Aspects of $(A)dS$ Extremal Surfaces and Entanglement Entropy

K. Narayan
Chennai Mathematical Institute

- Reviewing gauge/string realizations of Lifshitz & hyperscaling violation
- A lightlike limit of entanglement entropy
- de Sitter space, $dS/CFT$ and extremal surfaces

Based mainly on arXiv:1408.7021, 1501.03019, 1504.07430, and in progress.
Gauge/gravity duality, entanglement

Many explorations of $AdS/CFT$ over the years: e.g. nonrelativistic systems (holographic condmat), time-dependent systems, cosmology, …

Holographic handle on strongly coupled gauge theory (CFT) physics.

A striking example is entanglement entropy: entropy of reduced density matrix of subsystem. Ryu-Takayanagi bulk prescription for EE: area of minimal surface in gravity dual.
Holographic Entanglement Entropy

Entanglement entropy: entropy of reduced density matrix of subsystem.

EE for spatial subsystem A, \( S_A = -tr\rho_A \log \rho_A \), with partial trace \( \rho_A = tr_B \rho \).

Quantum Field Theory: in general difficult to compute EE. Corr’ns strongest near interface \( \rightarrow \) leading scaling, \( d\)-dim area law \( N_{dof} \frac{V_{d-2}}{\epsilon^{d-2}} \).

[\( \epsilon = \text{UV cutoff} \) (Bombelli, Koul, Lee, Sorkin; Srednicki) [except: 2d CFT, Fermi surfaces]

2d Conformal field theory (single interval): \( S_A = \frac{c}{3} \log \frac{l}{\epsilon} \) (\( c = \text{central charge} \))

(Holzhey,Larsen,Wilczek) \[\text{“replica”}: tr\rho^n_A = \frac{Z^n}{(Z_1)^n}, S^{EE}_A = -\lim_{n \to 1} \partial_n tr\rho^n_A \] (Calabrese,Cardy)

\( d\)-dim free QFT (strip, width \( l \), infinitely long): \( S_A \sim N_{dof} \left( \frac{V_{d-2}}{\epsilon^{d-2}} - \# \frac{V_{d-2}}{ld-2} \right) \).

[More progress recently (in part from interplay with holography): 2d CFT, spheres, \ldots]
Holographic Entanglement Entropy

Ryu-Takayanagi: \[ EE = \frac{A_{\text{min.surf.}}}{4G} \] [motivated by black hole entropy]

Codim-2 minimal surface in gravity dual.

Substantial evidence by now (see recent Lewkowycz, Maldacena).

Non-static situations: extremal surfaces.

( Hubeny, Rangamani, Takayanagi)

EE a bulk surface probe [akin to correlation fns (geodesics), Wilson loops (bulk strings), ...]
Holographic Entanglement Entropy

Ryu-Takayanagi: \[ EE = \frac{A_{\text{min.surf.}}}{4G} \]

(i) Define boundary spatial subsystem on const time slice ,
(ii) corresponding const time slice in bulk, surface bounding subsystem,
(iii) extremize codim-2 surface area functional \( \rightarrow \) minimal area.

Example: CFT ground state = empty \( AdS_{d+1} \),

\[ ds^2 = \frac{R^2}{r^2} (dr^2 - dt^2 + dx_i^2) \]

Strip, width \( \Delta x = l \), infinitely long. Bulk surface \( x(r) \). Turning point \( r_* \).

\[ S_A \sim \frac{R^{d-1}}{G_{d+1}} \left( \frac{V_{d-2}}{\epsilon^{d-2}} - c_d \frac{V_{d-2}}{l^{d-2}} \right), \quad \frac{R^3}{G_5} \sim N^2 \quad [4d], \quad \frac{R^2}{G_4} \sim N^{3/2} \quad [3d]. \]

\[ S_A = \frac{R}{2G_3} \log \frac{l}{\epsilon}, \quad \frac{3R}{2G_3} = c \quad [2d]. \]

CFT thermal state (AdS black brane): minimal surface wraps horizon.

\[ S_{\text{fin}} \sim N^2 T^3 V_{d-2} l \]

Spherical extremal surfaces: subleading log-div. \( \rightarrow \) anomaly. Casini, Huerta, Myers derive EE.

\[ [S_A = \frac{1}{4G_{d+1}} \int_{-\infty}^{\infty} \prod_{i=1}^{d-2} \frac{R dy_i}{r} \int R \sqrt{dr^2 + dx^2} - \frac{V_{d-2} R^{d-1}}{4G_{d+1}} \int \frac{dr}{r^{d-1}} \sqrt{1 + (\partial_r x)^2} \rightarrow \]

\[ \frac{l}{2} = \int_0^{r_*} \frac{dr}{\sqrt{1 - (r/r_*)^{2d-2}}}, \quad S = \frac{V_{d-2} R^{d-1}}{4G_{d+1}} \int_0^{r_*} \frac{dr}{r^{d-1}} \frac{2}{\sqrt{1 - (r/r_*)^{2d-2}}}. \]

Aspects of \( (A) dS \) extremal surfaces and entanglement entropy, K. Narayan, CMI – p.5/29
Nonrelativistic Holography

Generalizations of $AdS/CFT$ with reduced symmetries.

**Lifshitz spacetime:** $ds^2 = -\frac{dt^2}{r^{2z}} + \frac{dx_i^2 + dr^2}{r^2}$.

(Kachru, Liu, Mulligan; Taylor)

scaling $t \rightarrow \lambda^z t$, $x_i \rightarrow \lambda x_i$ [dynamical exponent $z$ ($z > 1$)] $t$, $x_i$-translations, $x_i$-rotations

[smaller than Schrodinger symm e.g. Galilean boosts] [gravity, $\Lambda < 0$, massive gauge field]
Nonrelativistic Holography

Generalizations of \( AdS/CFT \) with reduced symmetries.

Lifshitz spacetime: \( ds^2 = -\frac{dt^2}{r^{2z}} + \frac{dx_i^2 + dr^2}{r^2} \). \(^{(Kachru,Liu,Mulligan; Taylor)}\)

scaling \( t \to \lambda^z t, \ x_i \to \lambda x_i \) [dynamical exponent \( z \ (z > 1) \)] \( t, x_i \)-translations, \( x_i \)-rotations [smaller than Schrodinger symm e.g. Galilean boosts] [gravity, \( \Lambda < 0 \), massive gauge field]

More general gravity phases: \( ds^2 = r^{2\theta/d_i} \left( -\frac{dt^2}{r^{2z}} + \frac{dx_i^2 + dr^2}{r^2} \right) \).

