NOTES ON THE KEMPF-LASCOUX-WEYMAN GEOMETRIC TECHNIQUE

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These are edited notes from a seminar on the geometric technique of Kempf-Lascoux-Weyman for computing syzygies. These follow Weyman's book [Wey03] closely. K. V. Subramaniam lectured on Section 4.1 and Appendix B. V. Balaji lectured on the Kempf rational resolution of singularities of Schubert varieties, although it is not included in these notes; we only use the fact that determinantal varieties are normal and have rational singularities.

Comments are welcome!

Notation. Let $\Bbbk$ denote a field.

1. Motivation

Let's consider the set-up of the Lascoux resolution as an example of the technique, before plunging into the details. This is a graded free resolution of the determinantal variety, describing the locus of $m \times n$ matrices of rank at most r inside the space of all $m \times n$ matrices.

Let X the denote the space of all all $m \times n$ matrices (over $\Bbbk$). Take two $\Bbbk$-vector spaces F of rank m and G of rank n. Think of X as $\operatorname{Hom}_{\Bbbk}(F, G) \simeq F^* \otimes_{\Bbbk} G$. The coordinate ring of X is $A = \Bbbk[\{\phi_{ij} : 1 \leq i \leq m, 1 \leq j \leq n]$, where we can think of the ϕ_{ij} as follows: let $\{f_i : 1 \leq i \leq m\}$ and $\{g_j : 1 \leq j \leq n\}$ be $\Bbbk$-bases of F and G respectively; then $\phi_{ij} = f_i \otimes g_j^* \in F \otimes_{\Bbbk} G^*$ where * denotes the dual basis.

Let $\Phi = [\phi_{ij}]$. Let $0 \leq r < \min\{m, n\}$. Then the space Y_r of matrices of rank at most r is defined by $I_{r+1}(\Phi)$, the ideal generated by the $(r+1) \times (r+1)$ minors of Φ. Our aim is to find a (locally) free A-resolution of its coordinate ring $\Bbbk[Y_r]$. (We are looking for a graded resolution, so a locally free A-resolution is a free resolution.)

Let $V = \operatorname{Grass}(r, G)$, the Grassmannian of r-dimensional subspaces of G. Think of $X \times_{\Bbbk} V \xrightarrow{p} V$ (projection to the second factor) as the total space of the trivial vector bundle with fibre $F^* \otimes G$ over V. Write $q : X \times_{\Bbbk} V \longrightarrow X$ for the projection. Let $Z = \{(\phi, R) \in X \times V : \operatorname{Im} \phi \subseteq R\}$. Now the following diagram

$$\begin{tikzcd}
Z \arrow[d, "q"] & \subseteq & X \times_{\Bbbk} V \arrow[r, "p"] \arrow[d, "q"] & V \\
Y_r = q(Z) & \subseteq & X &
\end{tikzcd}$$

is commutative; it additionally shows that $q : Z \longrightarrow Y_r$ is a desingularization, since, for a generic map $\phi \in Y_r$, its preimage is $\{(\phi, \operatorname{Im} \phi)\}$.

There is a Koszul complex on $X \times V$ that resolves $\mathscr{O}_Z$ as a $\mathscr{O}_{X \times V}$-module. The q_* of this complex gives a graded free A-resolution of $\Bbbk[Y_r]$. This is the geometric technique of computing syzygies, which we will soon get to. We need some background on Grassmannians first.

2. Grassmannians

Let E be an n-dimensional $\Bbbk$-vector space. Fix a basis $e_1, \ldots, e_n$ of E. Fix $1 \leq r \leq n$ and write $G = \operatorname{Grass}(r, E)$.

2.1. **Standard covering.** Let $R \in G$. Let $z_1, \ldots, z_r$ be a basis of R. Write $z_i = \sum_{j=1}^n z_{ij} e_j, 1 \leq i \leq r$. Think of the z_{ij} as an $r \times n$ matrix Z.

Let $1 \leq i_1 < \ldots < i_r \leq n$. Define $p(i_1, \ldots, i_r)(R)$ to be the determinant of the $r \times r$ submatrix of $[z_{ij}]$ consisting of the columns $i_1 < \ldots < i_r$. If we change the basis $z_1, \ldots, z_r$, then $p(i_1, \ldots, i_r)(R)$ is multiplied by the determinant of an invertible $r \times r$ matrix, so we get a map

$$p : \mathrm{Grass}(r, G) \longrightarrow \mathbb{P}(\bigwedge^r E)$$

sending $R \mapsto (p(i_1, \ldots, i_r)(R))_{1 \leq i_1 < \ldots < i_r \leq n}$, called the *Plücker embedding* of $\mathrm{Grass}(r, G)$.

Let $I = \{i_1 < \ldots < i_r\} \subseteq \{1, \ldots, n\}$. Define

$$U_I := \{R \in \mathrm{Grass}(r, E) : p(i_1, \ldots, i_r)(R) \neq 0\}.$$

Then $\{U_I : |I| = r, I \subseteq \{1, \ldots, n\}\}$ is an open covering of G.

Remark 2.1. Each U_I is an $r(n-r)$-dimensional affine space. Write Z_I for the $r \times r$ submatrix consisting of the columns $i_1, \ldots, i_r$ of Z and all the rows. It is an invertible matrix. A canonical choice for a point in U_I is $(Z_I)^{-1} Z$. The remaining $n - r$ columns (with the r rows) give an $r(n-r)$-dimensional affine space. This shows that the Grassmannian is a rational smooth projective variety.

