Aspects of $(A)dS$ Extremal Surfaces and Entanglement Entropy

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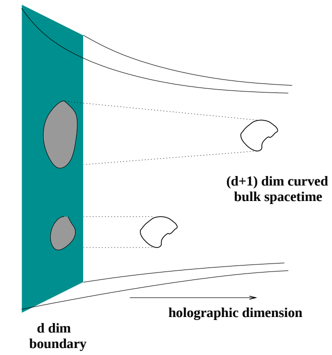
- Reviewing gauge/string realizations of Lifshitz & hyperscaling violation
- A lightlike limit of entanglement entropy
- de Sitter space, dS/CFT and extremal surfaces

Based mainly on arXiv:1408.7021, 1501.03019, 1504.07430, and in progress.

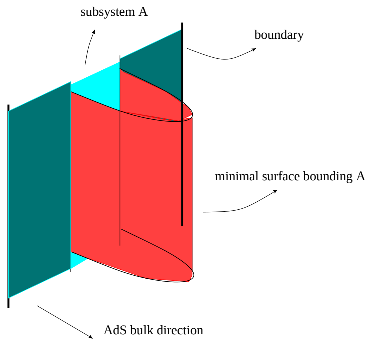
Gauge/gravity duality, entanglement

Many explorations of AdS/CFT over the years:
$e.g.$ nonrelativistic systems (holographic condmat),
time-dependent systems, cosmology, ...



Holographic handle on strongly coupled gauge theory (CFT) physics.

A striking example is entanglement entropy :
entropy of reduced density matrix of subsystem.
Ryu-Takayanagi bulk prescription for EE:
area of minimal surface in gravity dual.



Holographic Entanglement Entropy

Entanglement entropy: entropy of reduced density matrix of subsystem.

EE for spatial subsystem A, $S_A = -tr\rho_A \log\rho_A$, with partial trace $\rho_A = tr_B\rho$.

Quantum Field Theory: in general difficult to compute EE. Corrl'ns strongest near interface $\rightarrow$ leading scaling, d-dim area law $\mathcal{N}_{dof}\frac{V_{d-2}}{\epsilon^{d-2}}$.

[ϵ = UV cutoff] (Bombelli, Koul, Lee, Sorkin; Srednicki) [except: 2d CFT, Fermi surfaces]

2d Conformal field theory (single interval): $S_A = \frac{c}{3}\log\frac{l}{\epsilon}$ (c = central charge)

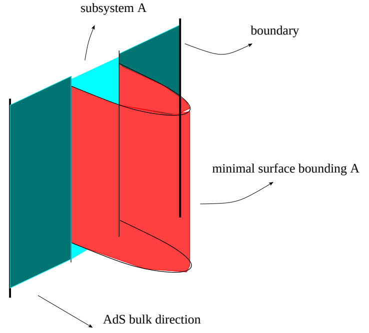
(Holzhey,Larsen,Wilczek) ["replica": $tr\rho_A^n = \frac{Z_n}{(Z_1)^n}$, $S_A^{EE} = -\lim_{n\rightarrow 1}\partial_n tr\rho_A^n$ (Calabrese,Cardy)]

d-dim free QFT (strip, width l, infinitely long): $S_A \sim \mathcal{N}_{dof}\left(\frac{V_{d-2}}{\epsilon^{d-2}} - \#\frac{V_{d-2}}{l^{d-2}}\right)$.

[More progress recently (in part from interplay with holography): 2d CFT, spheres, ...]

Holographic Entanglement Entropy

Ryu-Takayanagi: $EE = \frac{A_{min.surf.}}{4G}$ [motivated by black hole entropy]



Codim-2 minimal surface in gravity dual.

Substantial evidence by now (see recent Lewkowycz, Maldacena).

Non-static situations: extremal surfaces.

(Hubeny, Rangamani, Takayanagi)

EE a bulk surface probe [akin to correlation fns (geodesics), Wilson loops (bulk strings), . . .]

Holographic Entanglement Entropy

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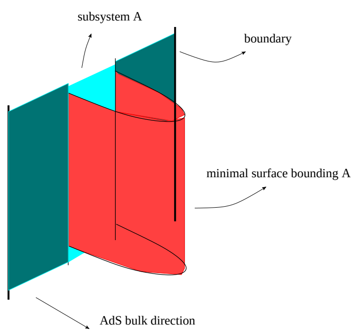

(i) Define boundary spatial subsystem on const time slice ,

(ii) corresponding const time slice in bulk, surface bounding subsystem,

(iii) extremize codim-2 surface area functional $\rightarrow$ minimal area.

Example: CFT ground state = empty AdS_{d+1}, $ds^2 = \frac{R^2}{r^2}(dr^2 - dt^2 + dx_i^2)$.

Strip, width $\Delta x = l$, infinitely long. Bulk surface $x(r)$. Turning point r_*.

$S_A \sim \frac{R^{d-1}}{G_{d+1}}(\frac{V_{d-2}}{\epsilon^{d-2}} - c_d \frac{V_{d-2}}{l^{d-2}})$, $\frac{R^3}{G_5} \sim N^2$ [4d], $\frac{R^2}{G_4} \sim N^{3/2}$ [3d].

$S_A = \frac{R}{2G_3} \log \frac{l}{\epsilon}$, $\frac{3R}{2G_3} = c$ [2d].

CFT thermal state (AdS black brane): minimal surface wraps horizon. $S^{fin} \sim N^2 T^3 V_{d-2} l$

Spherical extremal surfaces: subleading log-div. $\rightarrow$ anomaly. Casini, Huerta, Myers derive EE.

[$S_A = \frac{1}{4G_{d+1}} \int_{-\infty}^{\infty} \prod_{i=1}^{d-2} \frac{R dy_i}{r} \int \frac{R\sqrt{dr^2+dx^2}}{r} = \frac{V_{d-2}R^{d-1}}{4G_{d+1}} \int \frac{dr}{r^{d-1}} \sqrt{1+(\partial_r x)^2} \rightarrow$

$\frac{l}{2} = \int_0^{r_*} \frac{dr\ (r/r_*)^{d-1}}{\sqrt{1-(r/r_*)^{2d-2}}}$, $S = \frac{V_{d-2}R^{d-1}}{4G_{d+1}} \int_\epsilon^{r_*} \frac{dr}{r^{d-1}} \frac{2}{\sqrt{1-(r/r_*)^{2d-2}}}$.]

