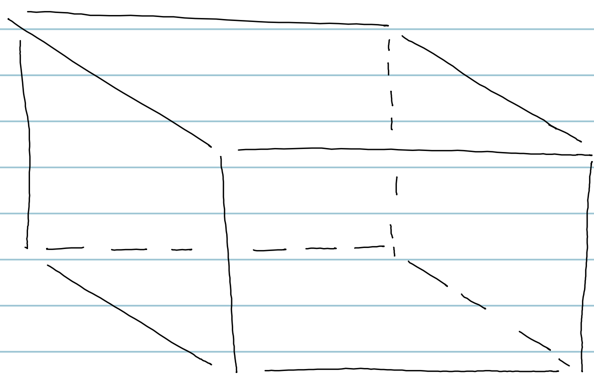
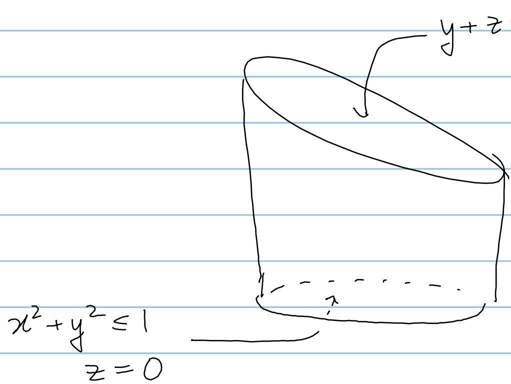
Testing. I am testing

So this is the first page.

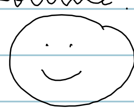
Should inaugurate (sp?) with some pictures. Here is one:



That isn't so bad. How about this:



Find the volume.



Manoj Kummini
Syzygies.

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Lecture 3

§5.1 from the book: K field

V projective variety. $X = \mathbb{A}^N_K$

Think of $X \times V$ = total sp. of the trivial rk N v.b. $\mathcal{E}_0$ on V.

$\mathcal{S} \subseteq \mathcal{E}_0$ subbundle.

$Z \subseteq X \times V$ total space of $\mathcal{S}$

$$\begin{tikzcd}
Z \arrow[r, hook] \arrow[rr, bend left, "p"] \arrow[dr] \arrow[d, bend right, "q"'] & X \times V \arrow[r, "p"] \arrow[d, "q=\text{proj}"] & V \arrow[d] \\
Y = q(Z) \arrow[r, phantom, "\subseteq"] & X \arrow[r] & \operatorname{Spec} k
\end{tikzcd}$$

Have short exact seq:

$$0 \to \mathcal{S} \to \mathcal{E} \to \mathcal{T} \to 0$$

$s = \operatorname{rk} \mathcal{S}$, $t = \operatorname{rk} \mathcal{T}$, $A = K[X]$
= poly ring in N variables$/K$.

Set $\xi = \mathcal{T}^*$

Proposition:

(a) A locally free resolution of $\mathcal{O}_Z$ as an $\mathcal{O}_{X \times V}$-module is given by

$$K_\bullet(\xi):\ 0 \to \Lambda^t(p^*\xi) \to \cdots \to \Lambda^2(p^*\xi) \to (p^*\xi) \to \mathcal{O}_{X \times V} \to$$

The differentials are homogeneous of degree 1 in the coordinate functions of X.

(b) $p_* \mathcal{O}_Z$ can be identified with the sheaf of alg $\operatorname{Sym}(\mathcal{S}^*)$.

Proof:

(b) is obvious.

(a) Identify X with an N-dim k.v.s. E.
By the universal property of Grassmanians $\exists!$ map

$$f : V \longrightarrow \mathrm{Grass}(s, E)$$

s.t. $\quad \mathcal{S} = f^* \mathcal{R}. \qquad \mathcal{R}$ tant sub-bdle

On $E \times \mathrm{Grass}(s, E)$, $\mathcal{O}_{\mathcal{R}}$ is resolved as follows:

$$K_\circ(\sigma) \qquad \to \Lambda^2 p^* Q^* \to p^* Q^* \longrightarrow \mathcal{O}_{E \times \mathrm{Grass}(s, E)} \to 0$$

where σ is the section of $p^* Q$ given by the picture

$$\begin{tikzcd} p^* Q \arrow[r] \arrow[d] & Q \arrow[d] \\ E \times \mathrm{Grass} \arrow[r, "p"'] \arrow[u, bend left, "\sigma"] \arrow[ur] & \mathrm{Grass}(s, E) \end{tikzcd}$$

Define $\quad K(\xi) = f^* K(\sigma)$
$K_\circ(\xi)$ resolves $\mathcal{O}_Z$ as an $\mathcal{O}_{X \times V}$-module. $\blacksquare$

Notation: $\quad K_\circ(\xi, \mathcal{V}) = K_\circ(\xi) \otimes p^* \mathcal{V}$ for $\mathcal{V}$ a locally free sheaf on V.