2.2. **Tautological bundles.** Let $\mathcal{R} = \{(x, R) \in E \times \mathrm{Grass}(r, E) : x \in R\}$ and $p : \mathcal{R} \longrightarrow G$ the projection to the second factor. Then p gives $\mathcal{R}$ the structure of a vector bundle over G, which is a sub-bundle of the trivial bundle $E \times G$ on G. We call $\mathcal{R}$ the *tautological sub-bundle* of the trivial bundle with fibre E over G. The quotient of the trivial bundle by $\mathcal{R}$ is called the *tautological quotient-bundle*. There is an exact sequence of vector bundles

$$0 \longrightarrow \mathcal{R} \longrightarrow E \times G \longrightarrow \mathcal{Q} \longrightarrow 0. \tag{2.2}$$

Proposition 2.3. *Let X be a $\Bbbk$-scheme, $\mathcal{E}$ a rank-r sub-bundle of the trivial bundle $E \times X$. Then there exists $f : X \longrightarrow \mathrm{Grass}(r, E)$ such that $f^* \mathcal{R} \simeq \mathcal{E}$.*

Sketch. Define $f : X \longrightarrow G$ by $x \mapsto \mathcal{E}_x$. ☐

(In general, for a morphism $f : X \longrightarrow Y$ and a bundle $\mathcal{F}$ on Y, $f^*\mathcal{F}$ is the product $X \times_Y \mathcal{F}$.) Let $p : E \times G \longrightarrow G$ be the projection to the second factor. There is a commutative diagram (of $\Bbbk$-varieties)

$$\begin{tikzcd} p^* \mathcal{Q} \arrow[r] & \mathcal{Q} \arrow[d] \\ E \times G \arrow[u, "\exists s"] \arrow[ur, two heads, "q"] \arrow[r, "p"'] & G \end{tikzcd}$$

(The map $q : E \times G \twoheadrightarrow \mathcal{Q}$ is the surjective morphism from (2.2). It lifts to a map $E \times G \longrightarrow (E \times G) \times_G \mathcal{Q} =: p^* \mathcal{Q}$ because of the universal property of products.) This gives a section s of $p^* \mathcal{Q}$ on $E \times G$, from which we get the Koszul complex (see Section A.2)

$$K_\bullet(s) : \qquad 0 \longrightarrow \bigwedge^{n-r}(p^* \mathcal{Q}^*) \longrightarrow \cdots \longrightarrow \bigwedge^2(p^* \mathcal{Q}^*) \longrightarrow (p^* \mathcal{Q}^*) \longrightarrow \mathscr{O}_{E \times G} \longrightarrow 0. \tag{2.4}$$

Proposition 2.5. *The Koszul complex in (2.4) is a locally free resolution of $\mathscr{O}_\mathcal{R}$ as an $\mathscr{O}_{E \times G}$-module. The maps are linear in the entries of coordinate functions of E.*

Sketch. The Koszul complex is acyclic if the zero locus $Z(s)$ of s has the expected codimension which is $\mathrm{rk}(p^* \mathcal{Q}) = n - r$. The vanishing of s defines $\mathscr{O}_\mathcal{R}$. To see that $\mathcal{R}$ is defined locally by $n - r$ equations, We work locally on the affine open subsets in the standard covering of G. Fix $I := \{1 \leq i_1 < \cdots < i_r \leq n\}$. Over U_I, a point $(x, R) \in E \times U_I$ is in $\mathcal{R}_{U_I}$ if and only if x belongs to the subspace generated by the rows of the matrix $(Z_I)^{-1} Z$, which gives $n - r$ conditions on the coordinate functions of E. These conditions are linear. See [Wey03, 3.3.3] for the details. ☐

3. The geometric technique

Let $\Bbbk$ be field of arbitrary characteristic. Let V be a projective variety over $\Bbbk$, and $X := \mathbb{A}^N_{\Bbbk}$. Write A for the coordinate ring $\Bbbk[X]$, which is an N-dimensional polynomial ring over $\Bbbk$. Denote its maximal ideal corresponding to the origin in X (i.e., the one generated by the variables) by $\mathfrak{m}_A$. Think of $X \times_{\Bbbk} V \xrightarrow{p} V$ (projection to the second factor) as the total space of the trivial vector bundle $\mathcal{E}$ over V. Write $q : X \times_{\Bbbk} V \longrightarrow X$ for the projection. Let Z be the total space of a subbundle $\mathcal{S}$ of $\mathcal{E}$. Now the following diagram

$$\begin{tikzcd} Z \arrow[d, "q"] & \subseteq & X \times_{\Bbbk} V \arrow[r, "p"] \arrow[d, "q"] & V \\ Y = q(Z) & \subseteq & X & \end{tikzcd} \tag{3.1}$$

We have an exact sequence of locally free sheaves

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{E} \longrightarrow \mathcal{T} \longrightarrow 0 \tag{3.2}$$

Let $s = \operatorname{rk} \mathcal{S}$ and $t = \operatorname{rk} \mathcal{T}$. Write $\xi := \mathcal{T}^*$.

Proposition 3.3. *With notation as above, we have the following:*

(a) A locally free resolution of $\mathscr{O}_Z$ as an $\mathscr{O}_{X\times V}$-module is given by a Koszul complex

$$K_\bullet(\xi) : \qquad 0 \longrightarrow \bigwedge^t (p^*\xi) \longrightarrow \bigwedge^{t-1}(p^*\xi) \longrightarrow \cdots \longrightarrow (p^*\xi) \longrightarrow \mathscr{O}_{X\times V} \longrightarrow 0.$$

(b) The direct image $p_* \mathscr{O}_Z$ can be identified with the sheaf of $\mathscr{O}_V$-algebras $\operatorname{Sym}(\mathcal{S}^*)$.

Proof. (a): Identify X with an N-dimensional $\Bbbk$-vector space E. By the universal property of Grassmannians, there exists $f : V \longrightarrow \operatorname{Grass}(s, E)$ such that $\mathcal{S} = f^*\mathcal{R}$, where $\mathcal{R}$ is the tautological subbundle (Proposition 2.3). In fact, applying f^* to the tautological sequence (2.2) yields the sequence (3.2). Define $K_\bullet(\xi) := f^*K_\bullet(\sigma)$, where $K_\bullet(\sigma)$ is the Koszul complex (2.4) resolving $\mathscr{O}_{\mathcal{R}}$ as an $\mathscr{O}_{E\times \operatorname{Grass}(s,E)}$ and σ is the section $E \times \operatorname{Grass}(s, E) \longrightarrow p^*Q$ in the diagram (3.1). It is acyclic and is a resolution of $p^* \mathscr{O}_{\mathcal{R}} = \mathscr{O}_Z$ since the zero-locus of the section $p^*\sigma$ has the expected codimension $t = \operatorname{rk} \mathcal{T}$.

