Cosmologies with Big-Bang Singularities and Their Gauge Theory Duals

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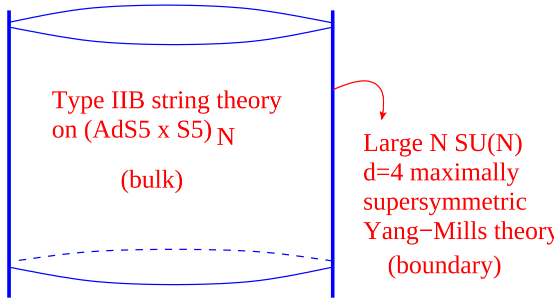
- AdS/CFT with Big-Bang cosmological singularities
- Holographic gauge theory duals
- Null: towards nonsingularity, bulk point of view . . .
- Spacelike Big-Bangs, FRW cosmologies etc.

Time dependence in string theory

- Usually broken spacetime supersymmetry $\Rightarrow$ time dependence. Metastable/unstable stringy vacua (tachyon dynamics etc).
- Time in string theory ? Beginning/end of time (Big Bang/Crunch) in string theory models ? Good approximations to our Universe ?
- General Relativity breaks down at singularities: so want "stringy" description. Smooth quantum (stringy) completion of classical spacetime geometry ? For example, various "stringy phases" arise in $e.g.$ 2D linear sigma model worldsheet descriptions, Matrix theory, etc.

Note: Big-bang singularities somewhat different from black holes: no horizon cloaking.

AdS/CFT, modes, deformations



Nice stringy playground: AdS/CFT. Bulk string theory on $AdS_5 \times S^5$ with dilaton (scalar) $\Phi = const$, and metric

$$ds^2 = \frac{1}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2) + ds^2_{S^5} \ ,$$

(Poincare coords) with 5-form field strength, dual to boundary $d = 4$ $\mathcal{N}{=}4$ (large N) $SU(N)$ Super Yang-Mills theory.

Deformations of AdS/CFT: either growing towards boundary (non-normalizable) or subleading at boundary (normalizable). These are dual to either sources for or expectation values of CFT operators.

Time-dependent deformations: cosmologies

Start with $AdS_5 \times S^5$ and turn on non-normalizable deformations for the metric and dilaton:

$$\begin{aligned} ds^2 &= \frac{1}{z^2}(\tilde{g}_{\mu\nu}dx^\mu dx^\nu + dz^2) + ds^2_{S^5} \ , \\ \Phi &= \Phi(x^\mu) \ , \qquad \text{also nontrivial } 5-\text{form} \ . \end{aligned}$$

This is a solution in string theory if

$$\tilde{R}_{\mu\nu} = \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi \ , \qquad \frac{1}{\sqrt{\tilde{g}}}\,\partial_\mu(\sqrt{-\tilde{g}}\;\tilde{g}^{\mu\nu}\partial_\nu\Phi) = 0 \ ,$$

$i.e.$ if it is a solution to a 4-dim Einstein-dilaton system.

The general solutions

Harmonic function $Z = Z(x^m)$ (and appropriate 5-form)

$$ds^2 = Z^{-1/2}\tilde{g}_{\mu\nu}dx^\mu dx^\nu + Z^{1/2}g_{mn}dx^m dx^n \ , \quad \Phi = \Phi(x^\mu).$$

$g_{mn}(x^m)$ is Ricci flat, and $\tilde{g}_{\mu\nu} = \tilde{g}_{\mu\nu}(x^\mu)$. $[\mu = 0123, m = 4\ldots 9.]$
Dilaton satisfies its EOM. And the Type IIB sugra EOM is

$$R_{MN} = \frac{1}{6}F_{MABCD}F_N{}^{ABCD} + \frac{1}{2}\partial_M\Phi\partial_N\Phi.$$

If $\tilde{g}_{\mu\nu}$ were flat, and dilaton constant, then such solutions are well-known ($e.g.$ coincident D3-branes at a conical singularity with base space $g^\perp_{mn}$). 5-form effectively acts as 5D cosmological constant. New contribution to R_{MN} is $\tilde{R}_{\mu\nu}$, so

$$\tilde{R}_{\mu\nu} = \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi.$$

Time-dependent/Null cosmologies

Spacelike: Consider $\tilde{g}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \sum_{i=1}^{3} t^{(2p_i)} dx^i dx^i$ and $e^\Phi = t^\alpha$. We obtain solutions generalizing Kasner-like cosmologies if $\sum_i p_i = 1$, $\frac{\alpha^2}{2} = 1 - \sum_i p_i^2$, from the R_{tt}, R_{ii}, Φ EOM. Restrictive. Can be generalized to other $\tilde{g}_{\mu\nu}$. More later. Contain $spacelike$ cosmological singularities.