\( \theta = \) hyperscaling violation exponent; \( d_i = \) boundary spatial dim \((x_i)\).

[ Conformally Lifshitz. Effective Einstein-Maxwell-Dilaton theories \(^{(Trivedi et al; Kiritsis et al, \ldots)}\) ]

\( S \sim T^{(d_i-\theta)/z} \). Thermodynamics \( \sim \) space dim \( d_{eff} = d_i - \theta \): actual space is \( d_i \)-dim. 

\( \theta = d_i - 1 \): entanglement entropy \( \sim \) \( \log l \), logarithmic behaviour.

Gravity duals of Fermi surfaces? \(^{(Ogawa,Takayanagi,Ugajin; Huijse,Sachdev,Swingle)}\)

\( d_i - 1 \leq \theta < d_i \): EE area law violations. \(^{(Dong,Harrison,Kachru,Torroba,Wang)}\)

[Energy conditions: \( (d_i - \theta)(d_i(z - 1) - \theta) \geq 0, \ (z - 1)(d + z - \theta) \geq 0.\)]
Lif/h.v., gauge/string realizations

Narrow gravity parameter space. Identify recognizable CFT deformations and regimes.

Various string constructions involve $x^+$-dimensional reduction of

$$ds^2 = \frac{R^2}{r^2}(-2dx^+dx^- + dx_i^2 + dr^2) + R^2 g_{++}(dx^+)^2 + R^2 d\Omega_5^2.$$

i.e. $AdS + g_{++}$, where $g_{++} > 0$. In lower dim’nal theory, time is $t \equiv x^-$. 

(i) \( z = 2 \) Lifshitz \hspace{1cm} \text{(Balasubramanian,KN; Donos,Gauntlett; …)}:

[Non-normalizable deformations] \quad g_{++} \sim r^0 \xrightarrow{x^+\text{-dim. redn.}} z = 2 \text{ Lifshitz.}

\( g_{++} \) sourced by lightlike matter, \text{e.g.} \( g_{++} \sim (\partial_+ c_0)^2 \) with lightlike axion \( c_0 = K x^+ \):

$$ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + K^2 R^2 (dx^+)^2 \quad \rightarrow \quad ds^2 = -\frac{dt^2}{r^4} + \sum_{i=1}^d \frac{dx_i^2 + dr^2}{r^2}.$$ 

(ii) Hyperscaling violation: \( AdS_{d+1} \) plane waves \hspace{1cm} \text{(KN)}

[Normalizable \( g_{++} \)] \quad ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + R^2 Q r^{d-2} (dx^+)^2 \quad \rightarrow 

$$ds^2 = r^{\frac{2\theta}{d_i}} \left(-\frac{dt^2}{r^{2z}} + \sum_{i=1}^{d_i} \frac{dx_i^2 + dr^2}{r^2}\right), \quad z = \frac{d-2}{2} + 2, \quad \theta = \frac{d-2}{2}, \quad d_i = d - 2.$$ 

Anisotropic CFT excited state, energy-momentum density \( T_{++} = Q \).

\( AdS_5 \) plane wave: \( d = 4, \ d_i = 2, \ \theta = 1, \ z = 3 \). Logarithmic behaviour of EE.

Highly boosted limit of black branes \hspace{1cm} \text{(Singh)}.
Entanglement, $AdS$ plane waves

$$ds^2 = \frac{R^2}{r^2} [-2dx^+ dx^- + dx_i^2 + dr^2] + R^2 Q r^{d-2} (dx^+)^2,$$

dual to CFT state, $T_{++} \sim Q$ EE, spacelike strips (width $l$, $\Delta x^+ > 0 > \Delta x^-$).

$$(KN, \text{Takayanagi, Trivedi)}$$

Non-static spacetime $\rightarrow$ extremal surfaces.

Spacelike subsystem, UV cutoff $\epsilon$:
leading divergence is area law $\sim \frac{V_{d-2}}{\epsilon^{d-2}}$

**Case A:** width direction $x_i$. Strip along energy flux.

Finite cutoff-independent part of EE: size-dependent measure of entanglement $S^{fin} \sim N^2 \sqrt{Q V_2} \log(lQ^{1/4})$ [d=4].

[ground st] $-N^2 \frac{V_2}{l^2} < S^{fin} < N^2 T^3 V_2 l$ [thermal entr]

**Case B:** Strip $\perp$ flux.

Phase transition (no connected surface if $\Delta x^+ > 0 > \Delta x^-$).

$S_A$ saturated for $l \gtrsim Q^{-1/4}$.

[Boosted black branes (Maldacena,Martelli,Tachikawa): large boost $\lambda$, low temperature $r_0$ limit (Singh)

$$ds^2 = \frac{R^2}{r^2} \left( -2dx^+ dx^- + \frac{r_0^4 r_0^4}{2} (\lambda dx^+ + \lambda^{-1} dx^-)^2 + \sum_i dx_i^2 \right) + \frac{R^2 dr^2}{r^2 (1-r_0^4 r_0^4)}. \right]$$

More general plane wave states: *e.g.* M2-brane plane waves
EE$^{finite} \sim \sqrt{QL} \sqrt{l} \sqrt{N^{3/2}}$,

nonconformal $Dp$-brane plane waves, ...
D-brane plane waves, EE

\[ ds^2 = \frac{R^2}{r^2} (-2dx^+ dx^- + dx_i^2 + dr^2) + \frac{G_{d+1} Q}{R^{d-3}} r^{d-2} (dx^+)^2 + R^2 d\Omega^2 \]

\[ AdS_{d+1} \text{ plane wave excited states: } EE^{finite} = \pm \sqrt{Q} V_{d-2} l^{2-d/2} \sqrt{\frac{R^{d-1}}{G_{d+1}}} \]

[\pm : d \geq 4] \quad V_{2N} \log(l Q^{1/4}) (D3), \quad V_l \sqrt{Q} \sqrt{N^{3/2}} (M2), \quad -\sqrt{Q} V_4 \sqrt{N^3} (M5).

3d, 4d: finite entanglement grows with width \( l \) (strip along flux direction).

[spacelike strip: leading divergence, area law, \( \frac{V_2}{\epsilon^2} (4d), \frac{V_4}{\epsilon} (3d) \) ] \quad [\perp \text{ flux: phase transition.}]

[EE^{finite} scaling estimates \leftarrow \text{approximate} \ r_*, S^{finite} \text{ for large } Q, l \text{ from EE area functional}]

\[ G_5 \sim G_{10} R_{D3}^5, G_{4,7} \sim G_{11} R_{M2,M5}^{7,4}, \text{with } R_{D3}^4 \sim g_s N l_s^4, R_{M2}^6 \sim N l_P^6, R_{M5}^3 \sim N l_P^3 \]

Nonconformal Dp-brane plane waves \rightarrow \theta = \frac{p^2 - 6p + 7}{p - 5}, \quad z = \frac{2(p-6)}{p-5}.