(b). Follows from noting that $Z \simeq \mathbf{Spec}(\operatorname{Sym}(\mathcal{S}^*))$, so $\mathscr{O}_Z \simeq \operatorname{Sym}(\mathcal{S}^*)$; see [Har77, Exercise II.5.18]. ☐

Notation 3.4. Let $\mathcal{V}$ be a vector bundle on V. Set $K_\bullet(\xi, \mathcal{V}) := K_\bullet(\xi) \otimes_{\mathscr{O}_{X\times V}} p^*\mathcal{V}$. Since $p^*\mathcal{V}$ is a vector bundle, $K_\bullet(\xi, \mathcal{V})$ is a locally free resolution of the $\mathscr{O}_Z$-module $M(\mathcal{V}) := \mathscr{O}_Z \otimes p^*\mathcal{V}$. ☐

Theorem 3.5 (Main Theorem)**.** *With notation as above, we have the following:*

(a) There exists a minimal complex $F_\bullet(\mathcal{V})$ with terms

$$F_i(\mathcal{V}) = \bigoplus_{j\geq 0} \mathrm{H}^j(V, \bigwedge^{i+j} \xi \otimes \mathcal{V}) \otimes_{\Bbbk} A(-i-j)$$

and differentials $d_i : F_i(\mathcal{V}) \longrightarrow F_{i-1}(\mathcal{V})$ of degree 0. In particular, $F_\bullet(\mathcal{V})$ is a finite graded complex of free A-modules of finite rank.

(b) For all i, $\mathrm{H}_{-i}(F_\bullet(\mathcal{V})) \simeq \mathbf{R}^i q_* M(\mathcal{V})$. Thus, $F_\bullet(\mathcal{V})$ is exact in positive degrees.

(c) For $i \geq 0$, $\mathbf{R}^i q_* M(\mathcal{V})$ is the sheaf associated to $\mathrm{H}^i(Z, M(\mathcal{V}))$ (on X) and can be identified with $\mathrm{H}^i(V, \operatorname{Sym}(\mathcal{S}^*) \otimes \mathcal{V})$.

Remark 3.6. Suppose that $\mathbf{R}^i q_* M(\mathcal{V}) = 0$ for all $i \geq 0$. Then $F_{-i}(\mathcal{V}) = 0$ for all $i > 0$, i.e, $F_\bullet(\mathcal{V})$ looks like

$$F_\bullet(\mathcal{V}) : \qquad \cdots \longrightarrow F_2(\mathcal{V}) \longrightarrow F_1(\mathcal{V}) \longrightarrow F_0(\mathcal{V}) \longrightarrow 0.$$

To prove the above assertion, let, by way of contradiction, $n > 0$ be maximum such that $F_{-n}(\mathcal{V}) \neq 0$. Then we have a complex

$$F_{-n+1}(\mathcal{V}) \xrightarrow{d_{-n+1}} F_{-n}(\mathcal{V}) \longrightarrow 0$$

such that $\mathrm{H}_{-n} F_\bullet(\mathcal{V}) \simeq \operatorname{coker} d_{-n+1} = 0$. Hence $\operatorname{Im} d_{-n+1} = F_{-n}(\mathcal{V})$. By the minimality of d_{-n+1}, $\operatorname{Im} d_{-n+1} \subseteq \mathfrak{m}_A F_{-n}(\mathcal{V})$, so $F_{-n}(\mathcal{V}) = \mathfrak{m}_A F_{-n}(\mathcal{V})$. By the Nakayama lemma, $F_{-n}(\mathcal{V}) = 0$, contradicting the choice of n. Hence $F_{-i}(\mathcal{V}) = 0$ for all $i > 0$. As a consequence, we note that since $F_\bullet(\mathcal{V})$ is exact in positive degrees and $\mathrm{H}_0(F_\bullet(\mathcal{V})) \simeq q_* M(\mathcal{V})$, $F_\bullet(\mathcal{V})$, now, is a minimal graded A-free resolution of the finitely generated (graded) A-module $\Gamma(X, q_* M(\mathcal{V}))$. $\square$

Let W be a $\Bbbk$-variety. A *resolution of singularities* or *desingularization* of W is a proper birational morphism $f : W' \longrightarrow W$ where W' is smooth. Say that f is a *rational resolution* if the following conditions hold: (a) W is normal, i.e., $\mathscr{O}_W \longrightarrow f_* \mathscr{O}_{W'}$ is an isomorphism, (b) $\mathbf{R}^i f_* \mathscr{O}_{W'} = 0$ for all $i > 0$, and, (c) $\mathbf{R}^i f_* \omega_{W'} = 0$ for all $i > 0$ (where $\omega_{W'}$ is a canonical sheaf for W'). The last condition is redundant in characteristic zero. If $\operatorname{char} \Bbbk = 0$, say that W has *rational singularities* if W admits a rational resolution. If W has rational singularities, then every desingularization of W is rational. See [Wey03, Section 1.2].

Notation 3.7. If $\mathcal{V} = \mathscr{O}_Z$, we write $F_\bullet$ for $F_\bullet(\mathcal{V})$.

Theorem 3.8. *Suppose that $q : Z \longrightarrow Y$ is birational (hence a desingularization) and take $\mathcal{V} = \mathscr{O}_V$.*

(a) $q_* \mathscr{O}_Z$ is the normalization of $\Bbbk[Y]$.