Null: Consider $\tilde{g}_{\mu\nu}dx^\mu dx^\nu = e^{f(X^+)}(-2dX^+dX^- + dx^i dx^i)$, and $\Phi = \Phi(X^+)$, where $X^+ =$ lightlike coord. These are solutions if

$$\frac{1}{2}(\partial_+\phi)^2 = \tilde{R}_{++} = \frac{1}{2}(f')^2 - f'' \,. \qquad \left(f' = \frac{\partial f}{\partial X^+}\right)$$

Dilaton EOM $\partial_\mu(\sqrt{-\tilde{g}}\,\tilde{g}^{\mu\nu}\partial_\nu\Phi) = 0$ automatically satisfied since $\Phi = \Phi(X^+) \Rightarrow$ infinite family of solutions parametrized by $\Phi(X^+)$. Null singularities here. 8 lightcone supercharges preserved.

Prototypical example

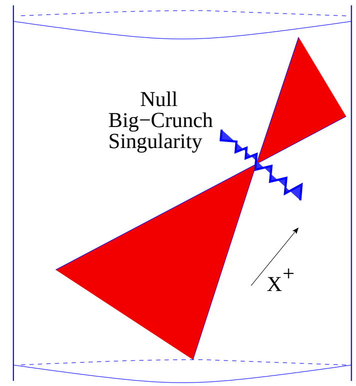
Consider $e^f = \tanh^2 X^+$

$$d\tilde{s}^2 = \tanh^2 X^+ (-2dX^+ dX^- + dx_2^2 + dx_3^2),$$
$$e^\Phi = g_s \left| \tanh \frac{X^+}{2} \right|^{\sqrt{8}}.$$

Far past/future: $AdS_5 \times S^5$ with dilaton constant. As $X^+ \to 0$, singularity (at finite affine time) as $e^f \to 0$, with $R_{++} = \frac{4}{\sinh^2 X^+}$.
EOM satisfied for $X^+ \neq 0$ and continuous at $X^+ = 0$.
With g_s small, dilaton can be made small everywhere.
Note: only solution with everywhere constant dilaton is $e^f = \frac{1}{(X^+)^2}$, which is flat space $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$ using

$$x_i = X^+ Y_i, \quad X^- = Y^- - X^+(Y_2^2 + Y_3^2), \quad X^+ = -\frac{1}{Y^+}.$$

Nature of null Big-Bang/Crunch singularity



These contain null Big-Bang (Crunch) cosmological singularities when the transverse space shrinks as $e^f \to 0$, at say $X^+ = 0$. Then curvature along infalling null geodesics ξ^μ diverges

$$\tilde{R}_{ab}\xi^a\xi^b = \tilde{R}_{++}e^{-2f} \to \infty.$$

These are infalling null geodesics at constant X^-, x^2, x^3. Affine parameter $\lambda = \text{const.} \int e^{f(X^+)} dX^+$ along geodesics. Singularity reached in finite affine time.

Nature of singularity cont'd

Other invariants R, $R_{AB}R^{AB}$ etc as in AdS_5.

Diverging compressional tidal forces along infalling null geodesic congruence: two nearby geodesics ξ^μ, displaced in say $x^i = x^2, x^3$, have relative acceleration

$$a^i = -R^i_{+i+}(\xi^+)^2 = -\frac{1}{2}\left(\frac{1}{2}(f')^2 - f''\right)e^{-2f}.$$

Physical distance $\Delta = \frac{e^{f/2}}{z}\sqrt{(x_i)^2}$ between two such geodesics reflects this tidal force

$$\frac{d^2\Delta}{d\lambda^2} = -\frac{1}{2}\left(\frac{1}{2}(f')^2 - f''\right)\Delta.$$

Consider $e^f = \tanh^2 X^+$ as limit of $e^f = (|\tanh X^+| + \epsilon)^2$. Then $e^\Phi \sim g_s(\epsilon)^{\sqrt{8}}$. Curvature, affine parameter: continuous, nonsingular.

