Cosmological singularities, gauge theory duals and strings

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[arXiv:0909.4731, KN; arXiv:0904.4532, Kallingalthodi Madhu, KN; arXiv:0807.1517, Adel Awad, Sumit Das, Suresh Nampuri, KN, Sandip Trivedi; arXiv:0711.2994, Awad, Das, KN, Trivedi; hep-th/0602107, hep-th/0610053, Das, Jeremy Michelson, KN, Trivedi; and work in progress.]

- AdS/CFT with cosmological singularities: gauge theories with time-dep couplings and spacelike singularities, BKL etc
- worldsheet: null singularities and free strings

See also

arXiv:0906.3275, Awad, Das, Ghosh, Oh, Trivedi.

Related references:

Craps et al; Tseytlin et al; ...: worldsheet investigations of plane waves with singularities.

Craps, Hertog, Turok; Chu, Ho; Hertog, Horowitz, ...: cosmological generalizations of AdS/CFT framework.

Craps, Sethi, Verlinde, and followup work by various people: Matrix theory duals of cosmological singularities.

...

Cosmology, time dependence, ...

Tempting to think very early Universe has deep repercussions on various aspects of physics.

- Big Bang singularities, time, in string theory models?
  Understand spacelike, null singularities — events in time.

General Relativity breaks down at singularities: curvatures, tidal forces divergent. Want "stringy" description, eventually towards smooth quantum (stringy) completion of classical spacetime geometry.

Previous examples: "stringy phases" in $e.g.$ 2-dim worldsheet (linear sigma model) descriptions (including time-dep versions, e.g. tachyon dynamics in (meta/)unstable vacua), dual gauge/Matrix theories, ...

In what follows, we'll use (i) the AdS/CFT framework,
(ii) worldsheet string spectrum analysis near singularity.

AdS/CFT and deformations

Nice stringy playground: AdS/CFT. Bulk string theory on $AdS_5 \times S^5$ with dilaton (scalar) $\Phi = const$, and metric (Poincare coords)

$$ds^2 = \frac{1}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2) + ds^2_{S^5} \ ,$$

with 5-form field strength, dual to boundary $d = 4$ $\mathcal{N}$=4 (large N) $SU(N)$ Superconformal Yang-Mills theory, coupling $g^2_{YM} = e^\Phi$.

Known: symmetries, mode/operator correspondence, correlators, ...
Deeper decoding of hologram (spacetime emergence etc): desirable.

Assume AdS/CFT: study *time-dependent* deformations of AdS/CFT.
Bulk subject to time-dependent sources classically evolves in time (thro Einstein eqns), eventually giving rise to a cosmological singularity, and breaks down. Avoid any bulk investigation near singularity.
Boundary: Gauge theory dual is a sensible Hamiltonian quantum system in principle, subject to time-dependent sources. Response ?

AdS cosmologies

Start with $AdS_5 \times S^5$ and turn on non-normalizable deformations for the metric and dilaton (also nontrivial 5-form):

$$ds^2 = \frac{1}{z^2}(\tilde{g}_{\mu\nu}dx^\mu dx^\nu + dz^2) + ds^2_{S^5} \ , \qquad \Phi = \Phi(x^\mu) \ .$$

This is a solution in string theory if

$$\tilde{R}_{\mu\nu} = \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi \ , \qquad \frac{1}{\sqrt{-\tilde{g}}}\,\partial_\mu(\sqrt{-\tilde{g}}\;\tilde{g}^{\mu\nu}\partial_\nu\Phi) = 0 \ ,$$

$i.e.$ if it is a solution to a 4-dim Einstein-dilaton system.
Time dep: $\Phi = \Phi(t)$ or $\Phi = \Phi(x^+)$.
More later on cosmological solutions.

General family of solutions: ($Z(x^m)$ harmonic function)

$$ds^2 = Z^{-1/2}\tilde{g}_{\mu\nu}dx^\mu dx^\nu + Z^{1/2}g_{mn}dx^m dx^n \ , \quad \Phi = \Phi(x^\mu),$$

$g_{mn}(x^m)$ is Ricci flat, and $\tilde{g}_{\mu\nu} = \tilde{g}_{\mu\nu}(x^\mu)$. $[\mu = 0123, m = 4\ldots 9.]$

AdS cosmologies cont'd

In many cases, possible to find new coordinates such that boundary metric $ds_4^2 = \lim_{z\to 0} z^2 ds_5^2$ is flat, at least as an expansion about the boundary ($z = 0$) if not exactly.

These are Penrose-Brown-Henneaux (PBH) transformations: subset of bulk diffeomorphisms leaving metric invariant (in Fefferman-Graham form), acting as Weyl transformation on boundary.

E.g. null cosmologies $ds^2 = \frac{1}{z^2}(e^{f(x^+)}\eta_{\mu\nu}dx^\mu dx^\nu + dz^2)\ ,\quad \Phi(x^+)$.

The coord. transf. $w = ze^{-f/2},\quad y^- = x^- - \frac{w^2 f'}{4}$, gives

$$ds^2 = \frac{1}{w^2}[-2dx^+dy^- + dx_i^2 + \frac{1}{4}w^2(\Phi')^2(dx^+)^2] + \frac{dw^2}{w^2}\ ,$$

using $R_{++} = \frac{1}{2}(f')^2 - f'' = \frac{1}{2}(\Phi')^2$, the constraint on these solutions.

Now boundary at $w = 0$ manifestly flat 4D Minkowski spacetime.

Gauge theories, time-dep couplings

Thus dual gauge theory lives on flat space. So sharp sub-question: Gauge theory with time-dependent coupling $g_{YM}^2 = e^{\Phi}$. Response?

We would like to study sources that are trivial in the far past (bulk is $AdS_5 \times S^5$) and smoothly turn on: this means the gauge theory begins in vacuum state and is subject to Hamiltonian time evolution through this external time-dependent source. Basic expectation: time-dep source excites vacuum to higher energy state.

Want to consider sources that approach $e^{\Phi} \to 0$ at some finite point in time: e.g. $g_{YM}^2 = e^{\Phi} \ \to \ (-t)^p \ , \ p > 0 \ \ [t < 0]$.

We'd specially like to understand gauge theory response near $t = 0$.