Dual to strongly coupled Yang-Mills theories with constant energy flux \( T_{++} \) \( (KN) \) \( (Singh) \).

\[ ds^2_{st} = \frac{r^{(7-p)/2}}{R_p^{(7-p)/2}} dx^2 + \frac{G_{10} Q_p}{R_p^{(7-p)/2}} \frac{(dx^+)^2}{r^{(7-p)/2}} + R_p^{(7-p)/2} \frac{dr^2}{r^{(7-p)/2}} + R_p^{(7-p)/2} \frac{r(p-3)/2}{r^{(7-p)/2}} d\Omega^2_{8-p} \]

\[ e^\Phi = g_s \left( \frac{R_{7-p}}{R_{7-p}} \right)^{\frac{3-p}{4}}, \quad g_{YM}^2 \sim g_s \alpha'(p-3)/2, \quad R_p^{7-p} \sim g_{YM}^2 N \alpha'^5-p \sim g_s N \alpha'(7-p)/2. \]

EE leading divergence \( N_{eff}(\epsilon) \frac{V_{d-2}}{\epsilon^{d-2}} \) as for ground states (area law).

\[ EE^{finite}: \quad \frac{\sqrt{N_{eff}(l)}}{3-p} V_{p-1} \sqrt{Q} \frac{l^{p-3/2}}{(l(p-3)/2)}, \quad N_{eff}(l) = N^2 \left( \frac{g_{YM}^2 N}{l^{p-3}} \right)^{\frac{p-3}{5-p}} \]

Consistent with Dp-brane phase diagram, RG flows.

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Mutual Information


Holographic mutual information: find extremal surface for $A \cup B$.

Subsystems far, two disjoint minimal surfaces: $MI = 0$.

Subsystems nearby, connected surface has lower area.

Ryu-Takayanagi $\Rightarrow$ MI disentangling transition (Headrick).

[This is large $N$: expect softer subleading decay for MI.]

Similar disentanglement for thermal states (Fischler,Kundu,Kundu): $\frac{x_c}{l} \sim 0$ (for $x, l \gg \frac{1}{T}$).

(Mukherjee, KN) MI for $AdS$ plane wave excited states $\rightarrow$ critical separation $\frac{x_c}{l}$ between subsystems smaller than in ground state.

Mutual information disentangling occurs faster.

Suggests energy density disorders system.

[e.g. $\frac{x_c}{l} \simeq 0.732$ (pure $AdS_5$) whereas $\frac{x_c}{l} \simeq 0.414$ ($AdS_5$ plane wave).]

[Wide strips ($Ql^d \gg 1$), critical $\frac{x_c}{l}$ independent of flux $Q$.] $[Ql^d \sim O(1)$: numerical study]

[Narrow strips $Ql^d \ll 1$: perturbative corrections $\Delta S$ ($\sim$ EE thermodynamics) $\rightarrow$ MI decreases.]
A lightlike limit of entanglement

$AdS_{d+1}$ null deformation: 

$$ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + R^2 g_{++}(dx^+)^2$$ (KN)

Recall that lower dim’nal theory after $x^+$-reduction has time $t \equiv x^-$

$\Rightarrow$ lower dim entangling surface lies on $x^- = \text{const}$ slice upstairs.

Strip subsystem: $x^+ = \alpha \chi, x^- = -\beta \chi, -\frac{l}{2} < x \leq \frac{l}{2}, -\infty < \chi, y_i < \infty.$

[Spacelike strip $\alpha = \beta = 1 \rightarrow \frac{x^+ + x^-}{\sqrt{2}} \equiv t = \text{const}$ surface. $S^{\text{div}} \sim \frac{V_{d-2}}{\epsilon^{d-2}}, \text{area law}.$]

EE, null time $x^-$ slice ($\beta = 0$) 

$$S \sim \frac{V_{d-2}R^{d-1}}{4G_{d+1}} \int_{\epsilon}^{r_*} \frac{dr}{r^{d-1}} \frac{2}{\sqrt{r^2 - A^2 r^{d-2}}}$$

$\Rightarrow$ Milder leading divergence $S^{\text{div}} \sim \frac{R^{d-1}}{4G_{d+1}} V_{d-2} \frac{\sqrt{g_{++}(\epsilon)}}{\epsilon^{d-3}}$

$g_{++} = 0$ (ground state) $\Rightarrow$ lightlike EE (on $x^-$ slices) vanishes.
A lightlike limit of entanglement

\( AdS_{d+1} \) null deformation:

\[ ds^2 = \frac{R^2}{r^2} [-2dx^+dx^- + dx^2_i + dr^2] + R^2 g_{++}(dx^+)^2 \]  

(KN)

Recall that lower dim’nal theory after \( x^+ \)-reduction has time \( t \equiv x^- \)  
\( \Rightarrow \) lower dim entangling surface lies on \( x^- = const \) slice upstairs.

Strip subsystem:  
\( x^+ = \alpha \chi, \quad x^- = -\beta \chi, \quad -\frac{l}{2} < x \leq \frac{l}{2}, \quad -\infty < \chi, y_i < \infty. \)

[Spacelike strip \( \alpha = \beta = 1 \) \( \Rightarrow \) \( \frac{x^+ + x^-}{\sqrt{2}} \equiv t = const \) surface. \( S^{div} \sim \frac{V_{d-2}}{\epsilon^{d-2}}, \) area law.]

EE, null time \( x^- \) slice (\( \beta = 0 \))  
\[ S \sim \frac{V_{d-2}R^{d-1}}{4G_{d+1}} \int_{\epsilon}^{r^*} \frac{dr}{r^{d-1}} \frac{2 + \lambda^2 g_{++} r^2}{\sqrt{2 + \lambda^2 g_{++} r^2 - A^2 r^2 d - 2}} \]

\Rightarrow Milder leading divergence  
\[ S^{div} \sim \frac{R^{d-1}}{4G_{d+1}} \frac{V_{d-2}}{\epsilon^{d-3}} \sqrt{g_{++}(\epsilon)} \]

\( g_{++} = 0 \) (ground state) \( \Rightarrow \) lightlike EE (on \( x^- \) slices) vanishes.

Lightlike limit \( \equiv \) highly boosted limit of EE for spacelike strips.

