Phi}\Big] \geq 0$$

Dim.redux $\rightarrow$ focus on 2-dim flows ending in IR at AdS_2.



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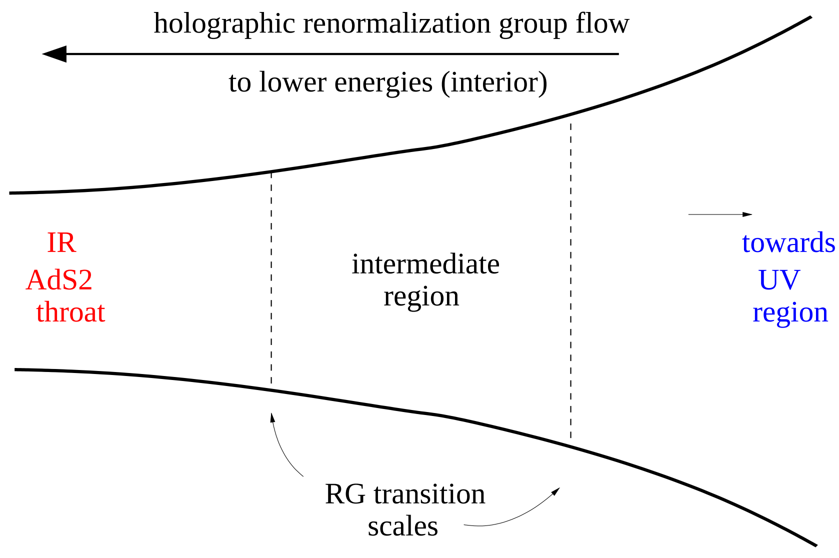

Extremal entropy $S_{BH} = \frac{g_{xx}^{(D-2)/2}|_h\, V_{D-2}}{4G_D} = \frac{\Phi_h^2}{4G_2}$ (compactified black branes)

Holographic c-function $\mathcal{C}(u) = \frac{\Phi^2(u)}{4G_2} = \frac{\Phi^2(u)\, V_{D-2}}{4G_D}$ $\qquad \Phi^2 = g_{xx}^{(D-2)/2}$

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Number of active degrees of freedom at scale u. $\mathcal{C}(u) \rightarrow S_{BH}$, number of BH microstates

[Studied by Goldstein,Jena,Mandal,Trivedi'05, nonsusy 4d BH attractors. Our context, details different]

2d holographic RG, NECs, c-function

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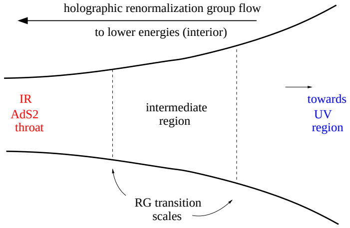
[c-theorem proof: define $\tilde{\Phi} = \Phi^{2/(D-2)}$. Then $NEC1 \Rightarrow -\frac{\tilde{\Phi}''}{\tilde{\Phi}} \geq 0$, $i.e.$ $\tilde{\Phi}'$ is decreasing along flow towards interior (IR). Can show $\tilde{\Phi}' \geq 0$ at boundary for hvLif bndry condns (NECs/sp.ht.+vity): broad family including AdS, nonconf branes etc $\Rightarrow \tilde{\Phi}' \geq 0$ all thro bulk]

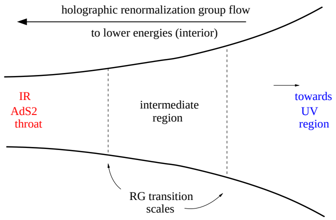
Dual: RG flow to lower energies in dual QM to IR CFT_1 with $\mathcal{C}_{IR} = S_{BH}$.



c-function, nonconformal Dp-branes

E.g.: nonconformal D2-M2 phases redux —

UV: 3d SYM $\rightarrow$ D2-sugra $\rightarrow$ M2-AdS_4 (IR)



D2-brane sugra:

$$ds^2_{st} = \frac{r^{5/2}}{R_2^{5/2}} dx_{||}^2 + \frac{R_2^{5/2}}{r^{5/2}}(dr^2 + r^2 d\Omega_6^2), \quad e^\phi = g_s \left(\frac{R_2^5}{r^5}\right)^{1/4}, \; g_{YM}^2 = \frac{g_s}{\sqrt{\alpha'}}, \; R_2^5 = \alpha'^3 g_{YM}^2 N$$

$$ds^2 = \frac{r^{7/2}}{R_2^{7/2}}(-dt^2 + dx_1^2 + dx_2^2) + \frac{R_2^{3/2}}{r^{3/2}} dr^2 = \left(\frac{\rho}{R_2}\right)^{1/3} \left[\frac{\rho^2}{R_2^2}(-dt^2 + dx_i^2) + \frac{R_2^2}{\rho^2} d\rho^2\right]$$

w nonconf Dp-brane supergravity radius/energy coord $\quad w = \frac{r^{3/2}}{R_2^{5/2}}, \quad \rho = \frac{r^{3/2}}{R_2^{1/2}}, \quad u = \frac{r^2}{R_2}$

$$\mathcal{C}(w) \sim \frac{V_2 \Phi^2}{G_4} = V_2\, w^{7/3} \frac{N^2}{(g_{YM}^2 N)^{1/3}} = V_2\, w^2\, N_{eff}(w)$$

M2-AdS_4: $\quad ds^2 = \frac{r^2}{R^2} dx_{||}^2 + \frac{R^2}{r^2} dr^2 + R^2 d\Omega_7^2, \quad R^6 \sim N l_p^6$

$$\mathcal{C}(w) = \frac{V_2}{G_4} \frac{r^2}{R^2} = N^{3/2}\, V_2\, w^2 \qquad [N^{3/2} \ll N_{eff}(w) \ll N^2]$$

IR AdS_2: $\mathcal{C} \rightarrow N^{3/2} \frac{V_2\, Q}{R^4} \quad \rightarrow$ doped CFT_3 redux (dopant density $\sigma_Q \equiv \frac{Q}{R^4}$) .