This point in time corresponds to a singularity in the bulk:

$R_{tt} = \frac{1}{2}\dot{\Phi}^2 \ \sim \frac{1}{t^2}$. Curvatures, tidal forces diverge near $t = 0$.

Gauge theories, time-dep couplings

Gauge theory kinetic terms $\int e^{-\Phi} F^2$ not canonical.
As in usual perturbation theory, try absorbing coupling $g_{YM}^2 = e^{\Phi}$ into the gauge field A_μ: now g_{YM} appears only in interaction terms.

First, consider toy scalar theory $L[\tilde{X}] = -e^{-\Phi}\left(\frac{1}{2}(\partial\tilde{X})^2 + \tilde{X}^4\right)$.
Redefining $\tilde{X} = e^{\Phi/2}X$: $\quad L \to -(\partial X)^2 - m^2(\Phi)X^2 - e^{\Phi}X^4$,
dropping a boundary term, and $m^2(\Phi) = \frac{1}{4}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\partial_\mu\partial^\mu\Phi$.

Time-dep $\Phi = \Phi(t)$: $e.g.$ $g_{YM}^2 = e^{\Phi} = (-t)^p$, $p > 0$ $[t < 0]$
gives $m^2(\Phi) = -\frac{1}{4}(\dot{\Phi})^2 + \frac{1}{2}\ddot{\Phi} = -\frac{p(p+2)}{4\,t^2}$.

Can study time-dep quantum mechanics of single momentum-k modes.
* X variables canonical: tachyonic divergent mass forces $X \sim \frac{1}{t^{p/2}}$.
Extra information required as $X \to \infty$: X description not good.
* $\tilde{X}$ variables finite near $t = 0$: interaction terms $e^{-\Phi}\tilde{X}^4|_{t\sim 0}$ large.

Time-dep field theory wave-fn

General field theory Schrodinger picture analysis possible near $t = 0$.

Lagrangian $\quad L = \int d^3x\, e^{-\Phi}(\frac{1}{2}(\partial_t\tilde{X})^2 - \frac{1}{2}(\partial_i\tilde{X})^2 - \tilde{X}^4)$.

Field theory Hamiltonian: $\quad H = e^{-\Phi}V[\tilde{X}] + e^{\Phi}\int d^3x(-\frac{1}{2}\frac{\delta^2}{\delta\tilde{X}^2})$,

where $V[\tilde{X}] = \int d^3x\, (\frac{1}{2}(\partial_i\tilde{X})^2 + \tilde{X}^4)$ [replacing $\Pi(x) \to \frac{1}{i}\frac{\delta}{\delta\tilde{X}}$] .

Schrodinger eqn: $\quad i\partial_t\psi[\tilde{X}(x), t] = H\psi[\tilde{X}(x), t]$.

Near $t = 0$, the potential term $e^{-\Phi}V$ dominates in the Hamiltonian $\Rightarrow$ $i\partial_t\psi \sim\ e^{-\Phi(t)}V[\tilde{X}(x)]\psi$. This gives the wave-fn (generic state)

$$\psi[\tilde{X}(x), t] = e^{-i(\int dt\, e^{-\Phi(t)})V[\tilde{X}(x)]}\ \psi_0[\tilde{X}(x)]\ .$$

Phase $\sim\ \frac{(-t)^{1-p}}{1-p}V[\tilde{X}(x)]$. If $p > 1$, “wildly” oscillating ($t \to 0$).

Energy diverges for generic states ($\langle V\rangle \neq 0$) [no time-dep in $\langle V\rangle$]

$$\langle H\rangle \simeq e^{-\Phi}\langle V\rangle = \frac{1}{(-t)^p}\ \int D\tilde{X}\ V[\tilde{X}]\ |\psi_0[\tilde{X}(x)]|^2\ .$$

The gauge theory

Scalars, fermions: no dilaton coupling in KE terms. Fermion Yukawa and scalar quartic terms come with powers of $g_{YM} = e^{\Phi/2}$, vanish near $t = 0$.

Gauge fields: KE terms have dilaton coupling $\int e^{-\Phi}\, \mathrm{Tr} F^2$.

Since $e^{\Phi} = (-t)^p$ near $t = 0$, the gauge field terms determine the behaviour of the system near $t \sim 0$. Focus on this.

Consider non-interacting theory first.

Convenient (Coulomb) gauge $A_0 = 0\ ,\ \partial_j A_j = 0$ (longitudinal part of gauge field time-indep from Gauss law: $\partial_0(\partial_j A_j) = 0$).

Residual action for two physical transverse components A^i becomes $\int e^{-\Phi}(\partial A^i)^2$, (i.e. two copies of the scalar theory earlier).

The gauge theory

Cubic/quartic interactions: no time derivatives.

Contribute only to potential energy terms (from magnetic field), not to KE terms (from electric field). PE $V[A^i(x)] = \frac{1}{4}\int d^3x\ \mathrm{Tr} F_{ij}^2$.

$$L_g = \frac{1}{4}\int d^3x\ e^{-\Phi}\ \mathrm{Tr}\left((\partial_t A^i)^2 - F_{ij}^2\right) .$$

Schrodinger quantization: $e^{-\Phi}\dot{A}^i = E^i \rightarrow \frac{1}{i}\frac{\delta}{\delta A^i}$. Then wave-fn

$$\psi[A^i(x),t] \sim e^{-i(\int dt\ e^{-\Phi})V[A^i(x)]}\ \psi_0[A^i(x)] \qquad (t \sim 0) .$$

Phase as before $\sim \frac{(-t)^{1-p}}{1-p} V[A^i]$: "wildly" oscillating (for $p > 1$).

Energy diverges $\langle H \rangle \simeq e^{-\Phi}\int DA^i\ V[A^i(x)]\ |\psi_0[A^i]|^2$ (if $\langle V \rangle \neq 0$) .

Note: this is not perturbation theory. Interactions important.

Thus if $g_{YM}^2 = e^{\Phi} \rightarrow 0$ strictly, gauge theory response singular.
For cutoff e^{Φ}, large energy production due to time-dep source.