Boost \( x^\pm \to \lambda^{\pm 1} x^\pm \) \( \Rightarrow \) \( \alpha = \lambda \) and \( \beta = \frac{1}{\lambda} \to 0 \)

\[ S = \frac{V_{d-2}R^{d-1}}{4G_{d+1}} \int_{\epsilon}^{r^*} \frac{dr}{r^{d-1}} \frac{2 + \lambda^2 g_{++} r^2}{\sqrt{2 + \lambda^2 g_{++} r^2 - A^2 r^2 d - 2}} \]  
(and width \( l \sim r_* \)).

Regime \( \lambda^2 g_{++}(\epsilon) \epsilon^2 \gtrsim 1: \) \( \Rightarrow \) EE on null time \( x^- \) slices (\( \beta = 0 \)).

[Similar structure for boosted black branes, nonconformal brane plane waves etc]
Null EE, $AdS_{d+1}$ plane waves

$$ds^2 = AdS_{d+1} + R^2 Q r^{d-2} (dx^+)^2, \quad T_{++} \sim Q: \text{ spacelike EE, area law, } S^{div} \sim \frac{V_{d-2}}{\epsilon^{d-2}}.$$

EE on null time $x^-$ slices if $\lambda^2 g_{++}(\epsilon)\epsilon^2 \gtrsim 1$, i.e. $\lambda^2 Q \epsilon^d \gtrsim 1$.

In bulk: UV surface $r = \epsilon$ dips in sufficiently to feel $g_{++}$ presence.

$$S \sim \frac{R^{d-1}}{4G_{d+1}} \frac{V_{d-2} \sqrt{\lambda^2 Q}}{\epsilon^{d-2}} \left( \frac{1}{\epsilon^2} - c_d \frac{1}{l^{d-2}} \right)$$

Milder leading divergence $S^{div} \sim \frac{R^{d-1}}{4G_{d+1}} \frac{V_{d-2}}{\epsilon^{d_{eff}-2}}$ \quad ($d_{eff} = d - 1 - \theta = \frac{d}{2}$)

Resembles spacelike EE in hyperscaling violating theory ($\theta = \frac{d-2}{2}$) from $x^+$-red’n.

$g_{++} = 0$ (ground state) $\Rightarrow$ lightlike EE (on $x^-$ slices) vanishes.

Reminiscent of ultralocality in lightcone QFT (Wall).

Ground state: $n$-pt functions (fields at distinct locations) vanish. Suggests vanishing EE.

Excited states, $P_+ \neq 0$: can show free-field correlators non-vanishing. Suggests EE nonzero.

Boundary space: $ds^2 = -2dx^+ dx^- + g^2 (dx^+)^2 + \sum_{i=1}^{d-2} dx_i^2$, with $g^2 = T_{++}\epsilon^d \gtrsim 1$.

Usual area law $S_{div} \sim N^2 \frac{V_{x^+} V_{y^i}}{\epsilon^{d-2}} = N^2 V_{d-2} \sqrt{T_{++} \epsilon^d} = N^2 \sqrt{Q} \frac{V_{d-2}}{\epsilon^{d_{eff}-2}}$. 

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de Sitter space
\[ ds^2 = \frac{R_{dS}^2}{\tau^2} (-d\tau^2 + dw^2 + dx_i^2) \].

Fascinating for various reasons.

\textit{dS/CFT}: fluctuations about \( dS \) encoded in dual Euclidean non-unitary CFT on boundary at future timelike infinity \( \mathcal{I}^+ \) (Strominger; Witten). Interesting to explore.

(Maldacena '02) analytic continuation \( r \rightarrow -i\tau, R_{AdS} \rightarrow -iR_{dS} \) from Eucl \( AdS \rightarrow \) Hartle-Hawking wavefunction of the universe \( \Psi[\varphi] = Z_{CFT} \).

Energy-momentum tensor \( \langle TT \rangle \) 2-pt fn \( \rightarrow \) dual CFT central charge
\[ C_d \sim i^{1-d} \frac{R_{dS}^{d-1}}{G_{d+1}} \] , negative or imaginary. \( C_3 \sim -\frac{R_{dS}^2}{G_4} \) for \( dS_4 \).

[ Bulk EAdS regularity conditions, deep interior \( \rightarrow \) Bunch-Davies initial conditions in deSitter, \( \varphi_k(\tau) \sim e^{i\kappa \tau} \), for large \( |\tau| \). \( Z_{CFT} = \Psi[\varphi] \sim e^{iS_{cl}[\varphi]} \) (semiclassical).

[Dual CFT: \( \langle O_k O_{k'} \rangle \sim \frac{\delta^2 Z}{\delta \varphi_k \delta \varphi_{k'}} \)] [Bulk expectation values \( \langle f_1 f_2' \rangle \sim \int D\varphi f_1 f_2' |\Psi|^2 \).]

Wavefunction \( \Psi[\varphi] \) not pure phase \( \rightarrow \) complex saddle points contribute to observables. ]
\[ ds^2_{d+1} = \frac{R_{dS}^2}{\tau^2} \left( -\frac{d\tau^2}{1 + \alpha \tau_0^d \tau^d} + (1 + \alpha \tau_0^d \tau^d)dw^2 + \sum_{i=1}^{d-1} dx_i^2 \right), \]

\( \alpha \) is a complex phase and \( \tau_0 \) real parameter of dimension energy, solves \( R_{MN} = \frac{d}{R_{dS}^2} g_{MN} \).

Regularity: Wick rotate \( \tau \to il \), demand resulting spacetime (thought of as saddle point in path integral) in Euclidean \((l, w)\)-plane has no conical singularity \( \Rightarrow \)

\[ \alpha = -(i)^d, \ l \geq \tau_0, \ w \simeq w + \frac{4\pi}{(d-1)\tau_0}. \] [analogous to interior regularity in \( AdS \)]

\[ \left[ \text{This is equivalent to analytic continuation } r \to -i\tau, \ R_{AdS} \to -iR_{dS} \text{ from } EAdS \text{ black brane } ds^2 = \frac{R_{AdS}^2}{r^2} \left( \frac{dr^2}{1-r_0^d r^d} + (1 - r_0^d r^d)d\theta^2 + \sum_{i=1}^{d-1} dx_i^2 \right). \right] \]

“Normalizable” metric modes \( \Rightarrow \) energy-momentum tensor vev.

\[ T_{ij} = \frac{2}{\sqrt{h}} \frac{\delta Z_{CFT}}{\delta h^{ij}} = \frac{2}{\sqrt{h}} \frac{\delta \Psi}{\delta h^{ij}} \propto i \frac{R_{dS}^{d-1}}{G_{d+1}} g_{ij}^{(d)} \rightarrow dS \text{ black brane.} \]

\([g_{ij}^{(d)} = \text{coefficient of } \tau^{d-2} \text{ in Fefferman-Graham expn}]. \) [\( dS/CFT: Z_{CFT} = \Psi \).]