(b) If $\mathbf{R}^i q_* \mathscr{O}_Z = 0$ for all $i > 0$, then $F_\bullet$ is a minimal finite A-free resolution of the normalization of $\Bbbk[Y]$.

(c) If $\operatorname{char} \Bbbk = 0$ and $\mathbf{R}^i q_* \mathscr{O}_Z = 0$ for all $i > 0$ and $\operatorname{rk} F_0 = 1$, then Y is normal and has rational singularities. In particular, $F_\bullet$ is a resolution of $\Bbbk[Y]$.

(d) Conversely, if Y is normal and has rational singularities, then $F_\bullet$ is a minimal graded A-free resolution of $\Bbbk[Y]$.

We note immediately that 3.8(b) follows from Remark 3.6 and 3.8(a). To see 3.8(c), we need to show that if $\operatorname{rk} F_0 = 1$, then $\Bbbk[Y]$ is normal. Write S for the normalization of $\Bbbk[Y]$. Then S is graded, and its $\Bbbk[Y]$-algebra generators are in positive degrees [ZS60, Chapter VII.2, Theorem 11]. Notice that $\operatorname{rk} F_0 = \dim_\Bbbk S/m_A S$ (by minimality of d_1) which is the minimum number of generators of S as an A-module which is also the same as the minimum number of generators of S as an $\Bbbk[Y]$-module. Hence $\operatorname{rk} F_0 = 1$ if and only if S is a quotient of $\Bbbk[Y]$, or equivalently, $S = \Bbbk[Y]$. If Y is normal, then the remaining hypotheses imply that Y has rational singularities. 3.8(d) follows immediately from 3.8(b). Hence we need to prove the Main Theorem and 3.8(a).

Proof of 3.8(a). Since Z is normal, q factors through the normalization $\tilde{Y}$ of Y. Now apply the following lemma with $Z' = Z$, $Y' = \tilde{Y}$ and $\phi : Z \longrightarrow \tilde{Y}$, then the morphism induced by q. $\square$

Lemma 3.9. *Let $\phi : Z' \longrightarrow Y'$ be a proper birational morphism of integral schemes, with Y' normal. Then $\phi_* \mathscr{O}_{Z'} \simeq \mathscr{O}_{Y'}$.*

Proof. The question is local on Y', so we may assume that $Y' = \operatorname{Spec} R$ for some normal domain R. Since ϕ is proper, $\phi_* \mathscr{O}_{Z'}$ is a coherent sheaf of $\mathscr{O}_{Y'}$-algebras, so $\Gamma(Y', \phi_* \mathscr{O}_{Z'}) = \Gamma(Z', \mathscr{O}_{Z'})$ is a finitely generated R-module, and, hence, $\Gamma(Z', \mathscr{O}_{Z'})$ is integral over R. Since R and $\Gamma(Z', \mathscr{O}_{Z'})$ are integral domains with the same field of fractions, and R is normal, $R = \Gamma(Z', \mathscr{O}_{Z'})$. $\square$

Proof of the Main Theorem (Theorem 3.5). Embed V into a projective space so that $\mathscr{O}_V(1)$ is very ample. By reembedding with $\mathscr{O}_V(k)$ for some $k \gg 0$, we may assume that $\mathrm{H}^i(V, \mathscr{O}_V(n)) = 0$ for all $i \geq 1$ and $n \geq 1$. Let R be the coordinate ring of V with this embedding.

Lemma 3.10. *There exists a resolution $0 \longrightarrow K_\bullet(\xi, \mathcal{V}) \longrightarrow P_{\bullet,\bullet}(\mathcal{V})$*

$$\begin{tikzcd}
0 \arrow[r] & P_{\bullet,t}(\mathcal{V}) \arrow[r] & \cdots \arrow[r] & P_{\bullet,0}(\mathcal{V}) \arrow[r] & 0 \\
0 \arrow[r] & K_\bullet(\xi, \mathcal{V}) \arrow[r] \arrow[u, hook] & \cdots \arrow[r] & K_\bullet(\xi, \mathcal{V}) \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

such that

(a) Each $P_{i,j}(\mathcal{V})$ is a direct sum of sheaves of the form $q^* \mathscr{O}_X(j) \otimes p^* \mathscr{O}_V(n)$ with $n \geq 1$, and, hence, q_*-acyclic.

(b) For each j, the jth column is a q_*-acyclic resolution of $K_\bullet(\xi, \mathcal{V})$ by coherent $\mathscr{O}_{X\times V}$-modules that arise as the tensor product of $q * A(-j)$ with a Γ-acyclic resolution of $p^*(\wedge^j \xi \otimes \mathcal{V})$.

Proof of Lemma. Consider the dual complex $K_\bullet(\xi, \mathcal{V})^*$: $K_j(\xi, \mathcal{V})^* = \bigwedge^{-j}(p^*\xi^*) \otimes_{\mathscr{O}_{X\times V}} p^* \mathcal{V}^*, -t \leq j \leq 0$. Apply $\Gamma_*(-) := \oplus_{i\in\mathbb{Z}}\Gamma(X \times V, - \otimes p^* \mathscr{O}_V(i))$ to this complex to obtain a bigraded complex of finitely generated graded $(A \otimes_{\Bbbk} R)$-modules. All the generators of the term in jth position $(-t \leq j \leq 0)$ has A-degree $-j$. Replace this complex with the complex obtained by cutting out all the graded pieces in nonpositive R-degrees. Thus we get a complex $C_\bullet(\mathcal{V})$ such that for all $-t \leq j \leq 0$, and for all the generators of the j term of $C_\bullet(\mathcal{V})$, the A-degree is $-j$ and R-degree is positive. Notice that $\widetilde{C_\bullet(\mathcal{V})}$ is $K_j(\xi, \mathcal{V})^*$. Take a minimal free resolution $\widehat{C_{\bullet,\bullet}(\mathcal{V})}$ of $C_\bullet(\mathcal{V})$. Sheafify this double complex to get a locally free resolution of $K_j(\xi, \mathcal{V})^*$, and then apply $\mathcal{H}om_{\mathscr{O}_{X\times V}}(-, \mathscr{O}_{X\times V})$ to obtain $P_{\bullet,\bullet}(\mathcal{V})$. (Since $K_j(\xi, \mathcal{V})^*$ is locally free for all j, the columns of $P_{\bullet,\bullet}(\mathcal{V})$ are acyclic.) Note that it has the desired properties. □

