

22 Sep 2011

Review :

We showed that any orientable 3-mfld M^3

$$M^3 = H_1 \cup_f H_2$$

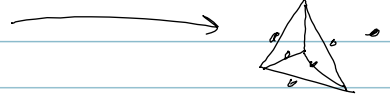
where H_i is a handle body.

Proof

Take a triangulation

Take 1-skeleton

K_1, \dots



~~$\mathbb{R}^2 \subset \mathbb{C}$~~ Take 1st barycentric subdivision

$K_2 =$ all ^{1-dim} simplices not meeting K_1 .

Take attached the



Take 2nd barycentric subdivision & look at $N(K_1)$ and $N(K_2)$ which are simplicial nbhd of K_1 .

Equivalently $K_2 \sim$ all 1-simplices satisfying the following property:

* $\left\{ \begin{array}{l} \text{such that } \sigma \text{ is a simplex in } \tilde{\tau} \text{ then} \\ \sigma \cap K_1 = \emptyset \end{array} \right.$ ^{$\tilde{\tau}$ - 2nd barycentric subdiv}

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Koszul complexes
((Aprodan - Nagel) Serre local alg)

Let k be a field, E f.d.v.s / k $r_E = r$.

Let $\delta \in V^* = \text{Hom}_k(V, k)$ $E^* = \text{Hom}_k(E, k)$

Using this can define a contraction
map

$$i_\delta: \Lambda^k E \longrightarrow \Lambda^{k-1} E, \quad (\text{of degree } -1)$$

defined inductively

$$i_\delta: \Lambda^k E \longrightarrow \Lambda^{k-1} E$$

$$e_0 \wedge e_1 \wedge \dots \wedge e_{k-1} \longmapsto \delta(e_0) (e_1 \wedge \dots \wedge e_{k-1}) \\ - e_0 \wedge i_\delta (e_1 \wedge \dots \wedge e_{k-1})$$

Fact: $i_\delta \circ i_\delta = 0$.

Hence get complex

$$K_*(\delta) \quad 0 \rightarrow \Lambda^r E \rightarrow \Lambda^{r-1} E \rightarrow \dots \rightarrow \Lambda^1 E \rightarrow E \otimes_k k \\ \rightarrow 0$$

Fact: $K_*(\delta)$ is exact.

Except for the statement about exactness,
everything else works even if we had taken
 k to be Noether. local ring, and E to be
a free k -module of finite rank r . Let e_1, \dots, e_r

be a basis of E and write $s = \sum_i f_i e_i^*$,
where $f_i \in k$.

Fact: If f_1 is a NZD on k and f_i is a NZD
on $k/(f_1, \dots, f_{i-1}) \quad \forall 2 \leq i \leq r$, then $K_0(s)$ is
acyclic. Conversely if $\langle f_1, \dots, f_r \rangle \neq k$ (k local) then
the converse holds, i.e. $K_0(s)$ is acyclic.

[Aside: A complex C_0 is acyclic if $C_i = 0 \quad i \geq 1$,
and $H_i(C_0) = 0$ for $i \neq 0$].

Rmk: Say that (f_1, \dots, f_r) has depth (sometimes
grade) or in k if f_1 is a NZD on k
and f_i is a NZD on $k/(f_1, \dots, f_{i-1}) \quad i=2, \dots, r$.

Since k is Noeth. local, (f_1, \dots, f_r) has depth r
implies $\text{codim}(f_1, \dots, f_r) = r$.

If k is a regular local ring (or even
Cohen-Macaulay) then every k ideal I will
satisfy $\text{codim } I = \text{depth } I$.

Now let X be a smooth variety, and E a
rank r vector bundle on X . Suppose there is
a non-zero section s of E^* . We can repeat
the construction from above to get a complex
(again called the Koszul complex)

$$K_0(s). \quad 0 \rightarrow \Lambda^r E \rightarrow \Lambda^{r-1} E \rightarrow \dots \rightarrow \Lambda^2 E \rightarrow E \rightarrow \mathcal{O}_X \rightarrow 0.$$

Denote the zero locus of s by $Z(s)$.

In physical (set theoretic) terms:

$$Z(s) = \{x \in X \mid s(x) = 0 \text{ in the fibre } (E^*)_x\}.$$

$K_0(Z)$ is acyclic if and only if for every $x \in Z(s)$, $Z(s)$ is defined (locally) in $\mathcal{O}_{x,x}$ ~~from~~ by a regular sequence, i.e. $Z(s)$ ~~is~~ has codim r in $\mathcal{O}_{x,x}$ (it is already defined by ~~the~~ r eqns).