Note \( i \) arising from the wavefunction of the universe \( \Psi \sim e^{iS_{cl}} \)

\( \Rightarrow \) energy-momentum real only if \( g_{ij}^{(d)} \) pure imaginary.

\( dS_4/CFT_3: \ \alpha = -i, \ T_{ww} = -\frac{R_{dS}^2}{G_4} \tau_0^3 \) with \( T_{ww} + (d-1)T_{ii} = 0. \)
de Sitter “bluewall”

\[ ds^2 = \frac{R_{dS}^2}{\tau^2} \left( -\frac{d\tau^2}{1-\tau^0 d\tau} + (1 - \tau_0^d \tau d)dw^2 + dx_i^2 \right) \]

Penrose diagram resembles AdS-Schwarzschild rotated by \( \frac{\pi}{2} \).

\([ -\infty \leq w \leq \infty ] \) Take \( \alpha = -1 \) earlier.

Equivalently, analytically continue \( \tau_0^d \) parameter too.

Using Kruskal coordinates: two asymptotic \( dS \) universes (\( \tau \to 0 \)).

Timelike singularities (\( \tau \to \infty \)). Cauchy horizons (\( \tau = \tau_0 \)).

\( \simeq \) interior of Reissner-Nordstrom black hole (or wormhole).

Trajectories in the de Sitter bluewall and the Cauchy horizon \( \to \)

Observers \( P_1 \) are static while \( P_2 \) has \( w \)-momentum \( p_w \),

crosses the horizon, turns around inside and appears

to re-emerge in the future universe.

Incoming lighttrays from infinity “crowd near” Cauchy horizon:

Late time infalling observers \( P_2 \) see early lighttrays blueshifted.

Infinite blueshift due to Cauchy horizon: instability.
A generalization of Ryu-Takayanagi to $dS \quad ds^2 = \frac{R^2_{dS}}{\tau^2}(-d\tau^2 + dw^2 + dx_i^2)$: Eucl time slice $w = \text{const}$, subregion at future timelike infinity $\rightarrow$ codim-2 extremal surfaces in de Sitter space.

$\rightarrow$ bulk analog of setting up entanglement entropy in dual Eucl CFT: consider boundary Euclidean time slice, construct spatial subsystem, trace over complement.

An obvious concern: $\mathcal{I}^+$ boundary spacelike $\Rightarrow$ real surfaces appear timelike, dipping inwards into past. Might imagine appropriate surfaces encoding EE should be spacelike.
de Sitter extremal surfaces

A generalization of Ryu-Takayanagi to $dS$ $ds^2 = \frac{R_{dS}^2}{\tau^2}(-d\tau^2 + dw^2 + dx_i^2)$:

Eucl time slice $w = const$, subregion at future timelike infinity $\rightarrow$ codim-2 extremal surfaces in de Sitter space.

$\rightarrow$ bulk analog of setting up entanglement entropy in dual Eucl CFT: consider boundary Euclidean time slice, construct spatial subsystem, trace over complement.

An obvious concern: $\mathcal{I}^+$ boundary spacelike $\Rightarrow$ real surfaces appear timelike, dipping inwards into past. Might imagine appropriate surfaces encoding EE should be spacelike.

Some possible expectations of extremal surface area for interpretation as entanglement entropy based on $Z_{CFT} = \Psi$:

- central charge coefficient in leading (area law) divergence must match dual CFT central charge earlier (from $\langle TT \rangle$ correlators).
- coefficient in logarithmic divergence must match conformal anomaly.
- expect finite cutoff-independent parts which are size-dependent measures of entanglement entropy in CFT.
de Sitter extremal surfaces

de Sitter (Poincare): \( ds_{d+1}^2 = \frac{R_{dS}^2}{\tau^2} (-d\tau^2 + dw^2 + dx_i^2) \rightarrow \) EE in dual Eucl CFT \( \rightarrow \)

bulk: Eucl time slice \( w = \text{const} \), subregion at future timelike infnty \( \rightarrow \) codim-2 extremal surface.

The above expectations and dual CFT central charge being negative or pure imaginary suggest real surfaces will not work \( \longrightarrow \)

[strip] \( S_{dS} = \frac{1}{4G_{d+1}} \int \prod_{i=1}^{d-2} \frac{R_{dS} dy_i}{\tau} \frac{R_{dS}}{\tau} \sqrt{dx_i^2 - d\tau^2} \)

\( \rightarrow \) \( S_{dS} \propto \frac{R_{dS}^{d-1} V_{d-2}}{4G_{d+1}} \int \frac{d\tau}{\tau^{d-1}} \sqrt{1 - \dot{x}^2}. \)

Extremize \( \rightarrow \) \( \dot{x}^2 = \frac{B^2 \tau^{2d-2}}{1 + B^2 \tau^{2d-2}}, \) \( B^2 = \text{const} \) is conserved quantity.
de Sitter extremal surfaces

de Sitter (Poincare): \[ ds^2_{d+1} = \frac{R^2_{dS}}{\tau^2} (-d\tau^2 + dw^2 + dx_i^2) \rightarrow \text{EE in dual Eucl CFT} \rightarrow \]
bulk: Eucl time slice \( w = \text{const} \), subregion at future timelike infnty \( \rightarrow \) codim-2 extremal surface.

The above expectations and dual CFT central charge being negative or pure imaginary suggest real surfaces will not work \( \rightarrow \)

\[ [\text{strip}] S_{dS} = \frac{1}{4G_{d+1}} \int \prod_{i=1}^{d-2} \frac{R_{dS} dy_i}{\tau} \frac{R_{dS}}{\tau} \sqrt{dx^2 - d\tau^2} \]
\[ \rightarrow S_{dS} \propto \frac{R_{dS}^{d-1} V_{d-2}}{4G_{d+1}} \int \frac{d\tau}{\tau^{d-1}} \sqrt{1 - \dot{x}^2} . \]

\[ \text{Extremize} \rightarrow \dot{x}^2 = \frac{B^2 \tau^{2d-2}}{1 + B^2 \tau^{2d-2}} , \quad B^2 = \text{const} \] is conserved quantity.

\[ \bullet \text{ Sign difference from } AdS \Rightarrow \text{ no real “turning point”}. \quad x(\tau) \text{ hyperboloid.} \]