Nonrelativistic Holography

Generalizations of AdS/CFT with reduced symmetries.

Lifshitz spacetime: $ds^2 = -\frac{dt^2}{r^{2z}} + \frac{dx_i^2+dr^2}{r^2}$. (Kachru,Liu,Mulligan; Taylor)

scaling $t \to \lambda^z t,\ x_i \to \lambda x_i$ [dynamical exponent z $(z > 1)$] t, x_i-translations, x_i-rotations

[smaller than Schrodinger symm $e.g.$ ~~Galilean boosts~~] [gravity, $\Lambda < 0$, massive gauge field]

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More general gravity phases: $ds^2 = r^{2\theta/d_i}\Big(-\frac{dt^2}{r^{2z}} + \frac{dx_i^2+dr^2}{r^2}\Big).$

θ = hyperscaling violation exponent; d_i = boundary spatial dim (x_i).

[Conformally Lifshitz. Effective Einstein-Maxwell-Dilaton theories (Trivedi et al; Kiritsis et al, ...)

$S \sim T^{(d_i-\theta)/z}$. Thermodynamics $\sim$ space dim $d_{eff} = d_i - \theta$: actual space is d_i-dim.]

$\theta = d_i - 1$: entanglement entropy $\sim$ $\log l$, logarithmic behaviour.

Gravity duals of Fermi surfaces? (Ogawa,Takayanagi,Ugajin; Huijse,Sachdev,Swingle)

$d_i - 1 \le \theta < d_i$: EE area law violations. (Dong,Harrison,Kachru,Torroba,Wang)

[Energy conditions: $(d_i-\theta)(d_i(z-1)-\theta) \ge 0, \quad (z-1)(d+z-\theta) \ge 0.$]

Lif/h.v., gauge/string realizations

Narrow gravity parameter space. Identify recognizable CFT deformations and regimes.

Various string constructions involve x^+-dimensional reduction of

$$ds^2 = \frac{R^2}{r^2}(-2dx^+dx^- + dx_i^2 + dr^2) + R^2 g_{++}(dx^+)^2 + R^2 d\Omega_S^2.$$

$i.e.$ AdS + g_{++}, where $g_{++} > 0$. In lower dim'nal theory, time is $t \equiv x^-$.

(i) $z = 2$ Lifshitz (Balasubramanian,KN; Donos,Gauntlett; ...):

[Non-normalizable deformations] $g_{++} \sim r^0 \xrightarrow{x^+-\mathrm{dim.redn.}} z = 2$ Lifshitz.

g_{++} sourced by lightlike matter, $e.g.$ $g_{++} \sim (\partial_+ c_0)^2$ with lightlike axion $c_0 = Kx^+$:

$$ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + K^2R^2(dx^+)^2 \longrightarrow ds^2 = -\frac{dt^2}{r^4} + \frac{\sum_{i=1}^{d_i} dx_i^2 + dr^2}{r^2}.$$

(ii) Hyperscaling violation: AdS_{d+1} plane waves (KN)

[Normalizable g_{++}] $ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + R^2Qr^{d-2}(dx^+)^2 \longrightarrow$

$$ds^2 = r^{\frac{2\theta}{d_i}}\left(-\frac{dt^2}{r^{2z}} + \frac{\sum_{i=1}^{d_i} dx_i^2 + dr^2}{r^2}\right), \quad z = \frac{d-2}{2} + 2, \quad \theta = \frac{d-2}{2}, \quad d_i = d - 2.$$

Anisotropic CFT excited state, energy-momentum density $T_{++} = Q$.

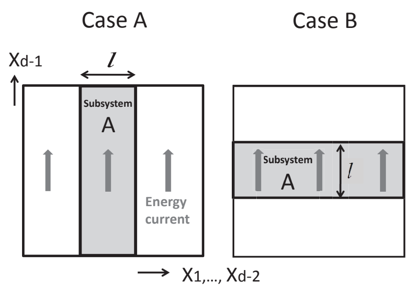
AdS_5 plane wave: $d = 4,\ d_i = 2,\ \theta = 1,\ z = 3$. Logarithmic behaviour of EE.

Highly boosted limit of black branes (Singh).

Entanglement, AdS plane waves

$ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + R^2 Q r^{d-2}(dx^+)^2$, dual to CFT state, $T_{++} \sim Q$

EE, spacelike strips (width l, $\Delta x^+ > 0 > \Delta x^-$). (KN, Takayanagi, Trivedi)



Non-static spacetime $\rightarrow$ extremal surfaces.
Spacelike subsystem, UV cutoff ϵ:
leading divergence is area law $\sim \frac{V_{d-2}}{\epsilon^{d-2}}$

Case A: width direction x_i. Strip along energy flux.

Finite cutoff-independent part of EE: size-dependent measure of entanglement $S^{fin} \sim N^2\sqrt{Q}V_2 \log(lQ^{1/4})$ [d=4].

[ground st] $-N^2\frac{V_2}{l^2} < S^{fin} < N^2T^3V_2 l$ [thermal entr]

Case B: Strip $\perp$ flux.

Phase transition (no connected surface if $\Delta x^+ > 0 > \Delta x^-$).

S_A saturated for $l \gtrsim Q^{-1/4}$.

[Boosted black branes (Maldacena,Martelli,Tachikawa): large boost λ, low temperature r_0 limit (Singh)

$$ds^2 = \frac{R^2}{r^2}\left(-2dx^+dx^- + \frac{r_0^4 r^4}{2}(\lambda dx^+ + \lambda^{-1}dx^-)^2 + \sum_i dx_i^2\right) + \frac{R^2 dr^2}{r^2(1-r_0^4 r^4)} .]$$

More general plane wave states: $e.g.$ M2-brane plane waves $\quad \text{EE}^{finite} \sim \sqrt{Q}L\sqrt{l}\,\sqrt{N^{3/2}}$,

nonconformal Dp-brane plane waves, ...