Renormalization effects

$Caveats$: For sufficiently high-frequency modes, $KE \sim e^{\Phi} k^2$ might not be negligible relative to $V[A^i]$ (with a regularization $e.g.$ $|t| \sim \epsilon$).

Introduce momentum cutoff Λ, consider renormalization effects for the Wilsonian effective action. Gauge theory conformal: RG effects are due to coupling time-dep. Expected to be proportional to $\dot{\Phi}$. Then effective potential could acquire additional terms $e.g.$ with operators $\mathcal{O}_i$

$$V_{eff} \sim \ e^{-\Phi} \left(\mathcal{O}_{bare} + c_1 e^{\Phi} \frac{\dot{\Phi}^2}{\Lambda^2} \mathcal{O}_1 + c_2 \left(e^{\Phi} \frac{\dot{\Phi}^2}{\Lambda^2} \right)^2 \mathcal{O}_2 + \ldots \right) .$$

Heuristically, with $\mathcal{O}_i \sim \mathcal{O}_{bare}$, this as a geometric series sums to

$$V_{eff} \sim \ e^{-\Phi} \frac{1}{1 - e^{\Phi} \frac{\dot{\Phi}^2}{\Lambda^2}} \ \sim \ e^{-2\Phi} \frac{\Lambda^2}{\dot{\Phi}^2} \ .$$

So if $V_{eff}[A^i, \Phi]$ is comparable to $KE \sim e^{\Phi}$, then potential might not dominate in wavefunction. For the heuristic calculation above, V_{eff} dominates over KE (as $t \rightarrow 0$) if $t^{2-2p} \gg t^p$, $i.e.$ if $p > \frac{2}{3}$.

AdS cosmologies with spacelike singularities

Recall: $ds^2 = \frac{1}{z^2}(\tilde{g}_{\mu\nu}dx^\mu dx^\nu + dz^2) + ds^2_{S^5}\ ,\quad \Phi = \Phi(x^\mu)\ .$

Solution if: $\tilde{R}_{\mu\nu} = \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi\ ,\qquad \frac{1}{\sqrt{-\tilde{g}}}\,\partial_\mu(\sqrt{-\tilde{g}}\,\tilde{g}^{\mu\nu}\partial_\nu\Phi) = 0\ .$

Solutions with spacelike Big-Bang (Crunch) singularities:

* $\quad ds^2 = \frac{1}{z^2}\left[dz^2 - dt^2 + \sum_{i=1}^3 t^{2p_i}(dx^i)^2\right]\ ,$

$\quad e^\Phi = |t|^{\sqrt{2(1-\sum_i p_i^2)}},\qquad \sum_i p_i = 1\ .$ [Kasner cosmologies]

* $ds^2 = \frac{1}{z^2}\left[dz^2 + |\sinh(2t)|(-dt^2 + \frac{dr^2}{1+r^2} + r^2(d\theta^2 + sin^2\theta d\phi^2))\right],$

$\quad e^\Phi = g_s\,|\tanh t|^{\sqrt{3}}\ .$ [$k=-1$ (hyperbolic) FRW boundary]

Dilaton bounded, approaching constant at early/late times: asymptotic spacetime is $AdS_5 \times S^5$ (using a coord transformation).

The $k=0$ (flat) FRW is the same as symmetric Kasner ($p_i = \frac{1}{3}$).

(There is also a $k=+1$ (spherical) FRW solution.)

AdS BKL-cosmologies

In fact, larger family of cosmological solutions where spatial metric is one of the homogenous spaces in the Bianchi classification:

$$ds^2 = \frac{1}{z^2}\left[dz^2 - dt^2 + \eta_{ab}(t)(e^a_\alpha dx^\alpha)(e^b_\beta dx^\beta)\right]\ , \quad e^\Phi = e^{\Phi(t)}\ .$$

$e^a_\alpha dx^\alpha$ are a triad of 1-forms defining symmetry directions. Spatially homogenous dilaton means spatial $R^a_{(a)}$ vanish, and $R^0{}_0 = \frac{1}{2}(\partial_0\Phi)^2$.

Bianchi-IX: $ds^2 = \frac{1}{z^2}\left[dz^2 - dt^2 + \eta_i^2(t)e^i_\alpha e^i_\beta dx^\alpha dx^\beta\right],\ e^\Phi = |t|^\alpha\ .$

Approximate Kasner-like solution $\eta_i(t) \simeq t^{p_i}$ with

$$\sum_i p_i = 1\ ,\ \sum_i p_i^2 = 1 - \frac{\alpha^2}{2}\ .$$

If all $p_i > 0$, cosmology "stable". Else, spatial curvatures force BKL bounces between distinct Kasner regimes. With each bounce, α increases — dilaton-driven attractor-like behaviour.

Attractor basin: generic Kasner-like solution with all $p_i > 0$.

Universal behaviour near singularities

Consider symmetric Kasner-like AdS BKL-cosmologies. Near singularity, spatial curvatures unimportant. Leading singular behaviour is essentially dilaton-driven, symmetric Kasner spacetime. Holographic stress tensor has similar leading behaviour ($T_{\mu\nu} \sim \frac{N^2}{t^4}$).

Consider families of such AdS cosmologies which are of the form of the symmetric Kasner-like solution i.e. $p_i = \frac{1}{3}$: (ds_3^2 spatial metric)

$$ds^2 = \frac{1}{z^2}\left[dz^2 + |2t|(-dt^2 + ds_3^2)\right] , \qquad e^\Phi = |t|^{\sqrt{3}} .$$

Ignoring subleading curvature effects, spatial metric approximately flat i.e. $ds_3^2 \sim flat$. Then boundary metric is conformally flat, to leading order. [we've used a different time coordinate here.]

Can use PBH transformations to recast boundary metric to be flat spacetime.

The gauge theory

Now $g_{YM}^2 = e^{\Phi} = (-t)^{\sqrt{3}}$ ($t < 0$). That is, $p = \sqrt{3} > 1$.

From earlier: wave-fn phase "wildly" oscillating, ill-defined.

Energy production divergent if coupling vanishes strictly near $t = 0$.

* In gauge theory, deform gauge coupling so that $g_{YM}^2 = e^{\Phi}$ is small but nonzero near $t = 0$. Now finite but large phase oscillation and energy production. $\dot{\Phi} \sim \frac{\dot{g}_{YM}}{g_{YM}}$ finite so bulk also nonsingular. Sugra may still not be valid of course.