Proof of Theorem cont. Notice that $\mathcal{P}_{\bullet,\bullet}(\mathcal{V})$ is a free resolution of a complex $\mathcal{P}_\bullet(\mathcal{V})$ that is a q_*-acyclic resolution of $M(\mathcal{V})$. Let $\mathscr{I}^\bullet$ be an injective resolution of $M(\mathcal{V})$. Then there is a comparison morphism $\gamma : \mathcal{P}_\bullet(\mathcal{V}) \longrightarrow \mathscr{I}^\bullet$.

$$\begin{tikzcd}
\mathcal{P}_{\bullet,\bullet}(\mathcal{V}) \arrow[r] & \mathcal{P}_\bullet(\mathcal{V}) \arrow[rr, "\exists\gamma"] & & \mathscr{I}^\bullet \\
K_\bullet(\xi, \mathcal{V}) \arrow[u] \arrow[rr] & & M(\mathcal{V}) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

(All the arrows are quasi-isomorphisms.) Since γ is a quasi-isomorphism, the mapping cone C_γ of γ is exact. Note that C_γ is q_*-acyclic. Hence the mapping cone $C_{q_*\gamma}$ of $q_*\gamma$ which is the same as q_*C_γ is exact, i.e., $q_*\gamma$ is a quasi-isomorphism. Let $G_\bullet(\mathcal{V})$ be the total complex of $q_*(\mathcal{P}_{\bullet,\bullet}(\mathcal{V}))$. Then $\mathrm{H}_{-i}(G_\bullet(\mathcal{V})) = \mathbf{R}^i\, q_*M(\mathcal{V})$.

Define $F_\bullet(\mathcal{V})$ to be the minimal part of $G_\bullet(\mathcal{V})$; then $\mathrm{H}_{-i}(F_\bullet(\mathcal{V})) = \mathbf{R}^i\, q_*M(\mathcal{V})$, giving Theorem 3.5(b). We need to find a description of the terms $F_i(\mathcal{V})$. By Proposition A.2, we need to determine $\mathrm{H}_*(G_\bullet(\mathcal{V}) \otimes_A \Bbbk)$, for which we look at $\mathrm{H}_*(q_*(\mathcal{P}_{\bullet,\bullet}(\mathcal{V})) \otimes_A \Bbbk)$. The horizontal maps in $q_*(\mathcal{P}_{\bullet,\bullet}(\mathcal{V})) \otimes_A \Bbbk$ are zero since the horizontal maps in $q_*(\mathcal{P}_{\bullet,\bullet}(\mathcal{V}))$ are matrices with entries of degree 1, by Lemma 3.10(a). Therefore we only need to compute the homology of each column $q_*(\mathcal{P}_{\bullet,j}(\mathcal{V}) \otimes_A \Bbbk)$. By Lemma 3.10(b), it is $\mathcal{P}_{\bullet,j}(\mathcal{V}) \otimes_A \Bbbk$ is a Γ-acyclic resolution of $\wedge^j\xi \otimes \mathcal{V}$; hence $\mathrm{H}^*(q_*(\mathcal{P}_{\bullet,j}(\mathcal{V}) \otimes_A \Bbbk)) = \mathrm{H}^*(V, \wedge^j\xi \otimes \mathcal{V}) \otimes_{\Bbbk} \Bbbk(-j)$. This yields

$$\mathrm{H}_i(G_\bullet(\mathcal{V}) \otimes_A \Bbbk) = \bigoplus_{j\geq 0} \mathrm{H}^j(V, \wedge^{i+j}\xi \otimes \mathcal{V}) \otimes_{\Bbbk} \Bbbk(-i-j)$$

for all i, j. Proposition A.2 now yields Theorem 3.5(a).

The first assertion of Theorem 3.5(c) follows from noting that X is affine. Now we want to show that $\mathrm{H}^i(Z, M(\mathcal{V})) = \mathrm{H}^i(V, \mathrm{Sym}(\mathcal{S}^*) \otimes \mathcal{V})$. Note that $p_*M(\mathcal{V}) = p_*(\mathscr{O}_Z \otimes p^* \mathcal{V}) = \mathrm{Sym}(S^*) \otimes \mathcal{V}$ by the projection formula [Har77, Exercise II.5.1(d)] and Proposition 3.3(b). Hence it suffices to show that $\mathrm{H}^i(V, p_*M(\mathcal{V})) = \mathrm{H}^i(X \times V, M(\mathcal{V}))$.

To this end, note that, being an affine morphism p is exact on quasi-coherent sheaves on $X \times V$. Let $\mathscr{I}^\bullet$ be a flasque resolution of $M(\mathcal{V})$ over $X \times V$. Then $p_* \mathscr{I}^\bullet$ is a flasque resolution of $p_*M(\mathcal{V})$. Hence

$$\mathrm{H}^i(V, p_*M(\mathcal{V})) = \mathrm{H}^i(\Gamma(V, p_* \mathscr{I}^\bullet)) = \mathrm{H}^i(\Gamma(X \times V, \mathscr{I}^\bullet)) = \mathrm{H}^i(X \times V, M(\mathcal{V})).$$ □

4. The Lascoux Resolution.

We return to the example discussed in Section 1. Since $V = \mathrm{Grass}(r, G)$, we need to compute the cohomology groups of exterior powers of $\xi = \mathcal{Q}^*$. (In reality, ξ is a direct sum of finitely many copies of $\mathcal{Q}^*$; we will make this precise later.)