Main Thm : (5.1.2) Let $F(\mathcal{V})_i := \bigoplus_{j\geq 0} H^j(V, \Lambda^{i+j}\xi \otimes \mathcal{V}) \otimes_k A(-i-j)$

(a) $\exists$ minimal diffs of degree 0 $d_i : F(\mathcal{V})_i \longrightarrow F(\mathcal{V})_{i-1}$ s.t.

$F(\mathcal{V})_\bullet$ is a cplx of f.g free graded A-modules with homology

$$H_{-i}(F(\mathcal{V})_\bullet) = R^i q_* M(\mathcal{V}) \quad \forall i \in \mathbb{Z}.$$

where $M(\mathcal{V}) = M := \mathcal{O}_Z \otimes p^* \mathcal{V}$. Note

$\mathcal{K}_\bullet(\xi, \mathcal{V})$ is a locally free resolution of $M(\mathcal{V})$

(b) $R^i q_* M(\mathcal{V}) = H^i(Z, M(\mathcal{V}))^\sim$ and can be identified with $H^i(V, \mathrm{Sym}(\delta^*) \otimes \mathcal{V})$.

Remark: $F(\mathcal{V})_\bullet$ looks like

$$F(\mathcal{V})_2 \xrightarrow{d_2} F(\mathcal{V})_1 \xrightarrow{d_1} F(\mathcal{V})_0 \xrightarrow{d_0} F(\mathcal{V})_{-1} \to \cdots$$

d_i is minimal if $\mathrm{Im}\, d_i \subseteq m_A \cdot F(\mathcal{V})_{i-1}$,

where m_A is the homogenous max'l ideal of $A = $ poly $/K$.

$F(\mathcal{V})_\bullet$ is exact in positive degrees.

Suppose that $R^i q_* M(\mathcal{V}) = 0 \quad \forall i > 0$. By minimality of the maps $d_0, d_{-1}, \ldots$ and using NAK, we see $F(\mathcal{V})_{-i} = 0 \quad \forall i > 0$.

Explanation: BWOC, suppose that $n > 0$ is max s.t $F(\mathcal{V})_{-n} \neq 0$. Then cplx looks like

$$F(\mathcal{V})_{-n+1} \xrightarrow{d_{-n+1}} F(\mathcal{V})_{-n} \to 0$$

$$F(\mathcal{V})_{-n} = \operatorname{Im} d_{-n+1}$$

Since $R^d q_* M(\mathcal{V}) = 0$.

By minimality. $\underset{F(\mathcal{V})_{-n}}{\overset{}{\operatorname{Im}}} \underset{''}{(d_{-n+1})} \subseteq M_A F(\mathcal{V})_{-n}$.

These are graded modules: By NAK, $F(\mathcal{V})_{-n} = 0$.

$\Rightarrow\Leftarrow$

If $\mathcal{V} = \mathcal{O}_V$, then we will write F_0 for $F(\mathcal{V})_0$

Theorem: Suppose that $q: Z \to Y$ is birational.

(a) $q_* \mathcal{O}_Z$ is the normalization of $k[Y]$ (coord ring of Y)

(b) $R^i q_* \mathcal{O}_Z = 0 \quad \forall i > 0$

$\Rightarrow$ F_0 is a finite free resolution of the normalization of $k[Y]$.

(c) $R^i q_* \mathcal{O}_Z = 0 \quad \forall i > 0$ and $\operatorname{rk} F_0 = 1$
Then Y is normal and has rational singularities

(d) Conversely if Y is normal and has rational singularities, then F_0 is a graded minimal free resoln of $k[Y]$.

Defn: 1) Z, Y varieties k, Z smooth, $f : Z \to Y$ proper, birational is called a desingularization or a resolution of singularities.

2) Say that f is a rational resolution if the following holds

(a) Y is normal, i.e. $f_* \mathcal{O}_Z \simeq \mathcal{O}_Y$

(b) $R^i f_* \mathcal{O}_Z = 0 \quad \forall i > 0$

(c) $R^i f_* \omega_Z = 0 \quad \forall i > 0$.

In characteristic zero (c) is not needed by Grauert-Riemenschneider.

We say Y has rational singularities if Y admits a rational resolution (eg every nodal singularity is rational).

Rmk: (b), (c), (d) of the second thm follow from (a) and the Main Theorem.

Under the hyp of (b), F_0 looks like (from Rmk)

$$F_2 \to F_1 \to F_0 \to 0$$

By Main Thm homology at 0^{th} spot is $q_* \mathcal{O}_Z$.
Part (a) implies above $=$ normalization $k[Y]$.