Eventual gauge theory endpoint ? Depends on details of energy production at coupling $\mathcal{O}(1)$. On long timescales, expect that gauge theory thermalizes: then reasonable to imagine that late-time bulk is AdS-Schwarzschild black hole.

See also arXiv:0906.3275, Awad, Das, Ghosh, Oh, Trivedi: slowly varying dilaton cosmologies and their gauge theory duals.

Null time-dependence

Null cosmologies: $\Phi = \Phi(x^+)$. No nonzero contraction so the mass term vanishes $i.e.$ $m^2(\Phi) = 0$.
Gauge theory (lightcone gauge for convenience): suppressing many details, but briefly, cubic/quartic interaction terms multiplied by powers of $g_{YM} = e^{\Phi/2}$, unimportant near $e^{\Phi} \to 0$.

Thus we obtain weakly coupled Yang-Mills theory at the location in null time ($x^+ = 0$) of the bulk singularity [e.g. $e^{\Phi} = g_s(-x^+)^p$].
This suggests that while classical bulk sugra variables are bad, lightcone Hamiltonian time evolution of the gauge theory is sensible.
Moreover: x^--translations are symmetries, so no particle production.
Suggests continuing past singularity at $x^+ = 0$ is OK, and late-time state is vacuum: $i.e.$ late-time bulk is $AdS_5 \times S^5$ (dual to $\mathcal{N}=4$ gauge theory vacuum, $\Phi \to const$ for large x^+).

Null singularities and strings

Bulk: since $e^{\Phi} \to 0$ near singularity, no large g_s effects. Rudimentary calculations suggest stringy effects (beyond GR) are important.
AdS string technically difficult. Possible to construct simpler toy models with no fluxes or dilaton, where the singularity is $purely$ $gravitational$ so more tractable by string worldsheet methods.

Consider $ds^2 = e^{f(x^+)}\left(-2dx^+dx^- + dx^i dx^i\right) + e^{h_m(x^+)}dx^m dx^m$, with $i = 1, 2,\ m = 3, \ldots, D-2$. Simple classes of null Kasner-like cosmological singularities at $x^+ = 0$ for (with two scale factors)

$$ds^2 = (x^+)^a\left(-2dx^+dx^- + dx^i dx^i\right) + (x^+)^b dx^m dx^m, \quad a > 0.$$

($a < 0$ solutions can be cast in this form by coord transf.)

No nonzero covariant contraction $\Rightarrow$ no local stringy corrections.

Ricci-flat solutions of Einstein equations if $R_{++} = 0$:

$$\frac{1}{2}(f')^2 - f'' + \frac{D-4}{4}(-2h'' - (h')^2 + 2f'h') = 0.$$

Null Kasner-like singularities

This gives $a^2 + 2a + \frac{D-4}{2}(-b^2 + 2b + 2ab) = 0$.

For $b \neq a$, we have $2a = -2 - (D-4)b \pm \sqrt{4 + (D-4)(D-2)b^2}$.

$a > 0 \Rightarrow$ positive radical.

Requiring unambiguous analytic continuation from $x^+ < 0$ to $x^+ > 0$ across singularity $\Rightarrow a, b$ are even integers.

More restrictive but such solutions do exist:

$(a, b) = (0, 2), (44, -2), (44, 92), (2068, -92) \ldots$, for $D = 26$ (bosonic string).

$(a, b) = (0, 2), (12, -2), (12, 28), (180, -28), (180, 390), \ldots$, for $D = 10$ (superstring).

No curvature invariants diverge in these null backgrounds.

Diverging tidal forces: from deviation of null geodesic congruences, the accelerations are $a^i, a^m \sim \frac{1}{(x^+)^{2a+2}}$.

String worldsheet theory

Closed string action $S = -\int \frac{d\tau d\sigma}{4\pi\alpha'} \sqrt{-h} h^{ab}\, \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X)$.

Lightcone gauge fixing gives

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \, ((\partial_\tau X^i)^2 - \tau^{2a} (\partial_\sigma X^i)^2 + \tau^{b-a} (\partial_\tau X^m)^2 - \tau^{b+a} (\partial_\sigma X^m)^2),$$

containing only physical transverse string degrees of freedom $X^I \equiv X^i, X^m$. Effectively solved constraints etc.

Quadratic worldsheet theory, external time-dependent coefficients.

[Lightcone gauge $x^+ = \tau$. Set $h_{\tau\sigma} = 0$, with $E(\tau,\sigma) = \sqrt{-\frac{h_{\sigma\sigma}}{h_{\tau\tau}}}$:

$S = \int \frac{d^2\sigma}{4\pi\alpha'} \, (E g_{IJ} \partial_\tau X^I \partial_\tau X^J - \frac{1}{E}\, g_{IJ} \partial_\sigma X^I \partial_\sigma X^J + 2E g_{+-} \partial_\tau X^-)$.

Setting lightcone momentum $p_- = \frac{E g_{+-}}{2\pi\alpha'} = -\frac{1}{2\pi\alpha'}$ to const by τ-independent σ-reparametrization invariance, we obtain $E = -\frac{1}{g_{+-}}$.]

Lightcone string wavefunctional

Lightcone Hamiltonian $H=-p_+$, satisfying physical state condition $m^2=-2g^{+-}p_+p_- - g^{II}(p_{I0})^2$:

$$H=\frac{1}{4\pi\alpha'}\int_0^{2\pi|p_-|\alpha'} d\sigma\Big((2\pi\alpha')^2(\Pi^i)^2+\tau^{2a}(\partial_\sigma X^i)^2$$
$$+(2\pi\alpha')^2\tau^{a-b}(\Pi^m)^2+\tau^{a+b}(\partial_\sigma X^m)^2\Big).$$

Understand string propagation across singularity by studying behaviour of lightcone string wavefunctional: the Schrodinger equation is

$$i\partial_{x^+}\Psi \equiv\ i\partial_\tau\Psi[X^I,\tau]=H[X^I,\tau]\Psi[X^I,\tau]\,,\quad \Pi^I[\sigma]=-i\frac{\delta}{\delta X^I[\sigma]}.$$

Coord modes decouple: $H=\sum_I H_I[X^I]\ \Rightarrow\ \ \Psi[X^I]=\prod_I \Psi_I[X^I]$.