Join two half-extremal-surfaces with cusp \( \rightarrow S_{dS} = 2 \frac{R_{dS}^{d-1}}{4G_{d+1}} V_{d-2} \int_{\epsilon}^{\tau_0} \frac{d\tau}{\tau^{d-1}} \sqrt{1 + B^2 \tau^{2d-2}} . \]
Minimize area: increase \( B \Rightarrow \) surface shape saturates, approaches \( \dot{x}^2 \rightarrow 1 \) as \( B \gg \frac{1}{\epsilon^{d-1}} \).

\[ \rightarrow \text{restriction of past lightcone wedge of subregion}. \quad x(\tau) \text{ null surface. Area vanishes.} \]

**Real codim-2 surfaces:** featureless, no apparent relation to EE.

[“outward bending” surfaces \( \rightarrow \) null, \( S_{dS} = 0 \)]

[surfaces \( x(\tau) = \text{const} \): \( B = 0 \), max area]

[Codim-1 surfaces: similar structure.]
de Sitter extremal surfaces

dS_{d+1}^2 = \frac{R_{dS}^2}{\tau^2} (-d\tau^2 + dw^2 + dx_i^2) \quad \rightarrow \quad \text{EE in dual Eucl CFT} \rightarrow \\
bulk: \text{Eucl time slice } w = \text{const}, \text{ subregion at future timelike infnty} \rightarrow \text{codim-2 extremal surface. }

$$\begin{align*}
\text{strip} & \quad S_{dS} = \frac{R_{dS}^{d-1}V_{d-2}}{4G_{d+1}} \int \frac{1}{\tau^{d-1}} \sqrt{dx^2 - d\tau^2} = \frac{R_{dS}^{d-1}V_{d-2}}{4G_{d+1}} \int \frac{d\tau}{\tau^{d-1}} \sqrt{\dot{x}^2 - 1} \\
\text{Extremize} & \quad (\partial_\tau x)^2 = \frac{-A^2 \tau^{2d-2}}{1 - A^2 \tau^{-2d-2}}. \quad [A^2 < 0 \text{ is earlier real-}\tau \text{ solution}] \end{align*}$$

Aspects of $(A)dS$ extremal surfaces and entanglement entropy, K. Narayan, CMI – p.22/29
de Sitter extremal surfaces

de Sitter (Poincare): \( ds^2_{d+1} = \frac{R^2_{dS}}{\tau^2} (-d\tau^2 + dw^2 + dx_i^2) \) \( \rightarrow \) EE in dual Eucl CFT \( \rightarrow \)
bulk: Eucl time slice \( w = const \), subregion at future timelike infty \( \rightarrow \) codim-2 extremal surface.

\[
S_{dS} = \frac{R^{d-1}_{dS} V_{d-2}}{4G_{d+1}} \int \frac{1}{\tau^{d-1}} \sqrt{dx^2 - d\tau^2} = \frac{R^{d-1}_{dS} V_{d-2}}{4G_{d+1}} \int \frac{d\tau}{\tau^{d-1}} \sqrt{x^2 - 1}
\]

Extremize \( \rightarrow \) \( (\partial_\tau x)^2 = \frac{-A^2\tau^{2d-2}}{1-A^2\tau^{2d-2}} \). \[A^2 < 0 \text{ is earlier real-\(\tau\) solution}\]

\[dS_4/CFT_3\]: consider \( A^2 > 0 \). Near \( \tau \rightarrow 0 \): \( \dot{x}^2 \sim -A^2\tau^4 \) \( i.e. \)
\( x(\tau) \sim \pm iA\tau^3 + x(0) \). This is spatial direction in Eucl CFT \( \Rightarrow \)
\( x(\tau) \text{ real-valued} \Rightarrow \tau = iT \) \[\text{[can show width \( \Delta x \) also real]}\]

\( x(\tau) \rightarrow \text{complex extremal surface}, \ \tau \text{ along imaginary path } \tau = iT \).
\( (\frac{dx}{dT})^2 = \frac{A^2T^4}{1-A^2T^4} \). Note turning point: \( T_* = \frac{1}{\sqrt{A}} \) \( \text{ (where } |\dot{x}|^2 \rightarrow \infty \).
de Sitter extremal surfaces

de Sitter (Poincare): \[ ds_{d+1}^2 = \frac{R^2_{dS}}{\tau^2} (-d\tau^2 + dw^2 + dx_i^2) \] \[ \rightarrow \quad \text{EE in dual Eucl CFT} \rightarrow \]

bulk: Eucl time slice \( w = \text{const} \), subregion at future timelike infinity \( \rightarrow \) codim-2 extremal surface.

\[ S_{dS} = \frac{R_{dS}^{d-1} V_{d-2}}{4 G_{d+1}} \int \frac{1}{\tau^{d-1}} \sqrt{dx^2 - d\tau^2} = \frac{R_{dS}^{d-1} V_{d-2}}{4 G_{d+1}} \int \frac{d\tau}{\tau^{d-1}} \sqrt{x^2 - 1} \]

Extremize \[ \rightarrow \quad (\partial_\tau x)^2 = \frac{-A^2 \tau^{2d-2}}{1 - A^2 \tau^{2d-2}} . \] \[ [A^2 < 0 \text{ is earlier real-} \tau \text{ solution}] \]

\( dS_4/CFT_3 \): consider \( A^2 > 0 \). Near \( \tau \rightarrow 0 \):

\[ x(\tau) \sim \pm iA\tau^3 + x(0) . \]

This is spatial direction in Eucl CFT \( \Rightarrow \)

\[ x(\tau) \text{ real-valued} \Rightarrow \tau = iT \]

[can show width \( \Delta x \) also real]

\[ x(\tau) \rightarrow \text{complex extremal surface,} \quad \tau \text{ along imaginary path} \quad \tau = iT. \]

\[ \left( \frac{dx}{dT} \right)^2 = \frac{A^2 T^4}{1 - A^2 T^4} . \quad \text{Note turning point:} \quad T_* = \frac{1}{\sqrt{A}} \quad (\text{where} \quad |\dot{x}|^2 \rightarrow \infty). \]

Can now smoothly join half-extremal-surfaces at turning point. \[ [\tau_{UV} = i\epsilon, \quad \tau_* \sim i\ell] \]

\[ \frac{\Delta x}{2} = \frac{l}{2} = \int_0^{\tau_*} d\tau \frac{iA\tau^2}{\sqrt{1 - A^2 \tau^4}} = \int_0^{T_*} \frac{(T^2/T_*^2) dT}{\sqrt{1 - (T^4/T_*^4)}} \sim T_* \]

\[ S_{dS_4} = -i \frac{R_{dS}^2}{4G_4} V_1 \int_{\tau_{UV}}^{\tau_*} \frac{d\tau}{\sqrt{1 - \tau^4/\tau_*^4}} = -\frac{R_{dS}^2}{4G_4} V_1 \int_\epsilon^l \frac{dT/T^2}{\sqrt{1 - T^4/T_*^4}} \sim -\frac{R_{dS}^2}{G_4} V_1 \left( \frac{1}{\epsilon} - c \frac{l}{l} \right) \]