[also other nonconformal Dp-branes]

Compare entropic c-function & Freedman,Gubser,Pilch,Warner: $\mathcal{C} \sim c_E V_{d_i} w^{d_i}, \; c_{FGPW} \sim c_E$

Holographic RG, radial flow, dVV

Various versions of the holographic renormalization group have been formulated: de Boer, Verlinde, Verlinde formulated this in terms of radial Hamiltonian flow using a radial ADM-type split.

[Not Wilsonian, data not just from integrating out bulk at higher scales.
Wilsonian formulations: Heemskerk,Polchinski; Faulker,Liu,Rangamani]

- $L_{bulk} \rightarrow H_{bulk} \rightarrow$ conjugate momenta $\rightarrow$ Hamiltonian constraints $[\pi^k, U, \dot{\psi}^k, \ldots]$.
- Imagine finding the boundary action on some radial slice as a function of boundary field values at that scale $\rightarrow$ Hamilton-Jacobi form for these conjugate momenta $\pi^k \sim \frac{\partial S}{\partial \psi^k}$.
- Segregate this boundary action into local and nonlocal parts in a derivative expansion $S_{bdy} = S_{loc} + \Gamma$. $\quad S_{loc} \sim \int \sqrt{-\gamma} W + \ldots$, with boundary potential W
- Hamiltonian constraints $\rightarrow$ relate H_{bulk} & S_{bdy} in derivative expansion: *e.g.* $U \leftrightarrow W \ \ldots$

2-dim dilaton-gravity-scalars: $\frac{U}{(8\pi G_2)^2} = \frac{2h^{IJ}}{\Phi^2} \frac{\partial W}{\partial \Psi^I} \frac{\partial W}{\partial \Psi^J} - \frac{W}{\Phi} \frac{\partial W}{\partial \Phi}$

Flow eqns: $\dot{\Phi} = \frac{(4\pi G_2)\, W}{\Phi}, \quad \dot{\gamma}_{tt} = (8\pi G_2) \frac{\gamma_{tt}}{\Phi} \frac{\partial W}{\partial \Phi}, \quad \dot{\Psi}^I = -\frac{(16\pi G_2) h^{IJ}}{\Phi^2} \frac{\partial W}{\partial \Psi^J}$

β-functions: $\gamma_{tt} = a^2 \hat{\gamma}_{tt}\ , \quad \beta^I \equiv a \frac{d}{da} \Psi^I = -\frac{4h^{IJ}}{\Phi \frac{\partial W}{\partial \Phi}} \frac{\partial W}{\partial \Psi^J}, \quad \beta_\Phi \equiv a \frac{d}{da} \Phi = \frac{W}{\frac{\partial W}{\partial \Phi}}$

NECs $\rightarrow$ constraints on 2d potential

NEC $T_{MN}\zeta^M\zeta^N \geq 0$ $(\zeta^t,\ \zeta^r \neq 0) \xrightarrow{redux}$ NEC in 2-dim (monotonicity of c-fn)

NEC $T_{MN}\xi^M\xi^N \geq 0$ $(\xi^t,\ \xi^x \neq 0)$; for static bgnds, $\partial_t\Psi^I = 0 \quad \longrightarrow$

$$\boxed{U - (D-2)\frac{\partial U}{\partial\Phi^2} \geq 0}$$

Nontrivial constraint on 2-dim effective potential & derivative.

Intrinsic origin/interpretation in 2-dim?

Sensible constraint for theories arising from dim.redux.

At AdS_2 criticality: $R = -\frac{2}{L^2} = \frac{\partial U}{\partial\Phi^2}\ ,\ \ U = 0$

dS_2? here $R > 0$ but $U = 0 \ldots$?

Conclusions, questions

- Various facets to nonrelativistic holography and AdS/CMT: string realizations for Lifshitz and hyperscaling violation.
- Nontrivial scalings for entanglement entropy. Viscosity bound satisfied for hvLif family with $z \leq 2 + d_i - \theta$.
- Adding charge $\rightarrow$ extremal black branes $\xrightarrow{redux}$ AdS_2 holography. Away from AdS_2 throat, nontrivial flow $\rightarrow$ holographic RG $\rightarrow$ dilatonic c-function in 2-dim dilaton-gravity-scalar theories.

??? Wilsonian RG in 2-dim theories

??? c-function formulation applies only for sufficiently symmetric transverse spaces. More generally?

??? RG in large N quantum mechanics?

??? NECs & constraints on 2-dim potential (intrinsically 2-dim)? dS_2?