This then simplifies to give $i\partial_\tau\Psi_I[X^I,\tau]=H_I[X^I]\Psi^I[X^I,\tau]$.

X^i modes: free (flat space) Schrodinger equation

$$i\partial_\tau\Psi_i[X^i,\tau]=\int\frac{d\sigma}{4\pi\alpha'}\left[(2\pi\alpha')^2(\Pi^i)^2+\tau^{2a}(\partial_\sigma X^i)^2\right]\Psi^i[X^i,\tau]$$
$$\rightarrow^{\tau\rightarrow 0}\ -\pi\alpha'\int d\sigma\frac{\delta^2}{\delta X^i[\sigma]^2}\Psi^i[X^i,\tau].$$

Lightcone string wavefunctional

X^m modes: $$i\partial_\tau \Psi_m[X^m,\tau] = \int \frac{d\sigma}{4\pi\alpha'}\Big[-(2\pi\alpha')^2\tau^{a-b}\frac{\delta^2}{\delta X^m[\sigma]^2} + \tau^{a+b}(\partial_\sigma X^m)^2\Big]\Psi^m[X^m,\tau].$$

$b>0$: kinetic term dominates giving
$$i\partial_\lambda \Psi_m[X^m,\lambda] = -\pi\alpha' \int d\sigma \frac{\delta^2}{\delta X^m[\sigma]^2}\Psi^m[X^m,\lambda],$$

Flat space Schrodinger eqn in time variable $\lambda = \int d\tau\ \tau^{a-b} = \frac{\tau^{2\nu}}{2\nu}$.

Alternatively $\Psi[X^m,\tau] \sim\ e^{i\pi\alpha' \frac{\tau^{2\nu}}{2\nu}\int d\sigma \frac{\delta^2}{\delta X^m[\sigma]^2}}\ \Psi[X^m]$,
well-defined for $2\nu = a+1-b \geq 0$.

$b<0$: potential term dominates, $i\partial_\lambda \Psi_m = \frac{1}{4\pi\alpha'}\int d\sigma(\partial_\sigma X^m)^2\Psi^m$.
Flat space Schrodinger equation in time variable $\lambda = \frac{\tau^{a-|b|+1}}{a+b+1} = \frac{\tau^{2\nu}}{2\nu}$.

Or $\Psi[X^m,\tau] \sim e^{-i\frac{\tau^{2\nu}}{8\pi\nu\alpha'}\int d\sigma(\partial_\sigma X^m)^2}\ \Psi[X^m]$ well-defined if $2\nu \geq 0$.

Wavefunctional nonsingular near $x^+ = 0$ for spacetimes with $2\nu \geq 0$.

Wavefunctional in other variables

$ds^2 = -2d\lambda dx^- + \lambda^{A_I}(dx^I)^2, \;\; A_I = \frac{a_I}{a+1}, \;\; \lambda = \frac{(x^+)^{a+1}}{a+1}$ (affine).

Hamiltonian $H = \int \frac{d\sigma}{4\pi\alpha'}((2\pi\alpha')^2\frac{(\Pi^I)^2}{\tau^{A_I}} + \tau^{A_I}(\partial_\sigma X^I)^2)$.

Wavefunctional $\;\Psi[X^I,\tau] \sim \; e^{-i\pi\alpha'\frac{\tau^{1-A_I}}{1-A_I}\int d\sigma\frac{\delta^2}{\delta X^{I\,2}}}\;\Psi[X^I]$, as $\tau \longrightarrow 0$

(for $e.g.\ A_i > 0$), with well-defined phase if $A_I < 1$.

Brinkman coordinates: $\;x^I = (x^+)^{-a_I/2}y^I, \;\; y^- = x^- + (\frac{\sum_I a_I(y^I)^2}{4(x^+)^{a+1}})$,

$ds^2 = -2d\lambda dy^- + \sum_I \chi_I(y^I)^2\frac{d\lambda^2}{\lambda^2} + (dy^I)^2, \quad \chi_I = \frac{A_I}{4}(A_I-2)\,.$

Anisotropic plane waves with singularities.

Hamiltonian $H = \int \frac{d\sigma}{4\pi\alpha'}\;((2\pi\alpha')^2(\Pi^I_y)^2 + (\partial_\sigma y^I)^2 - \sum_I\frac{\chi_I}{\tau^2}(y^I)^2)$,

Wavefunctional: $\;\;\Psi[y^I,\tau] \sim \; e^{-\frac{i}{\tau}\sum_I\chi_I(y^I)^2}\;\Psi[y^I]\,.$

Divergent mass term gives “wildly” oscillating phase as $\tau \longrightarrow 0$.

Wavefunctional difficult to interpret in Brinkman variables. Earlier Rosen coords better.

String mode functions

Classical string modes can be exactly solved for from worldsheet EOM:

$$f_n^I(\tau) = \sqrt{n\tau^{d_I}}\Big(c_{n1}^I J_{\frac{d_I}{2a+2}}(\tfrac{n\tau^{a+1}}{a+1}) + c_{n2}^I Y_{\frac{d_I}{2a+2}}(\tfrac{n\tau^{a+1}}{a+1})\Big),$$

with $d_I = 1, 2\nu$, for $I = i, m$, resp., $\nu = \frac{a+1-b}{2}$, and consts c_{n1}^I, c_{n2}^I.