D-brane plane waves, EE

$$ds^2 = \frac{R^2}{r^2}(-2dx^+dx^- + dx_i^2 + dr^2) + \frac{G_{d+1}Q}{R^{d-3}} r^{d-2}(dx^+)^2 + R^2 d\Omega^2$$

AdS_{d+1} plane wave excited states: EE^{finite} $\pm\sqrt{Q} V_{d-2} l^{2-\frac{d}{2}} \sqrt{\frac{R^{d-1}}{G_{d+1}}}$

[$\pm : d \gtrless 4$] $\sqrt{Q} V_2 N \log(lQ^{1/4})$ (D3), $\sqrt{Q} L\sqrt{l}\sqrt{N^{3/2}}$ (M2), $-\sqrt{Q}\frac{V_4}{l}\sqrt{N^3}$ (M5).

3d, 4d: finite entanglement grows with width l (strip along flux direction).

[spacelike strip: leading divergence, area law, $\frac{V_2}{\epsilon^2}$ $(4d)$, $\frac{V_1}{\epsilon}$ $(3d)$] [$\perp$ flux: phase transition.]

[EE^{fin} scaling estimates $\leftarrow$ approximate r_*, S^{fin} for large Q, l from EE area functional]

[$G_5 \sim G_{10} R_{D3}^5, G_{4,7} \sim G_{11} R_{M2,M5}^{7,4}$, with $R_{D3}^4 \sim g_s N l_s^4, R_{M2}^6 \sim N l_P^6, R_{M5}^3 \sim N l_P^3$]

Nonconformal Dp-brane plane waves $\rightarrow$ $\theta = \frac{p^2-6p+7}{p-5}$, $z = \frac{2(p-6)}{p-5}$.

Dual to strongly coupled Yang-Mills theories with constant energy flux T_{++} (KN) (Singh).

$$ds_{st}^2 = \frac{r^{(7-p)/2}}{R_p^{(7-p)/2}} dx_{\parallel}^2 + \frac{G_{10}Q_p}{R_p^{(7-p)/2}} \frac{(dx^+)^2}{r^{(7-p)/2}} + R_p^{(7-p)/2} \frac{dr^2}{r^{(7-p)/2}} + R_p^{(7-p)/2} r^{(p-3)/2} d\Omega_{8-p}^2$$

$$e^\Phi = g_s \left(\frac{R_p^{7-p}}{r^{7-p}}\right)^{\frac{3-p}{4}}, \quad g_{YM}^2 \sim g_s \alpha'^{(p-3)/2}, \quad R_p^{7-p} \sim g_{YM}^2 N \alpha'^{5-p} \sim g_s N \alpha'^{(7-p)/2}.$$

EE leading divergence $N_{eff}(\epsilon)\frac{V_{d-2}}{\epsilon^{d-2}}$ as for ground states (area law).

EE^{finite}: $\frac{\sqrt{N_{eff}(l)}}{3-p} \frac{V_{p-1}\sqrt{Q}}{l^{(p-3)/2}}$, $\quad N_{eff}(l) = N^2 \left(\frac{g_{YM}^2 N}{l^{p-3}}\right)^{\frac{p-3}{5-p}}$

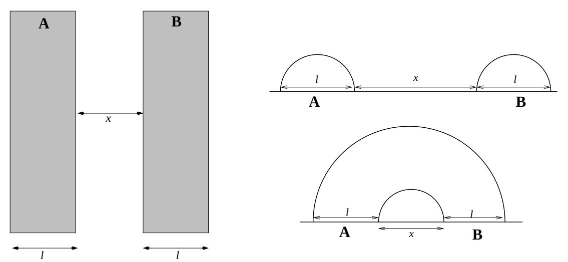
Consistent with Dp-brane phase diagram, RG flows.

Mutual Information

MI (disjoint subsystems A & B): $I[A,B] = S[A] + S[B] - S[A\cup B]$.

$I[A,B] \geq 0$. Cutoff-dependent divergences cancel. Gives bound for correlation fns.

Holographic mutual information: find extremal surface for $A\cup B$.



Subsystems far, two disjoint minimal surfaces: $MI = 0$.

Subsystems nearby, connected surface has lower area.

Ryu-Takayanagi $\Rightarrow$ MI disentangling transition (Headrick).

[This is large N: expect softer subleading decay for MI.]

Similar disentanglement for thermal states (Fischler,Kundu,Kundu): $\frac{x_c}{l} \sim 0$ (for $x, l \gg \frac{1}{T}$).

(Mukherjee, KN) MI for AdS plane wave excited states $\rightarrow$

critical separation $\frac{x_c}{l}$ between subsystems smaller than in ground state.

Mutual information disentangling occurs faster.

Suggests energy density disorders system.

[$e.g.$ $\frac{x_c}{l} \simeq 0.732$ (pure AdS_5) whereas $\frac{x_c}{l} \simeq 0.414$ (AdS_5 plane wave).]

[Wide strips ($Ql^d \gg 1$), critical $\frac{x_c}{l}$ independent of flux Q.] [$Ql^d \sim O(1)$: numerical study]

[Narrow strips $Ql^d \ll 1$: perturbative corrections ΔS ($\sim$ EE thermodynamics) $\rightarrow$ MI decreases.]

A lightlike limit of entanglement

AdS_{d+1} null deformation: $ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + R^2 g_{++}(dx^+)^2$ (KN)

Recall that lower dim'nal theory after x^+-reduction has time $t \equiv x^-$

$\Rightarrow$ lower dim entangling surface lies on $x^- = const$ slice upstairs.