Overall sign \( \rightarrow \) match with \( dS_4/CFT_3 \) central charge.
de Sitter extremal surfaces

de Sitter (Poincare): \( ds_{d+1}^2 = \frac{R_{dS}^2}{\tau^2} (-d\tau^2 + dw^2 + dx_i^2) \) \( \rightarrow \) EE in dual Eucl CFT \( \rightarrow \)
bulk: Eucl time slice \( w = const \), subregion at future timelike infnty \( \rightarrow \) codim-2 extremal surface.

\[
\begin{align*}
\text{[strip]} & \quad S_{dS} = \frac{R_{dS}^{d-1} V_{d-2}}{4 G_{d+1}} \int \frac{1}{\tau^{d-1}} \sqrt{dx^2 - d\tau^2} = \frac{R_{dS}^{d-1} V_{d-2}}{4 G_{d+1}} \int \frac{d\tau}{\tau^{d-1}} \sqrt{x^2 - 1}
\end{align*}
\]

Extremize \( \rightarrow \) \( (\partial_\tau x)^2 = \frac{-A^2 \tau^{2d-2}}{1 - A^2 \tau^{2d-2}} \). \[ A^2 < 0 \text{ is the earlier real solution} \]

\[\frac{dS_{d+1}}{CFT_d} \text{ (d even): near } \tau \rightarrow 0, \quad \dot{x} \sim \pm \sqrt{-A^2} \tau^{d-1} \text{ i.e.} \]

\[ x(\tau) \sim \pm \sqrt{-A^2} \tau^d + x(0). \] This is spatial direction in Eucl CFT

\[ \Rightarrow \quad \text{x(\tau) real-valued } \Rightarrow A^2 < 0, \quad \tau = iT \quad \text{[can show width } \Delta x \text{ also real]} \]

\[ x(\tau) \rightarrow \text{complex extremal surface}, \quad \tau \text{ along imaginary path } \tau = iT. \]

\[ (\frac{dx}{dT})^2 = \frac{A^2 T^{2d-2}}{1 + (-1)^{d-1} A^2 T^{2d-2}}. \] Note turning point: \( T_*^{2d-2} A^2 = 1. \)

\[ S_{dS} = -i \frac{R_{dS}^{d-1}}{4 G_{d+1}} V_{d-2} \int_{\tau_U}^{\tau_V} \frac{d\tau}{\tau^{d-1}} \sqrt{\frac{2}{1 + A^2 \tau^{2d-2}}} \]

\[ = i^{1-d} \frac{R_{dS}^{d-1}}{2 G_{d+1}} V_{d-2} \int_{\epsilon}^{T_\ast} \frac{dT/T_d^{d-1}}{\sqrt{1 + (-1)^{d-1} A^2 T^{2d-2}}} \sim i^{1-d} \frac{R_{dS}^{d-1}}{2 G_{d+1}} V_{d-2} (\frac{1}{\epsilon^{d-2}} - c_d \frac{1}{\epsilon^{d-2}}) \]
de Sitter extremal surfaces, \( dS/CFT \)

**Complex extremal surfaces:** compare \( dS_{d+1}/CFT_d \) central charges.

[Strip width real (CFT spatial direction) \( \Rightarrow \) path \( \tau = iT \) \( \rightarrow \) extremal surface with turning point.]

\( dS_4: \) area

\[
S_{dS_4} = -\frac{R_{dS}^2}{4G_4}V_1 \int_\epsilon^1 \frac{dT/T^2}{\sqrt{1-T^4/T_*^4}} \sim -\frac{R_{dS}^2}{G_4}V_1 \left( \frac{1}{\epsilon} - c_1 \right)
\]

\( dS_{d+1}, \) even \( d: \) area

\[
S_{dS} = i^{1-d} \frac{R_{dS}^{d-1}}{2G_{d+1}} V_{d-2} \int_\epsilon^{T_*} \frac{dT/T^{d-1}}{\sqrt{1+(-1)^{d-1}(T/T_*)^{2d-2}}}
\]

\[
\sim i^{1-d} \frac{R_{dS}^{d-1}}{2G_{d+1}} V_{d-2} \left( \frac{1}{\epsilon^{d-2}} - c_d \frac{1}{\epsilon^{d-2}} \right)
\]

\[\equiv \text{analytic continuation from AdS Ryu-Takayanagi extremization.}\]

\[S_{AdS}[R, x(r), \tau] = \frac{R^{d-1}}{4G_{d+1}} V_{d-2} \int \frac{dr}{r^{d-1}} \sqrt{1 + (\frac{dx}{dr})^2}, \quad (x')^2 = \frac{A^2 \tau^{2d-2}}{1-A^2 \tau^{2d-2}} \rightarrow \]

\[
\dot{x}^2 = \frac{-(A^2)^{d-1}(d-2)}{1-(A^2)^{d-1}A^2 \tau^{2d-2}}, \quad S_{dS} = -i\frac{R^{d-1}_{dS}}{4G_{d+1}} V_{d-2} \int \frac{d\tau}{\tau^{d-1}} \frac{1}{\sqrt{1+(-1)^{d-1}(d-1)A^2 \tau^{2d-2}}}
\]

- **leading “area law” divergence**

\[
C_d \frac{V_{d-2}}{\epsilon^{d-2}} \rightarrow
\]

central charges \( C_d = i^{1-d} \frac{R_{dS}^{d-1}}{G_{d+1}} \) match \( dS/CFT \) using \( Z_{CFT} = \Psi \).

- **finite cutoff-independent parts**

\[
\sim i^{1-d} \frac{R_{dS}^{d-1}}{G_{d+1}} \frac{V_{d-2}}{\epsilon^{d-2}}.
\]

- **Spherical extremal surfaces:** subleading log-div. Anomaly coeff exactly matches \( \Psi \) log-coeff.