Using basis modes $f_n^I(\tau)e^{in\sigma}$, mode expand worldsheet fields $X^I(\tau,\sigma)$. This gives the Hamiltonian $\left(k_n^I = \frac{i}{n}\sqrt{\frac{\pi\alpha'}{2|c_{n0}^I|(a+1)}}\right)$

$$\begin{aligned} H = {} & \tfrac{1}{2\alpha'}\left((\dot{X}_0^i)^2 + \tau^{b-a}(\dot{X}_0^m)^2\right) \\ & + \textstyle\sum_n \frac{|k_n^i|^2}{2\alpha'}\big((\{a_n^i, a_{-n}^i\} + \{\tilde{a}_n^i, \tilde{a}_{-n}^i\})(|\dot{f}_n^i|^2 + n^2\tau^{2a}|f_n^i|^2) \\ & - \{a_n^i, \tilde{a}_n^i\}((\dot{f}_n^i)^2 + n^2\tau^{2a}(f_n^i)^2) + c.c.\big) \\ & + \textstyle\sum_n \frac{|k_n^m|^2}{2\alpha'}\big((\{a_n^m, a_{-n}^m\} + \{\tilde{a}_n^m, \tilde{a}_{-n}^m\})(\tau^{b-a}|\dot{f}_n^m|^2 + n^2\tau^{b+a}|f_n^m|^2) \\ & - \{a_n^m, \tilde{a}_n^m\}(\tau^{b-a}(\dot{f}_n^m)^2 + n^2\tau^{b+a}(f_n^m)^2) + c.c.\big), \end{aligned}$$

Oscillator algebra: $[a_n^I, a_{-m}^J] = [\tilde{a}_n^I, \tilde{a}_{-m}^J] = n\delta^{IJ}\delta_{nm}$.

Mode asymptotics

Cutoff null surface $x^+ \equiv \tau = \tau_c$:

Low-lying (small n): $f_n^I \to \lambda_{n0}^I + \lambda_{n\tau}^I \tau_c^{d_I}$, $\frac{n\tau_c^{a+1}}{(a+1)} \lesssim 1$.

Highly stringy (large n): $\frac{n\tau_c^{a+1}}{(a+1)} \gg 1$,

$$f_n^i \sim \frac{1}{\tau_c^{a/2}}\, e^{-in\tau_c^{a+1}/l(a+1)}, \qquad f_n^m \sim \frac{1}{\tau_c^{b/2}}\, e^{-in\tau_c^{a+1}/(a+1)}.$$

These ultra-high frequency modes exist for any infinitesimal regularization of near-singularity region.

Then Hamiltonian for low-lying, highly stringy modes:

$$H_< = \pi\alpha'((p_{i0})^2 + \tau^{a-b}(p_{m0})^2) + \sum_n \frac{\pi}{2(a+1)n^2}\Big(\frac{1}{|c_{n0}^i|}(b_{n\tau}^{i\dagger} b_{n\tau}^i + n^2\tau^{2a} b_{n0}^{i\dagger} b_{n0}^i) + \frac{1}{|c_{n0}^m|}((2\nu)^2\tau^{a-b} b_{n\tau}^{m\dagger} b_{n\tau}^m + n^2\tau^{b+a} b_{n0}^{m\dagger} b_{n0}^m)\Big)$$

[with $b_{n0}^I = \lambda_{n0}^I a_n^I - \lambda_{n0}^{I*}\tilde{a}_{-n}^I,\ b_{n\tau}^I = \lambda_{n\tau}^I a_n^I - \lambda_{n\tau}^{I*}\tilde{a}_{-n}^I$],

$$H_> \sim \tau^a \sum_{I;\, n\gg n_c} \frac{1}{a+1}\,(a_{-n}^I a_n^I + \tilde{a}_{-n}^I \tilde{a}_n^I + n),$$

Oscillators and wavefunctional

The Schrodinger equation becomes

$$i\frac{\partial}{\partial\tau}|\Psi_i^<\rangle = H_i^<|\Psi_i^<\rangle$$
$$\sim \Big(\pi\alpha'(p_{i0})^2 + \sum_{n\lesssim n_c} \frac{\pi}{2(a+1)|c^i_{n0}|n^2} b^{i\dagger}_{n\tau} b^i_{n\tau}\Big)|\Psi_i^<\rangle\ ,$$

$$i\frac{\partial}{\partial\tau}|\Psi_m^<\rangle = H_m^<|\Psi_m^<\rangle \qquad [b>0]$$
$$\sim \tau^{a-b}\Big(\pi\alpha'(p_{m0})^2 + \sum_{n\lesssim n_c} \frac{(2\nu)^2\pi}{2(a+1)|c^m_{n0}|n^2} b^{m\dagger}_{n\tau} b^m_{n\tau}\Big)|\Psi_m^<\rangle, \quad ,$$
$$\sim \tau^{a+b}\Big(\sum_{n\lesssim n_c} \frac{\pi}{2(a+1)|c^m_{n0}|} b^{m\dagger}_{n0} b^m_{n0}\Big)\,|\Psi_m^<\rangle, \quad [b<0]\,,$$

$$i\frac{\partial}{\partial\tau}|\Psi_I^>\rangle = H_I^>|\Psi_I^>\rangle$$
$$\sim \tau^a\Big(\sum_{I;\ n\gg n_c} \frac{1}{a+1}\,(a^I_{-n}a^I_n + \tilde{a}^I_{-n}\tilde{a}^I_n + n)\Big)|\Psi_I^>\rangle.$$

Recovers earlier general Schrodinger wavefunctional analysis.

Length scales

No-scale property of these spacetimes manifest in Brinkman coords.
Then Rosen coords should have nontrivial length dimensions to maintain no-scale property: $dim\ \lambda \equiv L,\ \ dim\ x^+ \equiv L^{1/(a+1)}$

$$\Rightarrow\ \ dim\ x^i \equiv L^{1-a/(2(a+1))},\ \ dim\ x^m \equiv L^{1-b/(2(a+1))}.$$

Lightcone gauge: $dim\ \tau^{a+1} = dim\ \sigma = L$.
String coord length l: $\int d\sigma \equiv \int_0^{2\pi l} d\sigma$.
Lightcone momentum $p_- = -\frac{l}{2\pi\alpha'} < 0\ \Rightarrow\ l = 2\pi|p_-|\alpha'$.
$dim\ H = dim\ \frac{1}{\tau} = L^{-1/(a+1)}$.
A mode is highly stringy if $n \gg \frac{l}{\tau_c^{a+1}} \sim \frac{p_-\alpha'}{\tau_c^{a+1}}$.
Highly stringy state (instantaneous) masses: $m^2 \sim \frac{1}{\alpha'}(a_n^{i\dagger}a_n^i + \ldots)$
(Can also calculate low-lying spectrum)
Thus highly stringy oscillator states satisfying $\frac{p_-\alpha'}{\tau_c^{a+1}} \ll n \ll \frac{\alpha'}{\tau_c^{2a+2}}$ are light relative to typical energy scales ($a^i \sim \frac{1}{(x^+)^{2a+2}}$) near singularity.