Strip subsystem: $x^+ = \alpha\chi, \quad x^- = -\beta\chi, \quad -\frac{l}{2} < x \le \frac{l}{2}, \quad -\infty < \chi, y_i < \infty.$

[Spacelike strip $\alpha = \beta = 1 \rightarrow \frac{x^+ + x^-}{\sqrt{2}} \equiv t = const$ surface. $S^{div} \sim \frac{V_{d-2}}{\epsilon^{d-2}}$, area law.]

EE, null time x^- slice $(\beta = 0)$ $S \sim \frac{V_{d-2} R^{d-1}}{4G_{d+1}} \int_\epsilon^{r_*} \frac{dr}{r^{d-1}} \frac{\cancel{2\alpha}\, g_{++} r^2}{\sqrt{\cancel{2\alpha}\, g_{++} r^2 - A^2 r^{2d-2}}}$

$\Rightarrow$ **Milder leading divergence** $S^{div} \sim \frac{R^{d-1}}{4G_{d+1}} V_{d-2} \frac{\sqrt{g_{++}(\epsilon)}}{\epsilon^{d-3}}$

$g_{++} = 0$ (ground state) $\Rightarrow$ lightlike EE (on x^- slices) vanishes.

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EE, null time x^- slice ($\beta = 0$) $S \sim \frac{V_{d-2}R^{d-1}}{4G_{d+1}} \int_\epsilon^{r_*} \frac{dr}{r^{d-1}} \frac{\cancel{2+} g_{++}r^2}{\sqrt{\cancel{2+} g_{++}r^2 - A^2 r^{2d-2}}}$

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Lightlike limit $\equiv$ highly boosted limit of EE for spacelike strips.

Boost $x^\pm \rightarrow \lambda^{\pm 1} x^\pm \Rightarrow \alpha = \lambda$ and $\beta = \frac{1}{\lambda} \rightarrow 0$

$\rightarrow \quad S = \frac{V_{d-2}R^{d-1}}{4G_{d+1}} \int_\epsilon^{r_*} \frac{dr}{r^{d-1}} \frac{2+\lambda^2 g_{++}r^2}{\sqrt{2+\lambda^2 g_{++}r^2 - A^2 r^{2d-2}}}$ (and width $l \sim r_*$).

Regime $\lambda^2 g_{++}(\epsilon)\epsilon^2 \gtrsim 1$: $\rightarrow$ EE on null time x^- slices ($\beta = 0$).

[Similar structure for boosted black branes, nonconformal brane plane waves etc]

Null EE, AdS_{d+1} plane waves

$ds^2 = AdS_{d+1} + R^2 Q r^{d-2}(dx^+)^2$, $T_{++} \sim Q$: spacelike EE, area law, $S^{div} \sim \frac{V_{d-2}}{\epsilon^{d-2}}$.

EE on null time x^- slices if $\lambda^2 g_{++}(\epsilon)\epsilon^2 \gtrsim 1$, $i.e.$ $\lambda^2 Q \epsilon^d \gtrsim 1$.

In bulk: UV surface $r = \epsilon$ dips in sufficiently to feel g_{++} presence.

$$S \sim \frac{R^{d-1}}{4G_{d+1}} \frac{V_{d-2}\sqrt{\lambda^2 Q}}{d-4} \left(\frac{1}{\epsilon^{\frac{d}{2}-2}} - c_d \frac{1}{l^{\frac{d}{2}-2}} \right)$$

Milder leading divergence $S^{div} \sim \frac{R^{d-1}}{4G_{d+1}} \frac{V_{d-2}}{\epsilon^{d_{eff}-2}}$ $\quad$ ($d_{eff} = d - 1 - \theta = \frac{d}{2}$)

Resembles spacelike EE in hyperscaling violating theory ($\theta = \frac{d-2}{2}$) from x^+-red'n.

$g_{++} = 0$ (ground state) $\Rightarrow$ lightlike EE (on x^- slices) vanishes.

Reminiscent of ultralocality in lightcone QFT (Wall).

Ground state: n-pt functions (fields at distinct locations) vanish. Suggests vanishing EE.

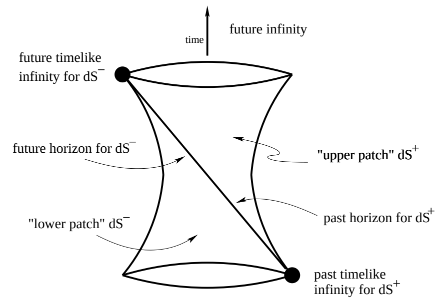
Excited states, $P_+ \neq 0$: can show free-field correlators non-vanishing. Suggests EE nonzero.

Boundary space: $ds^2 = -2dx^+dx^- + g^2(dx^+)^2 + \sum_{i=1}^{d-2} dx_i^2$, with $g^2 = T_{++}\epsilon^d \gtrsim 1$.

Usual area law $\quad S_{div} \sim N^2 \frac{V_{x^+} V_{y_i}}{\epsilon^{d-2}} = N^2 V_{d-2} \frac{\sqrt{T_{++}\epsilon^d}}{\epsilon^{d-2}} = N^2 \sqrt{Q} \frac{V_{d-2}}{\epsilon^{d_{eff}-2}}$.

de Sitter space and dS/CFT

de Sitter space $ds^2 = \frac{R_{dS}^2}{\tau^2}(-d\tau^2 + dw^2 + dx_i^2)$.
Fascinating for various reasons.
dS/CFT: fluctuations about dS encoded in dual Euclidean non-unitary CFT on boundary at future timelike infinity $\mathcal{I}^+$ (Strominger; Witten). Interesting to explore.



(Maldacena '02) analytic continuation $r \to -i\tau,\ R_{AdS} \to -iR_{dS}$ from Eucl $AdS \to$ Hartle-Hawking wavefunction of the universe $\Psi[\varphi] = Z_{CFT}$.

Energy-momentum tensor $\langle TT \rangle$ 2-pt fn $\to$ dual CFT central charge
$\mathcal{C}_d \sim i^{1-d} \frac{R_{dS}^{d-1}}{G_{d+1}}$, negative or imaginary. $\mathcal{C}_3 \sim -\frac{R_{dS}^2}{G_4}$ for dS_4.