Back to Grassmannians:

Recall the tautological sequence

$$0 \rightarrow \mathcal{R} \rightarrow E \times \text{Grass}(r, E) \rightarrow \mathcal{Q} \rightarrow 0 \quad \text{tautological sequence}$$

Here $rk E = n$, and $rk \mathcal{R} = r$.

Proposition: $\mathcal{O}_{\mathcal{R}}$ is resolved as an $\mathcal{O}_{E \times \text{Grass}(r, E)}$

$$\parallel$$

$$\mathcal{O}_{E \times G}$$

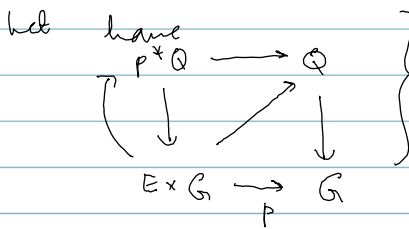
[where (for today) $G = \text{Grass}(r, E)$]

by the following Koszul complex

$$K_0(\mathcal{Q}^*): \Lambda^{n-r}(p^* \mathcal{Q}^*) \rightarrow \dots \rightarrow \Lambda^2(p^* \mathcal{Q}^*) \rightarrow p^* \mathcal{Q}^* \rightarrow \mathcal{O}_{E \times G} \rightarrow 0$$

$$p: E \times G \rightarrow G.$$

Proof: ~~Let~~ First here is Balaji's comment:



More precisely
 $E \times G \rightarrow Q$
 gives rise to a
 section $E \times G \rightarrow p^*Q$
 of $p^*Q \rightarrow E \times G$.

~~Here is the more local proof~~

This gives us the section of p^*Q on
 from which the Koszul $K_0(Q^*)$ is built.

Here is the way to get it via equations etc
 (a la Weyman).

Note ~~let~~ $\{U_I : I \neq \emptyset, I \subseteq \{1, \dots, n\}\}$ form an
 affine open cover of G .

The eqns of $R|_{U_I}$ in $E \times U_I$ are computed
 as follows:

Let $(x, R) \in E \times U_I$

Notation from last time

z_1, \dots, z_n is a basis of R

e_1, \dots, e_n " of E

$$Z = [z_{ij}]_{\substack{i=1, \dots, n \\ j=1, \dots, n}} \quad ; \quad z_i = \sum_{j=1, \dots, n} z_{ij} e_j$$

Symmetric in

$$I = \{i_1 < \dots < i_n\} \quad Z_I = \text{det of cols } z_{i_1}, \dots, z_{i_n} \text{ of } Z.$$

Z_I is invertible as an element.

$$(x, R) \in U_I \iff x \in \text{Span of rows of } (Z_I)^{-1} Z.$$

Write $x = \sum a_i e_i$

$$x = \sum_{u=1}^n a_{iu} w_u, \text{ where } w_u \text{ is the } u\text{th row of } Z.$$

$$(x = \sum a_i e_i).$$

The above implies

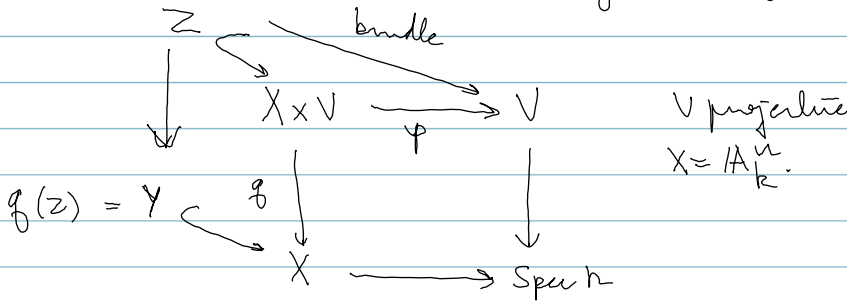
$$(*) \quad a_j = \sum_{t=1}^r a_{it} z'_{tj} \quad j \notin I, \text{ and the } z'_{ij} \text{ denote the entries of } (Z_I)^{-1} Z \text{ in col } i \text{ of } Z.$$

These define R on $E \times U_I$. Now we need to see that $(*)$ gives a regular sequence on $k[E \times U_I]$ is a polynomial ring of $\dim = n + r(n-r)$.

$$\dim R|_{U_I} = r + r(n-r).$$

Finally $(*)$ has r equations.

So it cuts out a regular sequence.



Do above for \mathcal{O}_Z & take $g_* k^0(s)$.

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