Regulating the singularity

Some natural regulators bad, violate energy conditions: $e.g.$ 4D scale factor $\quad e^f = L^a((\frac{x^+}{L})^2 + \epsilon^2)^{a/2}$

$$\Rightarrow \quad R^{(4)}_{++} = \frac{1}{2}(f')^2 - f'' \quad \longrightarrow^{x^+\to 0} \quad -\frac{a}{(L\epsilon)^2} < 0.$$

In terms of D-dim system, no natural solution to $R^{(D)}_{++} = 0$ whose 4D scale factor e^f is as above.

This is a universal near singularity $x^+ \to 0$ limit of many regulators $e.g.$ $e^f = L^a[1 - (1-\epsilon)e^{-(\frac{x^+}{L})^2}]^{a/2}$. Basic problem of regulators.

Consider $\quad e^f = L^a(\frac{|x^+|}{L} + \epsilon)^a, \quad e^h = L^b(\frac{|x^+|}{L} + \epsilon)^b.$

Now accelerations $\quad a^i,\ a^m \sim \frac{1}{L^{2a+2}(\frac{|x^+|}{L} + \epsilon)^{2a+2}}$.

Curvature scale: $L_c = (L\epsilon)^{a+1}$.

Although apparently non-analytic, the geodesics, affine parameter, curvature continuous.

Strings and regulated singularities

Primarily interested in approach to $x^+ = 0$ from early times

Worldsheet theory can again be exactly solved for mode functions

$$f_n^I(\tau) = \sqrt{\frac{nL^{d_I}}{l^{d_I/(a+1)}}\left(\frac{\tau}{L}+\epsilon\right)^{d_I}}\left[c_{n1}^I J_{\frac{d_I}{2a+2}}\left(\frac{nL^{a+1}(\frac{\tau}{L}+\epsilon)^{a+1}}{l(a+1)}\right) + c_{n2}^I Y_{\frac{d_I}{2a+2}}\left(\frac{nL^{a+1}(\frac{\tau}{L}+\epsilon)^{a+1}}{l(a+1)}\right)\right].$$

Can solve for Hamiltonian, string spectrum, (instantaneous) masses etc.

Highly string oscillators light if $\frac{p_-\alpha'}{L_c} \ll n \ll \frac{\alpha'}{L_c^2}$.

Implicitly requires $p_- \ll \frac{1}{L_c}$.

Number of such levels: $\frac{\alpha'}{L_c^2}(1 - p_- L_c)$.

Strings and regulated singularities

For any finite $p_- \ll \frac{1}{L_c}$, only finite set of highly stringy oscillators excited in regulated near singularity region. In singular limit $L_c \to 0$, all oscillator states light, number of excited oscillator states diverges.

With $L_c \sim l_s$, no highly stringy oscillators turned on in regulated region ($i.e.\ n \sim 1$ already not light).

With $L_c \sim l_p$, highest level oscillator is $n \sim (\frac{l_s}{l_p})^2$.

Large proliferation of light string states in near singularity region.

As $L_c \to 0$, we recover original no-scale spacetime. Now number of light oscillator levels diverges, all oscillators light.

Conclusions, open questions

*Spacelike: If $g^2_{YM}(t) \to 0$ strictly, then gauge theory response singular: energy diverges. Deform g^2_{YM} to be small but nonzero near $t = 0$. Now finite but large phase oscillation and energy production.
$\dot{\Phi} \sim \frac{\dot{g}_{YM}}{g_{YM}}$ finite now, so bulk also nonsingular.
Sugra may still not be valid of course.
* Explore AdS BKL-cosmologies/duals further.

*Null: Free string wavefunctional nonsingular for spacetimes with $2\nu \geq 0$. However, light string state production could be large. Also backreaction of modes could be divergent.
String states being light in near-singularity region suggests that interactions are non-negligible: dual to renormalization effects (for corresponding AdS cosmologies)?
2nd quantized (string field theory) framework?

...

* Energy divergence

Analyzing KE terms shows they are indeed subleading near $t = 0$.

This means energy pumped in by time-dep source diverges as

$$\langle H \rangle \simeq e^{-\Phi} \langle V \rangle = \frac{1}{(-t)^p} \int D\tilde{X}\ V[\tilde{X}]\ |\psi_0[\tilde{X}(x)]|^2 \ ,$$

since no time-dep in $\langle V \rangle$. Oscillating phase cancels in $|\psi_0|^2$.
This holds for generic states. For special states with $\langle V \rangle = 0$, energy may be finite (subleading KE terms do not diverge unless $p > 2$).

Even for these special states, $\langle H^2 \rangle$ will diverge (if $\langle V \rangle = 0$, generically $\langle V^2 \rangle$ does not vanish).
Thus fluctuations non-negligible about states with $\langle V \rangle = 0$.

Note: this is not perturbation theory. Interactions important.
Diverging energy since coupling strictly vanishes near $t = 0$.

* Time-dep quantum mechanics

In more detail: first ignore interactions, quantize $quadratic$ theory.

For a single momentum-k mode, this is time-dep quantum mechanics:

$S_k = \int dt\, (\frac{1}{2}\dot{X}^2 - \omega^2(t)X^2)\,, \;\; \omega^2(t) = k^2 + m^2(t) \;\longrightarrow^{t\to-\infty}\; \omega_0^2\,.$

Generic classical solutions: $X = \sqrt{-t}\,[AJ_\nu(-t) + BN_\nu(-t)]\,,$

$\nu = \frac{p+1}{2}$. Diverge as $t \to 0$: i.e. generic trajectory driven to large X.

Take $f(t) = \sqrt{\frac{\pi\omega_0}{2}}\sqrt{-t}H_\nu^1(-\omega_0 t)$ as the solution of $\ddot{f} + \omega^2 f = 0$,

with $f \to e^{-i\omega_0 t},\;\; t \to -\infty$. Expand $X = \frac{1}{\sqrt{2\omega_0}}[af(t) + a^\dagger f^*(t)]$.