[Bulk EAdS regularity conditions, deep interior $\to$ Bunch-Davies initial conditions in deSitter,
$\varphi_k(\tau) \sim e^{ik\tau}$, for large $|\tau|$. $Z_{CFT} = \Psi[\varphi] \sim e^{iS_{cl}[\varphi]}$ (semiclassical).
[Dual CFT: $\langle O_k O_{k'} \rangle \sim \frac{\delta^2 Z}{\delta\varphi_k^0 \varphi_{k'}^0}$] [Bulk expectation values $\langle f_1 f_2' \rangle \sim \int D\varphi f_1 f_2' |\Psi|^2$.]
Wavefunction $\Psi[\varphi]$ not pure phase $\to$ complex saddle points contribute to observables.]

dS/CFT at uniform energy density

(Sumit Das, Diptarka Das, KN)

$$ds^2_{d+1} = \frac{R^2_{dS}}{\tau^2}\Big(-\frac{d\tau^2}{1+\alpha\tau_0^d\tau^d} + (1+\alpha\tau_0^d\tau^d)dw^2 + \sum_{i=1}^{d-1} dx_i^2\Big),$$

α is a complex phase and τ_0 real parameter of dimension energy, solves $R_{MN} = \frac{d}{R^2_{dS}} g_{MN}$.

Regularity: Wick rotate $\tau \to il$, demand resulting spacetime (thought of as saddle point in path integral) in Euclidean (l, w)-plane has no conical singularity $\Rightarrow$

$\alpha = -(-i)^d,\ \ l \geq \tau_0,\ \ w \simeq w + \frac{4\pi}{(d-1)\tau_0}$. [analogous to interior regularity in AdS]

[This is equivalent to analytic continuation $r \to -i\tau,\ R_{AdS} \to -iR_{dS}$ from $EAdS$ black brane $ds^2 = \frac{R^2_{AdS}}{r^2}\big(\frac{dr^2}{1-r_0^d r^d} + (1-r_0^d r^d)d\theta^2 + \sum_{i=1}^{d-1} dx_i^2\big)$.]

"Normalizable" metric modes $\Rightarrow$ energy-momentum tensor vev.

$$T_{ij} = \frac{2}{\sqrt{h}}\frac{\delta Z_{CFT}}{\delta h^{ij}} = \frac{2}{\sqrt{h}}\frac{\delta\Psi}{\delta h^{ij}} \propto i\frac{R^{d-1}_{dS}}{G_{d+1}} g^{(d)}_{ij} \quad \to\ dS \text{ black brane.}$$

[$g^{(d)}_{ij}$ = coefficient of τ^{d-2} in Fefferman-Graham expn]. [dS/CFT: $Z_{CFT} = \Psi$].

Note i arising from the wavefunction of the universe $\Psi \sim e^{iS_{cl}}$

$\Rightarrow$ energy-momentum real only if $g^{(d)}_{ij}$ pure imaginary.

dS_4/CFT_3: $\alpha = -i,\quad T_{ww} = -\frac{R^2_{dS}}{G_4}\tau_0^3$ with $T_{ww} + (d-1)T_{ii} = 0$.

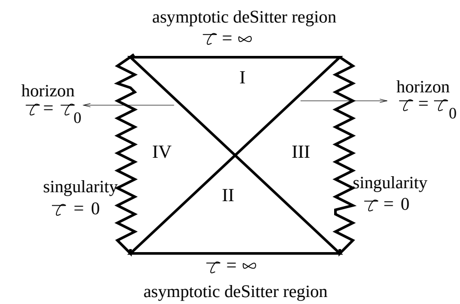
de Sitter "bluewall"

$$ds^2 = \frac{R_{dS}^2}{\tau^2}\Big(- \frac{d\tau^2}{1-\tau_0^d\tau^d} + (1-\tau_0^d\tau^d)dw^2 + dx_i^2\Big)$$

Penrose diagram resembles AdS-Schwarzschild rotated by $\frac{\pi}{2}$.

$[-\infty \le w \le \infty]$ Take $\alpha = -1$ earlier.

Equivalently, analytically continue τ_0^d parameter too.



Using Kruskal coordinates: two asymptotic dS universes $(\tau \to 0)$.

Timelike singularities $(\tau \to \infty)$. Cauchy horizons $(\tau = \tau_0)$.

$\simeq$ interior of Reissner-Nordstrom black hole (or wormhole).

Trajectories in the de Sitter bluewall and the Cauchy horizon $\rightarrow$

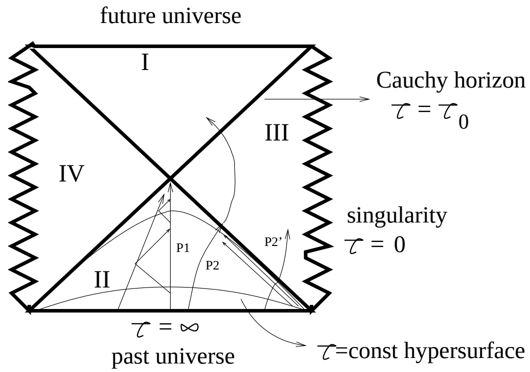
Observers P_1 are static while P_2 has w-momentum p_w,

crosses the horizon, turns around inside and appears

to re-emerge in the future universe.

Incoming lightrays from infinity "crowd near" Cauchy horizon:

Late time infalling observers P_2 see early lightrays blueshifted.



Infinite blueshift due to Cauchy horizon: instability.

de Sitter extremal surfaces

(KN)

A generalization of Ryu-Takayanagi to dS $ds^2 = \frac{R_{dS}^2}{\tau^2}(-d\tau^2 + dw^2 + dx_i^2)$: Eucl time slice $w = const$, subregion at future timelike infinity $\rightarrow$ codim-2 extremal surfaces in de Sitter space.

$\longrightarrow$ bulk analog of setting up entanglement entropy in dual Eucl CFT: consider boundary Euclidean time slice, construct spatial subsystem, trace over complement.