4.1. **Borel-Weil-Bott Theorem.** TBD

4.2. **Determinantal varieties.** TBD

Appendix A. Resolutions

Most of the things mentioned here can be found, in some way or the other, in [Eis95] and [BH93]. The discussion on Koszul complexes relies on [AN10, Chapter 1], [Eis95, Chapter 17] and [Ser00, Chapter IV].

A.1. **Free and locally free resolutions.** Let A be a Noetherian *graded* ring with $A_0 = \Bbbk$, i.e., there is a decomposition $A \simeq \oplus_{i \in \mathbb{N}} A_i$ as $\Bbbk$-vector spaces such that $A_0 = \Bbbk$, $A_i A_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{N}$ and such that there is a finite set of homogeneous elements $x_1, \ldots, x_n$ (of positive degree) that generate A as a $\Bbbk$-algebra. By $\mathfrak{m}_A$, we denote the unique homogeneous maximal A-ideal generated by A_1. Let M be a graded A-module, i.e., there is a decomposition $M \simeq \oplus_{i \in \mathbb{Z}} M_i$ as $\Bbbk$-vector spaces such that $A_i M_j \subseteq M_{i+j}$ for all $i \in \mathbb{N}$ and $j \in \mathbb{Z}$. Further assume that M is *finitely generated* as an A-module, i.e., there exists a finite set $\{m_1, \ldots, m_r\} \subseteq M$ of homogeneous elements such that $M = \sum_{i=1}^r A m_i$ (not necessarily a direct sum). Equivalently, by the Nakayama lemma, the images of the m_i form a spanning set of $M/\mathfrak{m}_A M$ as a $\Bbbk$-vector space. We can map a graded free A-module F_0 of finite rank surjectively on to M; we may assume that the map has degree zero. The kernel of this map is a finitely generated graded A-module, so we may repeat this construction, and obtain an exact sequence of finitely generated graded A-modules

$$\cdots \longrightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \longrightarrow M \longrightarrow 0$$

such that the F_i are free A-modules of finite rank and the morphisms are of degree zero. We will interchangeably call this exact sequence or the complex

$$(F_\bullet, \partial_\bullet): \qquad \cdots \longrightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \longrightarrow 0$$

a graded A-free resolution of M. More precisely, a *graded A-free resolution* of M is a complex

$$(G_\bullet, d_\bullet): \qquad \cdots \longrightarrow G_2 \xrightarrow{d_2} G_1 \xrightarrow{d_1} G_0 \longrightarrow 0$$

of finitely generated graded free A-modules of finite rank such that

$$\mathrm{H}_i(G_\bullet) = \begin{cases} M & i = 0 \\ 0 & i > 0 \end{cases}$$

(Complexes with at most H_0 nonzero are called *acyclic*.) For any finitely generated graded A-module N, let $\mu(N)$ be the least integer r such that there exists r homogeneous elements that generate N as an A-module; by the Nakayama lemma, $\mu(N) = \operatorname{rk}_\Bbbk N/\mathfrak{m}_A N$. Hence we can find a resolution $F_\bullet$ such that $\operatorname{rk} F_0 = \mu(M)$, $\operatorname{rk} F_1 = \mu(\ker(F_0 \longrightarrow M))$ and $\operatorname{rk} F_i = \mu(\ker(F_{i-1} \longrightarrow F_{i-2}))$ for $i \geq 2$, called a *minimal graded free resolution* of M. Among all free resolutions, minimal resolutions are characterized by the fact that the entries in the matrices that represent the maps $F_i \longrightarrow F_{i-1}$ belong to $\mathfrak{m}_A$. (In general, we will call a complex $\cdots \longrightarrow F_i \xrightarrow{\partial_i} F_{i-1} \longrightarrow \cdots$ of R-modules *minimal* if $\operatorname{Im} \partial_i \subseteq \mathfrak{m}_A F_{i-1}$ for all i.) Every free resolution $G_\bullet$ of M decomposes as $G'_\bullet \oplus G''_\bullet$ where $G'_\bullet$ is a minimal graded free resolution and $G''_\bullet$ is an exact sequence of free A-modules; see Proposition A.2 below. Given two minimal resolutions $(F_\bullet, \partial_\bullet)$ and $(G_\bullet, d_\bullet)$, there is an isomorphism of complexes $\phi_\bullet : (F_\bullet, \partial_\bullet) \longrightarrow (G_\bullet, d_\bullet)$, i.e., there is a diagram in which the rows squares are commutative and the vertical maps are isomorphisms.

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "\partial_2"] \arrow[d, "\phi_2"] & F_1 \arrow[r, "\partial_1"] \arrow[d, "\phi_1"] & F_0 \arrow[r] \arrow[d, "\phi_0"] & 0 \\
\cdots \arrow[r] & G_2 \arrow[r, "d_2"] & G_1 \arrow[r, "d_1"] & G_0 \arrow[r] & 0
\end{tikzcd}$$

Fix a minimal resolution $(F_\bullet, \partial_\bullet)$ of M. The *ith syzygy module* of M is $\operatorname{Im} \partial_i$. Choosing a different minimal resolution results in a syzygy module isomorphic to the original one.

Every finitely generated graded A-module has a minimal graded free resolution. Hilbert's Syzygy Theorem asserts that every minimal resolution has finite length; more precisely, M has a resolution of the form

$$0 \longrightarrow F_p \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

where F_p is nonzero A-module and $p \le n$. We call p the *projective dimension* of M and write $\operatorname{pd}_A M = p$.

Remark A.1. Except for the minimality of the resolution and the finiteness of projective dimension, all other assertions above work for arbitrary Noetherian rings and finitely generated modules over them, if we allow for projective modules in addition to free modules. If A is a quotient of a polynomial ring (with grading as above) then graded projective A-modules are free. Over a polynomial ring, every finitely generated projective module is free, which is a very difficult theorem of Quillen and Suslin, originally conjectured by Serre.