The gauge theory duals

- Conjecture: Type IIB string theory on these backgrounds is dual to $\mathcal{N}$=4 $d = 4$ SYM on a base space $\tilde{g}_{\mu\nu}$ with a time dependent gauge coupling $g_{YM}^2 = e^{\Phi}$.
  Natural extension of AdS/CFT for small perturbations $\delta\Phi,\ \delta g_{\mu\nu}$:
  $$S = \int d^4x\ [\frac{\delta\Phi(x^\mu)}{g_{\rm YM}^2} TrF^2 + \delta g_{\mu\nu} T^{\mu\nu}],$$
  $i.e.$ the dual is $\mathcal{N}$=4 SYM theory with these sources turned on.

- Analyzing the D-probe DBI action corroborates this. Imagine building up this spacetime by stacking D3-branes in a background $ds^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu + dx^m dx_m$ and dilaton $\Phi(x^\mu)$. This gives $ds^2 = Z^{-1/2}(x) \tilde{g}_{\mu\nu} dx^\mu dx^\nu + Z^{1/2}(x) dx^m dx_m$, with dilaton and appropriate 5-form. Now take near horizon limit.

$Questions$: Vanishing absorption cross-sections (near singularity)? D-brane boundary states in time-dependent backgrounds ?

The gauge theory duals cont'd.

Reverse question: time-dependent deformations of $\mathcal{N}=4$ SYM ? Start in $\mathcal{N}=4$ vacuum in the far past; turn on time-dependent gauge coupling and a time-dependent initially flat base space. Gauge theory response ? Well-posed problem: $\Phi, \tilde{g}_{\mu\nu}$ specify gauge theory data completely.

Supergravity dual ? If the dilaton and metric are related by the (IIB) equations earlier, then a sugra dual is straightforward to identify. Nontrivial sources turned on (for operators $TrF^2, T_{\mu\nu}$) are dilaton Φ and metric $\tilde{g}_{\mu\nu} \Rightarrow$ then sugra dual is this deformation of $AdS_5 \times S^5$.

Nonsingular gauge theory dual ?

Is the gauge theory dual to these cosmologies with null Big-Bang singularities $nonsingular$?

Gauge theory dual lives on base space $\tilde{g}_{\mu\nu} = e^{f(x^+)}\eta_{\mu\nu}$ conformal to flat space, and has null-time-dependent gauge coupling $g_{YM}^2 = e^{\Phi(x^+)}$.

These null cosmologies special for two reasons:

- The trace anomaly $T_\mu{}^\mu$ for these theories vanishes $\Rightarrow$ after a Weyl rescaling of the metric, partition function and correlations of conformally dressed operators $e^{f(x^+)\Delta/2}\mathcal{O}(x)$ are well-behaved.
- The gauge coupling $g_{YM}^2 = e^{\Phi}$ vanishes near the singularity. And there exist new variables $\tilde{A}_\mu = e^{-\Phi(x^+)/2}A_\mu$ which are weakly coupled near the singularity (and dual to nonlocal bulk operators), and encode a nonsingular description of the system.

The trace anomaly

The trace anomaly for a field theory on a curved background is

$$\begin{aligned} T_\mu{}^\mu &\propto c(C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}) - a(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2) \\ &\propto -R_{\alpha\beta}R^{\alpha\beta} + \frac{1}{3}R^2. \end{aligned}$$

For $SU(N)$ $\mathcal{N}=4$ SYM, we have $c = a = \frac{N^2-1}{4}$.
In the time dependent cosmologies, both terms $\propto \frac{1}{t^4}$.

Null cases: only nonzero R_{ab} is R_{++} $\Rightarrow$ $T_\mu{}^\mu = 0$ for any $f(X^+)$.
Time-varying dilaton effects: any additional term in $T_\mu{}^\mu$ must be generally covariant involving dilaton derivatives, and tensors made out of the metric. These vanish since only $\partial_+\Phi$ nonzero, and no tensor with two (or more) upper $+$ components.