An obvious concern: $\mathcal{I}^+$ boundary spacelike $\Rightarrow$ real surfaces appear timelike, dipping inwards into past. Might imagine appropriate surfaces encoding EE should be spacelike.

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Some possible expectations of extremal surface area for interpretation as entanglement entropy based on $Z_{CFT} = \Psi$:

- central charge coefficient in leading (area law) divergence must match dual CFT central charge earlier (from $\langle TT \rangle$ correlators).
- coefficient in logarithmic divergence must match conformal anomaly.
- expect finite cutoff-independent parts which are size-dependent measures of entanglement entropy in CFT.

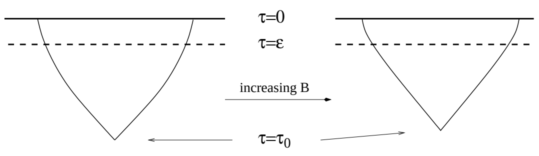
de Sitter extremal surfaces

de Sitter (Poincare): $ds^2_{d+1} = \frac{R^2_{dS}}{\tau^2}(-d\tau^2 + dw^2 + dx_i^2) \quad \rightarrow$ EE in dual Eucl CFT $\rightarrow$

bulk: Eucl time slice $w = const$, subregion at future timelike infnty $\rightarrow$ codim-2 extremal surface.

The above expectations and dual CFT central charge being negative or pure imaginary suggest real surfaces will not work $\longrightarrow$

[strip] $S_{dS} = \frac{1}{4G_{d+1}} \int \prod_{i=1}^{d-2} \frac{R_{dS}\, dy_i}{\tau} \frac{R_{dS}}{\tau} \sqrt{dx^2 - d\tau^2}$



$\longrightarrow \qquad S_{dS} \propto \frac{R^{d-1}_{dS} V_{d-2}}{4G_{d+1}} \int \frac{d\tau}{\tau^{d-1}} \sqrt{1 - \dot{x}^2}.$

$Extremize \rightarrow \dot{x}^2 = \frac{B^2\tau^{2d-2}}{1+B^2\tau^{2d-2}}\,, \quad B^2 = const$ is conserved quantity.

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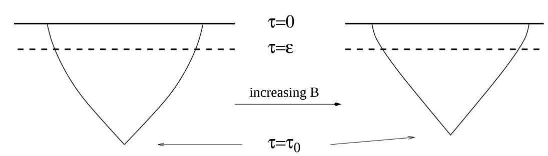

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• Sign difference from $AdS \;\Rightarrow$ no real "turning point". $\;x(\tau)$ hyperboloid.

Join two half-extremal-surfaces with cusp $\rightarrow \; S_{dS} = 2\frac{R^{d-1}_{dS}}{4G_{d+1}} V_{d-2} \int_\epsilon^{\tau_0} \frac{d\tau}{\tau^{d-1}} \frac{2}{\sqrt{1+B^2\tau^{2d-2}}}.$

Minimize area: increase $B \;\Rightarrow$ surface shape saturates, approaches $\dot{x}^2 \rightarrow 1$ as $B \gg \frac{1}{\epsilon^{d-1}}$.

$\rightarrow$ restriction of past lightcone wedge of subregion. $\;x(\tau)$ null surface. Area vanishes.

Real codim-2 surfaces: featureless, no apparent relation to EE.

["outward bending" surfaces $\rightarrow$ null, $S_{dS} = 0$] [surfaces $x(\tau) = const$: $B = 0$, max area]
[Codim-1 surfaces: similar structure.]

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$$\text{[strip]} \quad S_{dS} = \frac{R^{d-1}_{dS} V_{d-2}}{4G_{d+1}} \int \frac{1}{\tau^{d-1}} \sqrt{dx^2 - d\tau^2} = \frac{R^{d-1}_{dS} V_{d-2}}{4G_{d+1}} \int \frac{d\tau}{\tau^{d-1}} \sqrt{\dot{x}^2 - 1}$$

$$Extremize \quad \rightarrow \quad (\partial_\tau x)^2 = \frac{-A^2\tau^{2d-2}}{1 - A^2\tau^{2d-2}}\,. \qquad [A^2 < 0 \text{ is earlier real-}\tau \text{ solution}]$$

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$\underline{dS_4/CFT_3}$: consider $A^2 > 0$. Near $\tau \rightarrow 0$: $\dot{x}^2 \sim -A^2\tau^4$ $i.e.$

$x(\tau) \sim \pm iA\tau^3 + x(0)$. This is spatial direction in Eucl CFT $\Rightarrow$

$x(\tau)$ real-valued $\Rightarrow$ $\tau = iT$ [can show width Δx also real]

$x(\tau) \rightarrow$ complex extremal surface, τ along imaginary path $\tau = iT$.

$(\frac{dx}{dT})^2 = \frac{A^2T^4}{1-A^2T^4}$. Note turning point: $T_* = \frac{1}{\sqrt{A}}$ (where $|\dot{x}|^2 \rightarrow \infty$).

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Can now smoothly join half-extremal-surfaces at turning point. [$\tau_{UV} = i\epsilon, \ \ \tau_* \sim il$]

$$\frac{\Delta x}{2} = \frac{l}{2} = \int_0^{\tau_*} d\tau \frac{iA\tau^2}{\sqrt{1-A^2\tau^4}} = \int_0^{T_*} \frac{(T^2/T_*^2)\, dT}{\sqrt{1-(T^4/T_*^4)}} \sim T_*$$

$$S_{dS_4} = -i\frac{R^2_{dS}}{4G_4} V_1 \int_{\tau_{UV}}^{\tau_*} \frac{d\tau}{\tau^2} \frac{1}{\sqrt{1-\tau^4/\tau_*^4}} = -\frac{R^2_{dS}}{4G_4} V_1 \int_\epsilon^l \frac{dT/T^2}{\sqrt{1-T^4/T*^4}} \sim -\frac{R^2_{dS}}{G_4} V_1 (\frac{1}{\epsilon} - c\frac{1}{l})$$

Overall sign $\rightarrow$ match with dS_4/CFT_3 central charge.

de Sitter extremal surfaces

de Sitter (Poincare): $ds^2_{d+1} = \frac{R^2_{dS}}{\tau^2}(-d\tau^2 + dw^2 + dx_i^2) \quad \rightarrow \quad$ EE in dual Eucl CFT $\rightarrow$

bulk: Eucl time slice $w = const$, subregion at future timelike infnty $\rightarrow$ codim-2 extremal surface.