We say that a complex $G_\bullet$ is *bounded below* (respectively, *bounded above*, *bounded*) if $G_i = 0$ for all $i \ll 0$ (respectively, for all $i \gg 0$, for all $|i| \gg 0$).

Proposition A.2. *Let $G_\bullet$ be a bounded-below complex of finitely generated free graded A-modules. Then there exist a minimal complex $G'_\bullet$ of finitely generated free graded A-modules and an exact sequence $G''_\bullet$ such that $G_\bullet \simeq G'_\bullet \oplus G''_\bullet$. Moreover, for all i, $G'_i \simeq \mathrm{H}_i(G_\bullet \otimes_A \Bbbk) \otimes_\Bbbk A$.*

Sketch, with an example. The key idea is that, for each i, there is a decomposition $G_i = B_i \oplus U_i \oplus B'_i$ such that d_i is of the form

$$d_i = \begin{array}{c} \\ B_{i-1} \\ U_{i-1} \\ B'_{i-1} \end{array} \begin{array}{c} \begin{array}{ccc} B_i & U_i & B'_i \end{array} \\ \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & d'_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array}$$

where the entries in d'_i are in $\mathfrak{m}_A$. (Here we mean that $d_i|_{B_i} = 0$, that $\operatorname{Im} d'_i|_{U_i} \subseteq \mathfrak{m}_A U_{i-1}$, and that $d_i|_{B'_i}$ is an isomorphism to B_{i-1}.) Now let, for all i, $G'_i = U_i$ and $G''_i = B_i \oplus B'_i$. Note that

$$d_{i-1}d_i = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & d'_{i-1}d'_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} = 0$$

so, indeed, $G'_\bullet$ is a minimal complex. In the complex $G''_\bullet$, $\ker d''_i = B_i = \operatorname{Im} d''_{i+1}$, so $G''_\bullet$ is exact.

Let us look at an example. Let $A = \Bbbk[x, y]$.

$$G_\bullet : \qquad 0 \longrightarrow A(-2) \xrightarrow{\begin{bmatrix} x^2 \\ -x \\ -x+y \end{bmatrix}} \begin{matrix} A \\ \oplus \\ A(-1)^2 \end{matrix} \xrightarrow{\begin{bmatrix} 1 & x & 0 \\ 1 & y & x \end{bmatrix}} A^2 \longrightarrow 0$$

This is an acyclic complex. Denote the homogeneous bases (for which the differential have the given matrix representations) by e_1, e_2 for G_0 (both of degree zero), f_1, f_2, f_3 for G_1 (with $\deg f_1 = 0$ and $\deg f_2 = \deg f_3 = 1$ and g for G_2 (of degree 2). We then have $d(f_1) = e_1 + e_2$, $d(f_2) = xe_1 + ye_2$, $d(f_3) = xe_2$ and $d(g) = x^2 f_1 - xf_2 + (-x+y)f_3$. We change of bases of the G_i (such that the new bases are again homogeneous) as follows: $e'_1 = e_1 + e_2$, $e'_2 = e_2$, $f'_1 = f_3$, $f'_2 = f_2 - xf_1$ and $f'_3 = f_1$. With the new bases, we obtain $d(f'_1) = xe'_2$, $d(f'_2) = (-x+y)e'_2$, $d(f'_3) = e'_1$ and $d(g) = (-x+y)f'_1 - xf'_2$. After this change of bases, $G_\bullet$ looks like

$$G_\bullet : \qquad 0 \longrightarrow A(-2) \xrightarrow{\begin{bmatrix} -x+y \\ -x \\ 0 \end{bmatrix}} \begin{matrix} A(-1)^2 \\ \oplus \\ A \end{matrix} \xrightarrow{\begin{bmatrix} 0 & 0 & 1 \\ x & -x+y & 0 \end{bmatrix}} A^2 \longrightarrow 0$$

This splits as the direct sum of the following two complexes:

$$G'_\bullet: \qquad 0 \longrightarrow A(-2) \xrightarrow{\begin{bmatrix} -x+y \\ -x \end{bmatrix}} A(-1)^2 \xrightarrow{\begin{bmatrix} x & -x+y \end{bmatrix}} A \longrightarrow 0,$$

$$G''_\bullet: \qquad 0 \longrightarrow A \xrightarrow{1} A \longrightarrow 0.$$

Note that $G'_\bullet$ too is acyclic; it is a resolution of $\Bbbk$ as an A-module.

The proof of the key idea uses the fact that the above argument can be done consistently for all i. That $G'_i \simeq \mathrm{H}_i(G_\bullet \otimes_A \Bbbk) \otimes_\Bbbk A$ follows immediately from the fact that $G''_\bullet$ is exact. ☐

Proposition A.3. *Let $M_\bullet$ be a bounded-below complex of finitely generated graded A-modules. Then there exists a minimal complex $G_\bullet$ of finitely generated free graded A-modules and a map of complexes $\phi : G_\bullet \longrightarrow M_\bullet$ such that ϕ is a quasi-isomorphism.*

Proof. There is a double complex $C_{\bullet,\bullet}$ of finitely generated graded free A-modules living in the upper half-plane (i.e., $C_{\bullet,q} = 0$ for $q < 0$) called the *Cartan–Eilenberg resolution* of $M_\bullet$; see [Wei94, Section 5.7]. Define $G_\bullet$ to be the total complex of $C_{\bullet,\bullet}$. Then there exists a map of complexes $\phi : G_\bullet \longrightarrow M_\bullet$ that is a quasi-isomorphism; this can be seen using the spectral sequences associated to the double complex. Now apply Proposition A.2 to make $G_\bullet$ a minimal complex. ☐

One can extend these notions to locally free resolutions of coherent sheaves over projective schemes over Noetherian rings; see [Har77, Corollary II.5.18]. However, uniqueness (even upto isomorphism) is hard to come by. For example, let A be the polynomial ring as above and let M and N be finitely generated graded module with a graded morphism $\phi : M \longrightarrow N$ of degree zero such that both $\ker\phi$ and $\operatorname{coker}\phi$ have finite length. Since the sheafification functor is exact, we have an isomorphism $\widetilde{M} \simeq \widetilde{N}$ on $\operatorname{Proj} A = \mathbb{P}^n_\Bbbk$. However, it is very likely (i.e., it is not too difficult to show an example!) that the resolutions of M and N are not isomorphic to each other, so their sheafification gives non-isomorphic locally free resolutions of $\widetilde{M} \simeq \widetilde{N}$.