CFT in conformally flat bgnd

Consider the partition function (ignore time-dependent coupling)

$$Z[g_{\mu\nu}] = \int [D\varphi]_{[g_{\mu\nu}]} e^{iS[g_{\mu\nu},\varphi]},$$

S being the action over fields φ. Under metric variations $\delta g_{\mu\nu}$, $\delta Z = \int [D\varphi]_{[g_{\mu\nu}]} e^{iS[g_{\mu\nu},\varphi]} \left(i \int d^4x \sqrt{-g}\, \delta g_{\mu\nu} T^{\mu\nu} \right)$, $T^{\mu\nu}$ being the stress tensor. Thus under Weyl rescalings $g_{\mu\nu} \to e^{\delta\psi} g_{\mu\nu}$, $\delta \log Z = i \langle \int d^4x \sqrt{-g}\, T_\mu{}^\mu \delta\psi \rangle$.

Consider 1-parameter family of metrics $g_{\mu\nu} = e^{\alpha f(X^+)} \eta_{\mu\nu},\ \alpha \in [0,1]$. Then $T_\mu{}^\mu = 0 \ \Rightarrow\ \partial_\alpha Z[g_{\mu\nu}] = 0 \ \Rightarrow\ Z[e^f \eta_{\mu\nu}] = Z[\eta_{\mu\nu}]$. Similarly for correlation functions

$$\langle \prod_i e^{\frac{\alpha f(x_i)\Delta_i}{2}} \mathcal{O}(x_i) \rangle_{[e^f \eta_{\mu\nu}]} = \langle \prod_i \mathcal{O}(x_i) \rangle_{[\eta_{\mu\nu}]}.$$

Varying dilaton effects

For our prototypical example, far past/future state is the $\mathcal{N}=4$ SYM conformal vacuum. In general, time-varying interactions lead to particle production: null backgrounds ?

Consider conformal scalar with null-dependent interaction:

$$S = \int d^4x\sqrt{-g}\ \left[\frac{1}{2}(\partial\varphi)^2 + \frac{1}{6}R\varphi^2 + J(X^+)\varphi^3\right].$$

Lightcone quantization: modes $e^{-i(k_ix^i+k_-X^-+\frac{k_i^2}{2k_-}X^+)}$ are positive frequency ($k_+ = \frac{k_i^2}{2k_-} > 0$) w.r.t. $X^+ \ \Rightarrow\ k_- \geq 0$.

Mode expanding in the free theory gives

$$\varphi = e^{-\frac{f}{2}} \int d^2k \int_0^\infty \frac{dk_-}{\sqrt{(2\pi)^3 2k_-}} \left[a_k e^{-i(k_ix^i+k_-X^-+\frac{k_i^2}{2k_-}X^+)} + \text{c.c.}\right]$$

Conformal vacuum: $\quad a(k_i, k_-)|0\rangle = 0\ .$

Varying dilaton effects cont'd

Vacuum of the non-interacting theory remains unchanged with a null-dependent source $\Rightarrow$ no particle production.

Essentially, final interaction picture state (T_+: null-time ordering)
$|s\rangle = T_+ e^{-i\int d^4x\ e^{2f(X^+)} J(X^+)\ \varphi^3}|0\rangle$ remains unchanged.

$$\varphi = e^{-\frac{f}{2}} \int d^2k \int_0^\infty \frac{dk_-}{\sqrt{(2\pi)^3 2k_-}} \left[a_k e^{-i(k_i x^i + k_- X^- + \frac{k_i^2}{2k_-} X^+)} + \text{c.c.}\right]$$

so *e.g.* at first order, $\delta_1|s\rangle = -i\int d^4x e^{2f(X^+)} J(X^+)\varphi^3|0\rangle$.

With φ^3 normal ordered, the only term contributing is the $(a^\dagger)^3$ term.
But each $a^\dagger$ term comes with $e^{ik_- X^-}$, $k_- > 0$. Then since $J(X^+)$ is X^--independent, $\int dX^-$ gives a δ-function $\Rightarrow$ each k_- must vanish.
For the $k_- \neq 0$ sector, this constraint cannot be met.
Can generalize to higher orders.

Varying dilaton effects cont'd

Physically: analogous to space-varying source where time-translation invariance implies no particle production due to energy conservation. Here X^--translation invariance means P_- conserved since source $J(X^+)$ does not break this symmetry.

Thus the vacuum of the non-interacting theory remains unchanged with a null-dependent source $\Rightarrow$ no particle production.
$Caveat$: $k_- = 0$ subtleties.