[strip] $S_{dS} = \frac{R^{d-1}_{dS} V_{d-2}}{4G_{d+1}} \int \frac{1}{\tau^{d-1}} \sqrt{dx^2 - d\tau^2} = \frac{R^{d-1}_{dS} V_{d-2}}{4G_{d+1}} \int \frac{d\tau}{\tau^{d-1}} \sqrt{\dot{x}^2 - 1}$

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$\underline{dS_{d+1}/CFT_d}$ (d even): near $\tau \rightarrow 0$, $\quad \dot{x} \sim \pm\sqrt{-A^2}\, \tau^{d-1} \quad i.e.$

$x(\tau) \sim \pm\sqrt{-A^2}\, \tau^d + x(0)$. This is spatial direction in Eucl CFT

$\Rightarrow$ $\boxed{x(\tau) \text{ real-valued} \Rightarrow A^2 < 0,\ \tau = iT}$ [can show width Δx also real]

$x(\tau) \rightarrow$ complex extremal surface, τ along imaginary path $\tau = iT$.

$\left(\frac{dx}{dT}\right)^2 = \frac{A^2 T^{2d-2}}{1 + (-1)^{d-1} A^2 T^{2d-2}}$. Note turning point: $T_*^{2d-2} A^2 = 1$.

$$S_{dS} = -i \frac{R^{d-1}_{dS}}{4G_{d+1}} V_{d-2} \int_{\tau_{UV}}^{\tau_*} \frac{d\tau}{\tau^{d-1}} \frac{2}{\sqrt{1 + A^2 \tau^{2d-2}}}$$

$$= i^{1-d} \frac{R^{d-1}_{dS}}{2G_{d+1}} V_{d-2} \int_\epsilon^{T_*} \frac{dT/T^{d-1}}{\sqrt{1 + (-1)^{d-1} A^2 T^{2d-2}}} \sim i^{1-d} \frac{R^{d-1}_{dS}}{2G_{d+1}} V_{d-2} \left(\frac{1}{\epsilon^{d-2}} - c_d \frac{1}{l^{d-2}}\right)$$

de Sitter extremal surfaces, dS/CFT

Complex extremal surfaces: compare dS_{d+1}/CFT_d central charges.

[Strip width real (CFT spatial direction) $\Rightarrow$ path $\tau = iT$ $\rightarrow$ extremal surface with turning point.]

dS_4: area $S_{dS_4} = -\frac{R_{dS}^2}{4G_4} V_1 \int_\epsilon^l \frac{dT/T^2}{\sqrt{1-T^4/T_*^4}} \sim -\frac{R_{dS}^2}{G_4} V_1 (\frac{1}{\epsilon} - c\frac{1}{l})$

dS_{d+1}, even d: area $S_{dS} = i^{1-d} \frac{R_{dS}^{d-1}}{2G_{d+1}} V_{d-2} \int_\epsilon^{T_*} \frac{dT/T^{d-1}}{\sqrt{1+(-1)^{d-1}(T/T_*)^{2d-2}}}$

$$\sim i^{1-d} \frac{R_{dS}^{d-1}}{2G_{d+1}} V_{d-2} \left(\frac{1}{\epsilon^{d-2}} - c_d \frac{1}{l^{d-2}}\right)$$

$\equiv$ analytic continuation from AdS Ryu-Takayanagi extremization.

$S_{AdS}[R, x(r), r] = \frac{R^{d-1}}{4G_{d+1}} V_{d-2} \int \frac{dr}{r^{d-1}} \sqrt{1 + (\frac{dx}{dr})^2}$, $\quad (x')^2 = \frac{A^2 r^{2d-2}}{1-A^2 r^{2d-2}} \rightarrow$

$\dot{x}^2 = \frac{-(-1)^{d-1} A^2 \tau^{2d-2}}{1-(-1)^{d-1} A^2 \tau^{2d-2}}$, $\quad S_{dS} = -i \frac{R_{dS}^{d-1}}{4G_{d+1}} V_{d-2} \int \frac{d\tau}{\tau^{d-1}} \frac{1}{\sqrt{1-(-1)^{d-1} A^2 \tau^{2d-2}}}$.

- leading "area law" divergence $\mathcal{C}_d \frac{V_{d-2}}{\epsilon^{d-2}}$ $\rightarrow$

  central charges $\mathcal{C}_d = i^{1-d} \frac{R_{dS}^{d-1}}{G_{d+1}}$ match dS/CFT using $Z_{CFT} = \Psi$.
- finite cutoff-independent parts $\sim$ $i^{1-d} \frac{R_{dS}^{d-1}}{G_{d+1}} \frac{V_{d-2}}{l^{d-2}}$.
- Spherical extremal surfaces: subleading log-div. Anomaly coeff exactly matches Ψ log-coeff.
- dS_4 black brane, CFT_3 at uniform energy density: S_{dS}^{fin} resembles extensive thermal entropy.

Spherical extremal surfaces, dS/CFT

(KN)

$ds^2 = \frac{R_{dS}^2}{\tau^2}(-d\tau^2 + dw^2 + dr^2 + r^2 d\Omega_{d-2}^2) \to w = const$, sphere subregion. $0 \le r \le l$

$$S_{dS} = \frac{R_{dS}^{d-1}\Omega_{d-2}}{4G_{d+1}} \int \frac{d\tau}{\tau^{d-1}} r^{d-2} \sqrt{(\frac{dr}{d\tau})^2 - 1}\,, \quad \text{extremize: } r(\tau) = \sqrt{l^2+\tau^2}\,, \quad \dot{r} = \frac{\tau}{\sqrt{l^2+\tau^2}}$$

Real τ: outward-bending, $r(\tau) \ge l$. Timelike: $\dot{r} \le 1$. No "end" at finite τ.