- **\( dS_4 \) black brane, \( CFT_3 \) at uniform energy density:** \( S_{dS}^{fin} \) resembles extensive thermal entropy.
Spherical extremal surfaces, $dS/CFT$

$$ds^2 = \frac{R^2_{dS}}{\tau^2}(-d\tau^2 + dw^2 + dr^2 + r^2d\Omega^2_{d-2}) \to w = \text{const}, \text{ sphere subregion}. \quad 0 \leq r \leq l$$

$$S_{dS} = \frac{R_{dS}^{d-1}\Omega_{d-2}}{4G_{d+1}} \int \frac{d\tau}{\tau^{d-1}} r^{d-2} \sqrt{\left(\frac{dr}{d\tau}\right)^2 - 1}, \text{ extremize: } r(\tau) = \sqrt{l^2 + \tau^2}, \quad \dot{r} = \frac{\tau}{\sqrt{l^2 + \tau^2}}$$

Real $\tau$: outward-bending, $r(\tau) \geq l$. Timelike: $\dot{r} \leq 1$. No “end” at finite $\tau$.

$$\to \quad \epsilon < |\tau| < \infty \to S_{dS} \text{ real, no finite cutoff-indep parts.}$$

$$\tau = iT: \quad \text{now } 0 \leq r(\tau) < l \text{ and } \Delta r = l. \quad \text{Turning point } \tau_* = il.$$  

$$S_{dS} = \frac{R_{dS}^{d-1}\Omega_{d-2}}{4G_{d+1}} \int_{i\epsilon}^{\tau_*} \frac{d\tau}{\tau^{d-1}} (-il)(l^2 + \tau^2)^{(d-3)/2} \to \quad S_{dS4} = -\frac{\pi R^2_{dS}}{2G_4}\left(\frac{l}{\epsilon} - 1\right)$$

$d$ even: $\log \frac{l}{\epsilon}$ divergence. Coeff $\to -i\frac{R_{dS}}{2G_3} [dS_3]; \quad -i\frac{\pi R^3_{dS}}{2G_5} [dS_5], \ldots$

Free energy of $CFT_d$ on sphere: log-div, related to conformal anomaly.

Casini,Huerta,Myers: $-F_{CFT} = \log Z_{CFT} = a \log \epsilon + \ldots$, integ. trace anomaly $a = \int \langle T^k_k \rangle$.

$Z_{CFT} = e^{-F} = \Psi \sim e^{iS_{cl}}$ for auxiliary global $dS$. $T_{ij} \sim \frac{2}{\sqrt{h}} \frac{\delta(-F_{CFT})}{\delta h_{ij}} \sim i \frac{2}{\sqrt{h}} \frac{\delta S}{\delta h_{ij}}$

$\to \quad \log$-div coeff matches $\quad \to$ equivalent to analytic continuation from $AdS$.

$$[S_{CFT}^{EE} = -\lim_{n \to 1} \partial_n \frac{Z_n}{(Z_1)^n}; \quad \text{scale change } l \frac{\partial}{\partial l} S_{CFT}^{EE} \sim \int \langle T_{\mu}^{\mu} \rangle; \quad \text{here } S_{CFT}^{EE} = S_{dS}]$$

$$S_{cl} = \frac{2d\Omega_{d} R_{dS}^{d-1}}{16\pi G_{d+1}} \int \frac{dt}{R_{dS}} (\cosh \frac{t}{R_{dS}})^d \to \log$-div $[ds^2 = -dt^2 + R_{dS}^2 (\cosh \frac{t}{R_{dS}})^2 d\Omega^2_{d}]$$

Aspects of $(A)dS$ extremal surfaces and entanglement entropy, K. Narayan, CMI – p.27/29
$dS_4$ surfaces, negative EE

\[ dS_4: \quad S_A \sim -\frac{R_{ds}^2}{G_4} \left( \frac{V_1}{l} - \frac{V_1}{l} \right) \quad \text{[strip]} \]


- Consider two strip subregions, width $l_2$ and $l_1 > l_2$ ($l_1, l_2 \ll V_1$).
  
  Then 
  \[ S(l_1) - S(l_2) = -\frac{R_{ds}^2}{G_4} \left( \frac{V_1}{l_2} - \frac{V_1}{l_1} \right) < 0, \quad \text{i.e.} \]
  \[ S(l_1) < S(l_2) \Rightarrow \text{bigger subregion more ordered than smaller one.} \]

  [conventional unitary CFT: $S(l_1) > S(l_2)$, i.e. bigger subregion more disordered]

- Entropic c-function 
  \[ c(l) = l^{d-1} \frac{dS_A}{dl}. \quad c(l) \equiv \frac{l^2}{V_1} \frac{dS_A}{dl} = -\frac{R_{ds}^2}{G_4} < 0 \quad \text{i.e. as} \]
  \[ l \text{ increases, } S(l) \text{ decreases. Asymptotically } dS_4 \text{ spaces, negative areas } S_A \text{ of complex extremal surfaces imply } c'(l) > 0, \quad \text{i.e. as } l \text{ increases,} \]
  \[ c(l) \text{ increases. New degrees of freedom integrated in?} \]

| $|\tau_*|$ | $l$: increasing size $l \rightarrow$ going to larger $|\tau_*|$ (earlier times in past). |
Conclusions, questions

- Various gauge/string realizations of Lifshitz & hyperscaling violation involve $x^+$-reduction of $AdS$ deformations with $g_{++}$.
  Entanglement entropy, lower dim theory $\rightarrow$ null time $x^-$ slices upstairs $\rightarrow$ lightlike limit of EE. Lightcone QFT, ultralocality, . . . ?

- Deeper understanding of complex extremal surfaces in de Sitter space, EE as probe of $dS/CFT$.

Entanglement entropy in non-unitary (ghost) CFTs [tentative]

$[e.g. 2d \ bc$-CFTs with $(h_b, h_c) = (1, 0)$ have $c = -2$.
$SL(2)$ vacuum $|1\rangle$ ($L_0$ eigenvalue zero) satisfies $b_{m\geq0}|1\rangle = 0$, $c_{m\geq1}|1\rangle = 0$ appears to coincide with ghost ground state $|\downarrow\rangle$.
$Z_N$ orbifold $\rightarrow$ twist operator $\sigma_{k/N}$ dim $h_\sigma = -\frac{1}{2} \frac{k}{N} (1 - \frac{k}{N})$. Replica $\rightarrow$ $S = \frac{c}{3} \log \frac{1}{\epsilon} < 0$.
Lowest conformal dimension is $\Delta = 0 \Rightarrow$ it appears that $c_{eff} = c - 24\Delta = c < 0$.

$\rightarrow$ subsector of complex ghost $\partial\chi \bar{\partial}\bar{\chi}$ CFT with $c = -2$ with anticommuting scalars $\chi, \bar{\chi}$
$\rightarrow$ logarithmic CFT, but restricting to $\partial\chi, \bar{\partial}\bar{\chi}$ operators $\rightarrow$ negative EE (mapping to above).]