Varying dilaton: correlators

In perturbation theory in source $J(X^+)$, correlators with interaction $\int d^4X e^{\frac{f}{2}} J(X^+) e^{\frac{3f}{2}} \varphi^3(X)$ can be related to free correlators, and thus to flat space $dressed$ correlators, $e.g.$ 2-pt fn with interaction:
$G_F(x_1, x_2) = \langle 0| T_+ e^{\frac{f(x_1)}{2}} \varphi(x_1) e^{\frac{f(x_2)}{2}} \varphi(x_2) e^{-i \int d^4x \sqrt{g} J(X^+) \varphi^3(X)} |0\rangle.$

To leading order

$$-i \langle 0| T_+ e^{\frac{f(x_1^+)}{2}} \varphi(x_1) e^{\frac{f(x_2^+)}{2}} \varphi(x_2) \int d^4X \; e^{2f(X^+)} J(X^+) \varphi(X)^3 |0\rangle =$$
$$-i \int d^4X J(X^+) e^{\frac{f(X^+)}{2}} \langle T_+ e^{\frac{f(x_1^+)}{2}} \varphi(x_1) e^{\frac{f(x_2^+)}{2}} \varphi(x_2) e^{\frac{3f(X^+)}{2}} \varphi^3(X) \rangle.$$

Operator $\mathcal{O}(x)$ of conformal dimension Δ dressed as $e^{\frac{f\Delta}{2}} \mathcal{O}(x)$. For source $J(X^+)$ coupling to $\mathcal{O}(x)$, interaction damped if $J(X^+) e^{\frac{4-\Delta}{2} f(X^+)} \to 0$ as $X^+ \to 0$.

Field redefinitions: toy scalar

With varying dilaton, kinetic terms of gauge fields are nontrivial. Want redefinition to new variables with canonical kinetic terms.

$Toy\ model$: scalar field $\int d^4x\, e^{-\Phi(X^+)}(\partial\varphi)^2$. Redefine $\varphi = \epsilon(x)\tilde{\varphi}$:
$\int d^4x\, e^{-\Phi}\, \eta^{\mu\nu} \left(\epsilon^2\partial_\mu\tilde{\varphi}\partial_\nu\tilde{\varphi} + \epsilon\partial_\mu\epsilon\partial_\nu(\tilde{\varphi}^2) + (\partial_\mu\epsilon\partial_\nu\epsilon)\tilde{\varphi}^2\right)$.
Now if $\epsilon(x) = e^{\Phi(X^+)/2}$, first term canonical kinetic term. And ϵ is null $\Rightarrow$ (i) the third term vanishes, (ii) the second term is a total derivative $\partial_+(\epsilon^2)\partial_-(\tilde{\varphi}^2) = \partial_-[\partial_+(\epsilon^2)\tilde{\varphi}^2]$, which can be dropped.

Consider $interactions$:
$-\int d^4x\, e^{-\Phi(X^+)}[(\partial\varphi)^2 - \lambda\varphi^4] \;\rightarrow\; -\int d^4x[(\partial\tilde{\varphi})^2 - \lambda e^{\Phi(X^+)}\tilde{\varphi}^4]$.

Thus $\tilde{\varphi}$-variables have canonical kinetic terms. As $X^+ \rightarrow 0$, if $e^\Phi \rightarrow 0$, then $\tilde{\varphi}$-interaction damped $\Rightarrow$ nonsingular S-matrix. Theory well-defined, transparent in $\tilde{\varphi}$-variables used for defining asymptotic states.

$\mathcal{N}=4$ SYM: the tilde variables

$\mathcal{N}=4$ SYM: near singularity, $e^{\Phi} \rightarrow 0$, so kinetic terms singular: want well-defined variables $\int e^{-\Phi}\mathrm{tr}F^2 + \ldots \rightarrow \int \mathrm{tr}\tilde{F}^2 + \ldots$

For simplicity, work in lightcone gauge $A_- = 0$ (and flat metric).

Define $\tilde{A}_\mu = e^{-\Phi/2} A_\mu$. Then $S_{\mathrm{GF}} = -\frac{1}{4}\int d^4x\, e^{-\Phi}\ \mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}] \ \rightarrow$

$$-\int \frac{d^4x}{4}\,[\ \mathrm{Tr}(\partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu)^2 - 2ie^{\Phi/2}\mathrm{Tr}\{(\partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu)[\tilde{A}^\mu, \tilde{A}^\nu]\}$$

$$-e^{\Phi}\mathrm{Tr}([\tilde{A}_\mu, \tilde{A}_\nu])^2 - \partial_-\{(\partial_+\Phi)\tilde{A}_i\tilde{A}^i\}\ ]$$

The last term is a total derivative and does not affect the EOM.