$\to \quad \epsilon < |\tau| < \infty \quad \to S_{dS}$ real, no finite cutoff-indep parts.

$\tau = iT$: now $0 \le r(\tau) < l$ and $\Delta r = l$. Turning point $\tau_* = il$.

$$S_{dS} = \frac{R_{dS}^{d-1}\Omega_{d-2}}{4G_{d+1}} \int_{i\epsilon}^{\tau_*} \frac{d\tau}{\tau^{d-1}} (-il)(l^2+\tau^2)^{(d-3)/2} \quad \to \quad S_{dS_4} = -\frac{\pi R_{dS}^2}{2G_4}(\frac{l}{\epsilon} - 1)$$

d even: $\log\frac{l}{\epsilon}$ divergence. Coeff $\to \; -i\frac{R_{dS}}{2G_3} \; [dS_3]\,; \quad -i\frac{\pi R_{dS}^3}{2G_5} \; [dS_5], \; \ldots$

Free energy of CFT_d on sphere: log-div, related to conformal anomaly.

Casini,Huerta,Myers: $-F_{CFT} = \log Z_{CFT} = a\log\epsilon + \ldots$, integ. trace anomaly $a = \int\langle T^k{}_k\rangle$.

$Z_{CFT} = e^{-F} = \Psi \sim e^{iS_{cl}}$ for auxiliary global dS. $\quad T_{ij} \sim \frac{2}{\sqrt{h}}\frac{\delta(-F_{CFT})}{\delta h^{ij}} \sim i\frac{2}{\sqrt{h}}\frac{\delta S}{\delta h^{ij}}$

$\to$ log-div coeff matches $\to$ equivalent to analytic continuation from AdS.

$[S_{CFT}^{EE} = -\lim_{n\to 1}\partial_n \frac{Z_n}{(Z_1)^n}$; scale change $l\frac{\partial}{\partial l}S_{CFT}^{EE} \sim \int\langle T_\mu{}^\mu\rangle$; here $S_{CFT}^{EE} = S_{dS}]$

$S_{cl} = \frac{2d\;\Omega_d R_{dS}^{d-1}}{16\pi G_{d+1}} \int \frac{dt}{R_{dS}} \; (\cosh\frac{t}{R_{dS}})^d \to$ log-div $\; [ds^2 = -dt^2 + R_{dS}^2(\cosh\frac{t}{R_{dS}})^2 d\Omega_d^2]$

dS_4 surfaces, negative EE

dS_4: $S_A \sim -\frac{R_{dS}^2}{G_4}(\frac{V_1}{\epsilon} - \frac{V_1}{l})$ [strip]

• Disjoint strip subregions A, B: analog of mutual information $I[A,B] = S[A] + S[B] - S[A \cup B]$ negative definite for A, B sufficiently nearby (vanishes beyond critical separation).

• Consider two strip subregions, width l_2 and $l_1 > l_2$ $(l_1, l_2 \ll V_1)$.
Then $S(l_1) - S(l_2) = -\frac{R_{dS}^2}{G_4}(\frac{V_1}{l_2} - \frac{V_1}{l_1}) < 0$, $i.e.$
$S(l_1) < S(l_2)$ $\Rightarrow$ bigger subregion more ordered than smaller one.

[conventional unitary CFT: $S(l_1) > S(l_2)$, $i.e.$ bigger subregion more disordered]

• Entropic c-function $c(l) = \frac{l^{d-1}}{V_{d-2}}\frac{dS_A}{dl}$. $c(l) \equiv \frac{l^2}{V_1}\frac{dS_A}{dl} = -\frac{R_{dS}^2}{G_4} < 0$ $i.e.$ as l increases, $S(l)$ decreases. Asymptotically dS_4 spaces, negative areas S_A of complex extremal surfaces imply $c'(l) > 0$, $i.e.$ as l increases, $c(l)$ increases. New degrees of freedom *integrated in*?
$|\tau_*| = l$: increasing size $l \to$ going to larger $|\tau_*|$ (earlier times in past).

Conclusions, questions

- Various gauge/string realizations of Lifshitz & hyperscaling violation involve x^+-reduction of AdS deformations with g_{++}. Entanglement entropy, lower dim theory $\rightarrow$ null time x^- slices upstairs $\rightarrow$ lightlike limit of EE. Lightcone QFT, ultralocality, ...?

- Deeper understanding of complex extremal surfaces in de Sitter space, EE as probe of dS/CFT.
Entanglement entropy in non-unitary (ghost) CFTs [tentative]

[$e.g.$ 2d bc-CFTs with $(h_b, h_c) = (1, 0)$ have $c = -2$. $SL(2)$ vacuum $|1\rangle$ (L_0 eigenvalue zero) satisfies $b_{m\geq 0}|1\rangle = 0,\ \ c_{m\geq 1}|1\rangle = 0$ appears to coincide with ghost ground state $|\downarrow\rangle$.
Z_N orbifold $\rightarrow$ twist operator $\sigma_{k/N}$ dim $h_\sigma = -\frac{1}{2}\frac{k}{N}(1-\frac{k}{N})$. Replica $\rightarrow S = \frac{c}{3}\log\frac{l}{\epsilon} < 0$.
Lowest conformal dimension is $\Delta = 0 \ \Rightarrow$ it appears that $c_{eff} = c - 24\Delta = c < 0$.

$\rightarrow$ subsector of complex ghost $\partial\chi\bar{\partial}\bar{\chi}$ CFT with $c = -2$ with anticommuting scalars $\chi, \bar{\chi}$
$\rightarrow$ logarithmic CFT, but restricting to $\partial\chi,\ \partial\bar{\chi}$ operators $\rightarrow$ negative EE (mapping to above).]