Using the Schrodinger equation: the ground state wave-function is

$$\psi(t,x) = \frac{A}{\sqrt{f^*(t)}}e^{i(\frac{\dot{f}^*}{f^*})\frac{x^2}{2}}\;.$$

* Time-dep quantum mechanics

X: wave-fn $\psi(t,x) = \frac{A}{\sqrt{f^*(t)}} e^{i(\frac{\dot{f}^*}{f^*})\frac{x^2}{2}}$,

Wave-fn phase $\sim \frac{1}{t}$,   "wildly" oscillating near $t = 0$.

$t \longrightarrow 0^-$: $f \sim (-t)^{-p/2}$.

Probability density: $|\psi(t,x)|^2 = \frac{|A|^2}{|f|} e^{-\frac{\omega_0 x^2}{|f|^2}}$.

Gaussian, width $|f|^2 \rightarrow \infty$ as $t \rightarrow 0$.

Wave packet infinitely spread out as $t \rightarrow 0$.

X variables spread out infinitely: need extra information at $X \sim \infty$.

X description not good.

* Time-dep quantum mechanics

Original $\tilde{X} = e^{\Phi/2} X$ variables better defined: finite near $t = 0$.

$\tilde{X} = e^{\Phi/2}\sqrt{-t}[AJ_\nu(-t) + BN_\nu(-t)] \sim^{t\to 0} t^{p/2} t^{1/2} t^{-\nu/2}$.

Wave-fn, probability:

$$\psi(t,\tilde{x}) = \frac{A}{\sqrt{f^*(t)e^{\Phi/2}}} e^{i(\frac{\dot{f}^*}{f^*} + \frac{\dot{\Phi}}{2})\frac{\tilde{x}^2}{2e^{\Phi}}} , \qquad |\psi(t,\tilde{x})|^2 = \frac{|A|^2}{|f|e^{\Phi/2}} e^{-\frac{\omega_0 \tilde{x}^2}{|f|^2 e^{\Phi}}} .$$

$t \to 0^-$: $f \sim (-t)^{-p}$, $\quad |f|^2 e^{\Phi} \sim const$.

$\tilde{X}$: wave-fn phase $\sim \frac{1}{(-t)^{p-1}}$, prob. width $const$.

$p > 1$: wave-fn ill-defined near $t \sim 0$. "Wildly" oscillating phase.
$p < 1$: $\tilde{X}$ wave fn phase regular near $t \sim 0$, $|\psi(t,\tilde{x})|^2$ finite.

Quadratic approximation shows interactions are important near $t = 0$.
Perturbation theory insufficient.

* More on AdS BKL-cosmologies

Bianchi IX: symmetry algebra of $X_a = e_a^\alpha \partial_\alpha$ is $SU(2)$.

Spatial Ricci, decomposing along triad $R^a_{(a)} = R^a_\alpha e^\alpha_a$:

$R^1_{(1)} = \frac{\partial_t(\eta_2\eta_3\partial_t\eta_1)}{\eta_1\eta_2\eta_3} - \frac{1}{2(\eta_1\eta_2\eta_3)^2}[(\eta_2^2 - \eta_3^2)^2 - \eta_1^4] = 0 \,, \quad \ldots .$

Say $p_1 < 0$: then $\eta_1^4 \sim t^{-4|p_1|}$ non-negligible at some time. This forces metric to transit from one Kasner regime to another. As long as some $p_i < 0$, these bounces continue as:

$p_i^{(n+1)} = \frac{-p_-^{(n)}}{1+2p_-^{(n)}} \,, \quad p_j^{(n+1)} = \frac{p_+^{(n)}+2p_-^{(n)}}{1+2p_-^{(n)}} \,, \quad \alpha_{(n+1)} = \frac{\alpha_n}{1+2p_-^{(n)}} \,,$

for the bounce from the (n)-th to the $(n+1)$-th Kasner regime.

If $p_- < 0$, then $\alpha_{n+1} > \alpha_n$. Also $\alpha_{n+1} - \alpha_n = \alpha_n(\frac{-2p_-}{1+2p_-})$,

i.e., α increases slowly for small α: attractor-like behaviour. Finite number of bounces. If all $p_i > 0$, no bounce: cosmology “stable”.

For no dilaton ($\alpha = 0$), BKL bounces purely oscillatory.

* More on AdS BKL-cosmologies

Parametrization: $p_1 = x,\ p_{2,3} = \frac{1-x}{2} \pm \frac{\sqrt{1-\alpha^2+2x-3x^2}}{2}$.

Lower bound: $p_1 \geq \frac{1-\sqrt{4-3\alpha^2}}{3}$. Solution existence forces $\alpha^2 \leq \frac{4}{3}$.

Under bounces, α increases, window of allowed p_i shrinks. Lower bound hits $p_1 \geq 0 \Rightarrow \alpha^2 \geq 1$. Bounces stop, cosmology "stabilizes".

Attractor-like behaviour: e.g.: $\{p_1^0 = x_0 = 0.3,\ \alpha_0 = 0.001\}$, flows (initially slowly) to $\{p_i > 0\}$ after 15 oscillations ($\alpha_{15} = 1.0896$).

E.g.: $(-\frac{1}{5}, \frac{9}{35}, \frac{33}{35}) \rightarrow (-\frac{5}{21}, \frac{7}{21}, \frac{19}{21}) \rightarrow (-\frac{3}{11}, \frac{5}{11}, \frac{9}{11}) \rightarrow (-\frac{1}{5}, \frac{3}{5}, \frac{3}{5}) \rightarrow (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. [multiple flows with same endpoint]

Chaotic behaviour: 7% change to smallest exponent $-\frac{1}{5}$ gives $(-\frac{13}{70}, \frac{9}{35}, \frac{65}{70}) \rightarrow (-\frac{2}{11}, \frac{13}{44}, \frac{39}{44}) \rightarrow (-\frac{3}{28}, \frac{2}{7}, \frac{23}{28}) \rightarrow (\frac{1}{11}, \frac{3}{22}, \frac{17}{22})$, drastically different endpoint.

Note also that dilatonic ($\alpha \neq 0$) [attractor-like] and non-dilatonic ($\alpha = 0$) [oscillatory] flows drastically different.