Other interaction terms containing the dilaton:

$\int d^4x[e^{\frac{\Phi}{2}}J^{\mu a}\tilde{A}_{\mu a} + e^{\Phi}\mathrm{Tr}([\tilde{A}_\mu, \phi^\alpha][\tilde{A}^\mu, \phi^\alpha]) + e^{\Phi}\mathrm{Tr}([\phi^\alpha, \phi^\beta][\phi^\alpha, \phi^\beta])]$,

where $J^{\mu a}$ is the gauge current from scalars ϕ^α and fermions.

Note: Dilaton couples with positive powers.

$\mathcal{N}=4$ SYM tilde variables cont'd.

We have imposed $A_- = 0$ gauge: A_+ nondynamical (also therefore $\tilde{A}_+$). The A_- EOM gives a constraint $\partial_-(\partial \cdot A) = 0$, $i.e.$ $k_-(-k_- A_+ + k_i A^i) = 0$. Thus if $k_- \neq 0$, then $-k_- A_+ + k_i A^i = 0$. Now solve for A_+ in terms of A_i , $i.e.$ $A_+ = \frac{1}{k_-}(k_i A_i)$.

Note: $\partial_-(\partial \cdot A) = 0$ means $\partial \cdot A = F(X^+, x^i)$. Residual X^--independent gauge transformations $A'_\mu = A_\mu + \partial_\mu \lambda,\ \mu \neq X^-$, can be used to fix $\partial \cdot A = 0$, for $k_- \neq 0$.
Thus for $k_- \neq 0$ modes, we can fix gauge completely $\Rightarrow A_+, A_i$ are gauge-invariant.
In a general gauge: $\tilde{A}_\mu = e^{-\Phi/2}(A_\mu + \partial_\mu \chi)$, where $\chi = -\partial_-^{-1} A_-$ is uniquely defined if $k_- \neq 0$.

$Caveat$! $k_- = 0$ subtleties of lightcone gauge.

Tilde variables cont'd.

Curved metric $\tilde{g}_{\mu\nu} = e^f \eta_{\mu\nu}$: Dilaton couples to dimension $\Delta = 4$ operators, so no dressing factors since dressed source is $Je^{\frac{4-\Delta}{2}f}$.

Interaction terms are of the form $e^{k\Phi(X^+)}\mathcal{O}(x)$. From earlier arguments, $e^{\Phi(X^+)}$ is a lightlike source $\Rightarrow$ no particle production. Dilaton couples with positive powers $\Rightarrow$ all interactions die out when the dilaton vanishes. Thus as $X^+ \to 0$, the $\tilde{A}$ theory is becoming free.

Note: in the regulated theory, $e^f = (|\tanh X^+| + \epsilon)^2$, there is no singular behaviour at all. Only possible singular behaviour for $\epsilon \to 0$ arises near $X^+ \to 0$. But here, all interactions die.

Thus the gauge theory is nonsingular.

Other observables, bulk 2pt fn

* Operator relation: $\mathcal{O} \equiv e^{-\Phi}\,\mathrm{Tr}\,F^2 = \mathrm{Tr}\,\tilde{F}^2 - \frac{1}{2}\partial_-[(\partial_+\Phi)\tilde{A}_i\tilde{A}^i]$.
Then $\langle\mathcal{O}(x)\mathcal{O}(y)\rangle = \langle\mathrm{Tr}\,\tilde{F}^2(x)\,\mathrm{Tr}\,\tilde{F}^2(y)\rangle + divergent$,
since $\Phi \sim \sqrt{8}\log X^+ \;\Rightarrow\; \partial_+\Phi \sim \frac{1}{X^+}$ near $X^+ = 0$, so that
$\langle\mathcal{O}(x)\mathcal{O}(y)\rangle$ diverges. However, in our example, $\langle\mathrm{Tr}\,F^2(x)\,\mathrm{Tr}\,F^2(y)\rangle$
etc vanish as $X^+ \to 0$ since $e^\Phi \to 0$ faster than $\partial_+\Phi$ diverges.