A.2. **Koszul complexes.** Let E be a finite dimensional $\Bbbk$-vector space and denote its dual space by E^*. Let $s \neq 0 \in E^*$. Then s defines *contraction* map $i_s : \bigwedge E \longrightarrow \bigwedge E$ of degree -1, defined inductively by

$$i_s(e_0 \wedge e_1 \wedge \cdots \wedge e_{k-1}) = s(e_0)(e_1 \wedge \cdots \wedge e_{k-1}) - e_0 \wedge i_s(e_1 \wedge \cdots \wedge e_{k-1}).$$

Note that $(i_s \circ i_s)(e_0 \wedge \cdots \wedge e_{k-1}) = s(e_0) i_s(e_1 \wedge \cdots \wedge e_{k-1}) - s(e_0) i_s(e_1 \wedge \cdots \wedge e_{k-1}) + e_0 \wedge (i_s \circ i_s)(e_1 \wedge \cdots \wedge e_{k-1}) = 0$, again inductively. If $r = \operatorname{rk}_\Bbbk E$, then this yields a complex

$$K_\bullet(s): \qquad 0 \longrightarrow \textstyle\bigwedge^r E \longrightarrow \bigwedge^{r-1} E \longrightarrow \cdots \longrightarrow \bigwedge^2 E \longrightarrow E \longrightarrow \Bbbk \longrightarrow 0$$

called the *Koszul complex* on s. This is an exact sequence.

In fact, the construction of $K_\bullet(s)$ will work *verbatim* even if we take $\Bbbk$ to be a Noetherian local ring and E to be a free $\Bbbk$-module of finite rank. To determine when it would be acyclic, let $e_1, \ldots, e_r$ form a $\Bbbk$-basis of E and $e_1^*, \ldots, e_r^*$, its dual basis for E^*. Let $s = \sum_{i=1}^r f_i e_i^*$. Then $\mathrm{H}_i(K_\bullet(s)) = 0$ for all $i \geq 1$ (i.e., $K_\bullet(s)$ is acyclic) if f_1 is a non-zerodivisor on $\Bbbk$ and for all $2 \leq i \leq r$, f_i is a non-zerodivisor on $\Bbbk/(f_1, \ldots, f_{i-1})$. More precisely, if $(f_1, \ldots, f_r) = \Bbbk$, then $K_\bullet(s)$ is exact; apply, for instance, [Ser00, Proposition IV.A.1] recursively. Suppose that $(f_1, \ldots, f_r)$ is a proper ideal of $\Bbbk$. Then $K_\bullet$ is acyclic (it is never exact, in this situation) if and only if the $(f_1, \ldots, f_r)$ contains an R-regular sequence of length r. (We say that the *depth* (sometimes, also called *grade*) of $(f_1, \ldots, f_r)$ is r.) Note that the ideal $(f_1, \ldots, f_r)$ does not depend on the choice of the basis $\{e_1, \ldots, e_r\}$. If $K_\bullet(s)$ is acyclic, it is a minimal $\Bbbk$-free resolution of $\Bbbk/(f_1, \ldots, f_r)$.

Consider the ideal $(f_1, \ldots, f_r)$. If it is a proper ideal, then, by the Krull height theorem (see, e.g., [Eis95, Theorem 10.2]) its *height* (also called *codimension*) is at most r; further, if its depth is r, then its height is r. If $\Bbbk$ is a *Cohen–Macaulay* local ring (e.g., if it is a regular local ring or a complete intersection), then, for every $\Bbbk$-ideal I, its height and depth are the same. Hence, if $\Bbbk$ is Cohen–Macaulay, then $K_\bullet(s)$ is exact if and only if $\operatorname{height}(f_1, \ldots, f_r) = r$.

Let X be a smooth variety and E a vector bundle of rank r on X. Suppose that E^* has a nonzero global section s. Then the above construction can be extended to a Koszul complex

$$K_\bullet(s): \qquad 0 \longrightarrow \bigwedge^r E \longrightarrow \bigwedge^{r-1} E \longrightarrow \cdots \longrightarrow \bigwedge^2 E \longrightarrow E \longrightarrow \mathscr{O}_X \longrightarrow 0$$

The zero locus $Z(s)$ of s is the set $\{x \in X : s(x) = 0 \in (E^*)_x\}$ (where $(E^*)_x$ denotes the fibre of E^* at x). Assume that $Z(s)$ is non-empty; this is analoguous to the condition that the ideal $(f_1, \dots, f_r)$ is not the unit ideal, that we came across above. Then $K_\bullet(s)$ is acyclic (which can be checked locally, using the discussion above) if and only if the codimension of $Z(s)$ locally at every point $x \in Z(s)$ is $r = \operatorname{rk} E$, which is often referred to as its *expected codimension*. (For this observation, we need only that the local rings $\mathscr{O}_{X,x}$ are Cohen–Macaulay at all points $x \in Z(s)$.) Moreover, if it is acyclic, then

$$0 \longrightarrow \bigwedge^r E \longrightarrow \bigwedge^{r-1} E \longrightarrow \cdots \longrightarrow \bigwedge^2 E \longrightarrow E \longrightarrow 0$$

is a locally free resolution of the ideal sheaf $\mathscr{I}_{Z(s)}$.

Appendix B. Schur functors

TBD.

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