* For operator $\mathcal{O}$ dual to bulk scalar ($e.g.$ dilaton), bulk 2-pt function is
$\langle e^{\frac{f(x)\Delta}{2}}O(x)e^{\frac{f(x')\Delta}{2}}O(x')\rangle = e^{\frac{f(x)(\Delta-1)}{2}}e^{\frac{f(x')(\Delta-1)}{2}}(\frac{\Delta\lambda}{\Delta X^+})^{1-\Delta}\,\frac{1}{[(\Delta\vec{x})^2]^\Delta}$.
When $x \sim x'$, then $\frac{\Delta\lambda}{\Delta X^+} \sim \frac{d\lambda}{dX^+} = e^f$ giving $\frac{1}{[(\Delta\vec{x})^2]^\Delta}$.

However in singular backgrounds, for both $X^+, X'^+ \to 0$, this
depends on how the limit is taken: in our example, $\lambda \sim (X^+)^3$, giving
$(\frac{X^+}{(X')^+})^{\Delta-1}[(\frac{X^+}{(X')^+})^2 + \frac{X^+}{(X')^+} + 1]^{1-\Delta}\frac{1}{[(\Delta\vec{x})^2]^\Delta}$.

More on dual variables

This disagreement between bulk and gauge theory expectations is not surprising: bulk calculation fails near the singularity. Also these bulk modes, $e.g.$ dilaton, couple to operators made out of A_μ variables.

Good gauge theory variables are the $\tilde{A}_\mu$: bulk duals ? Hard to identify clearly. Operators such as $\tilde{F}_{ij} = e^{-\Phi/2} F_{ij}$ are local.
However operators $e.g.$ $\tilde{A}_\mu$, or $\tilde{F}_{+\nu} = e^{-\Phi/2} F_{+\nu} - \frac{1}{2} e^{-\Phi/2} (A_\nu \partial_+ \Phi)$, are not local in terms of $F_{\mu\nu}$, since A_μ cannot be expressed locally in terms of $F_{\mu\nu}$. A complete set of gauge invariant operators must include these. Thus the $\tilde{A}_\mu$ are possibly nonlocal, $i.e.$ their duals (good bulk variables) are stringy (recall $e.g.$ Wilson loop).
Also note: in usual AdS/CFT, $\alpha' \sim \frac{1}{g_{YM}^2 N} = \frac{1}{e^\Phi N}$.
This further suggests that α' (stringy) effects become important near the singularity (crude bulk worldsheet analysis corroborates this).

Revisiting the bulk: null case

We have seen so far various features of the gauge theory suggesting that it is nonsingular, and in fact lives on a flat spacetime with a time-dependent coupling. Can this be seen directly from the bulk ?

Consider the coordinate transformation

$$w = ze^{-f/2} , \qquad y^- = x^- - \frac{w^2 f'}{4} .$$

This transforms the bulk to the form

$$ds^2 = \frac{1}{w^2}[-2dx^+dy^- + dx_i^2 + \frac{1}{4}(\Phi')^2 w^2 (dx^+)^2] + \frac{dw^2}{w^2} ,$$

using $\frac{1}{2}(f')^2 - f'' = \frac{1}{2}(\Phi')^2$, the constraint on these solutions.

Now boundary at $w = 0$ manifestly flat 4D Minkowski spacetime. This gives further evidence that the gauge theory is in fact sensible.

Revisiting bulk, cont'd

Note that the conformal factor e^f does not appear at all. This recasting of the bulk metric in terms of an arbitrary dilaton function Φ is useful. For example, in the earlier formulation, the constraint forced the dilaton to be non-analytic ($e.g.\ e^\Phi = |\tanh(\frac{x^+}{2})|^{\sqrt{8}}$) if the conformal factor were analytic ($e.g.\ e^f = (\tanh(x^+))^2$) – this is generic.

Since the dilaton is the gauge coupling $g_{YM}^2 = e^\Phi$, one might worry about possible ambiguities in time evolution via analytic continuation past $x^+ = 0$ in the gauge theory.

With the above new coordinates, one can choose the dilaton to be analytic $e.g.\ e^\Phi = (\tanh(x^+))^2$. Then the bulk has a singularity at $x^+ = 0$ but the gauge theory from our earlier arguments appears well-behaved.

PBH transformations, boundary metrics

These coordinate transformations are in fact a class of Penrose-Brown-Henneaux transformations: subset of bulk diffeomorphisms leaving metric invariant (in Fefferman-Graham form) and acting as a Weyl transformation on boundary.

Star wth metric in FG form:

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(x,\rho) dx^i dx^j \ .$$

General infinitesimal form of the PBH transformations:

$$\rho = \rho e^{-\sigma(x',\rho')}, \qquad x^i = x^{i'} + a^i(x',\rho') \ .$$

Demanding that the $dx'd\rho'$ cross term vanishes gives

$$\partial_\rho a^i = \frac{1}{4} g^{ij} \partial_j \sigma \ .$$

This gives $\qquad \delta g_{ij} = \sigma(1 - \rho\partial_\rho) g_{ij} + \nabla_{(i} a_{j)} \ .$

Above, we have found $finite$ PBH transformations for the null case.

Cosmologies with spacelike Big-Bang singularities

We have described cosmologies with null Big-Bang singularities so far.

We will now describe solutions with spacelike Big-Bang singularities.

More restrictive. However in addition to Kasner cosmologies earlier

$$ds^2 = \frac{1}{z^2} \left[dz^2 - dt^2 + \sum_{i=1}^{3} t^{(2p_i)} dx^i dx^i \right] ,$$

$$e^\Phi = |t|^{\sqrt{2(1-\sum_i p_i^2)}}, \qquad \sum_i p_i = 1 ,$$

we find solutions with boundaries being FRW cosmologies with spacelike Big-Bang singularities, having metric and dilaton

$$ds^2 = \frac{1}{z^2} \left[dz^2 + e^{f(t)} \left(-dt^2 + \frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + sin^2\theta d\phi^2) \right) \right] ,$$

$$e^{f(t)} = c_1 \sin(2\sqrt{k}\, t) + c_2 \cos(2\sqrt{k}\, t),$$

$$e^\Phi = e^{\sqrt{3} \int dt\ e^{-f(t)}} .$$

$k = 0, \pm 1$ corresponds to flat, spherical or hyperbolic FRW universe.

Spacelike Big-Bangs cont'd

In particular, the $k = -1$ solution with $e^{f(t)} = \sinh(2t)$ is

$$ds^2 = \sinh(2t)\left(-dt^2 + \frac{dr^2}{1+r^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)\right),$$

$$e^\Phi = |\tanh t|^{\sqrt{3}}.$$

Thus the dilaton is in fact bounded, approaching constant at early/late times: asymptotic spacetime is $AdS_5 \times S^5$. This gives hope that perhaps these gauge theories admit some interesting description.

Spacelike Big-Bangs cont'd

The $k = 0$ case is in fact the symmetric Kasner solution

$$ds^2 = \frac{1}{z^2}\left[dz^2 + \frac{2t}{3}(-dt^2 + dx^i dx^i)\right] ,$$
$$e^{\Phi} = |t|^{\sqrt{3}} .$$

It is possible to find PBH transformations that transform this flat FRW case to a spacetime with flat Minkowski boundary, along similar lines as the null cosmologies:

$$z = \frac{32wT^{\frac{5}{2}}}{\sqrt{6}}\frac{1}{16T^2-w^2}, \qquad t = T\ \left(\frac{16T^2+5w^2}{16T^2-w^2}\right)^{\frac{2}{3}} ,$$

$$ds^2 = \frac{1}{w^2}\left[dw^2 - \frac{(16T^2-5w^2)^2}{256T^4}dT^2 + \frac{(16T^2-w^2)^{\frac{4}{3}}(16T^2+5w^2)^{\frac{2}{3}}}{256T^4}dx^i dx^i\right]$$

Spacelike Big-Bangs cont'd

There are further coordinate transformations that make the boundary conformally flat for the other FRW solutions too. This suggests that there should again be PBH transformations for these cases.
We have not found exact PBH transformations here, but can find them in an expansion about the boundary.

Using these and going to the new coordinates with a flat boundary metric, we can calculate the stress tensors for these spacetimes using holographic RG techniques. These generically diverge near the singularity for the spacelike Big-Bangs.
However: for the $k = 0$ flat FRW, the stress tensor vanishes in the earlier coordinates (with a conformally flat boundary).

Open questions

- Work in progress: more detailed understanding of $\mathcal{N}=4$ SYM with this time-dependent coupling, loop amplitudes, etc.
  Spacelike singularities ?
- What is the bulk resolution of these null singularities ?
  Stringy physics near these singularities ?

  D-brane dynamics in time-dependent backgrounds ?
- More severe (spacelike) cosmological singularities ? Realistic cosmologies ?