ALEXANDER DUALITY AND SERRE'S PROPERTY (S_i) FOR SQUARE-FREE MONOMIAL IDEALS

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Abstract. In this note, we study Serre's property (S_i), and its relation to Alexander duality for monomial ideals in a polynomial ring over a field. We describe ideals that define the non-Cohen-Macaulay- and the non-(S_i)-loci of finitely generated modules over regular rings, and show that minimal prime ideals in these loci are homogeneous, in the graded case. We show that a square-free monomial ideal has property (S_i) if and only if its Alexander dual has a linear resolution up to homological degree $i-1$. We prove that for square-free monomial ideals, having property (S_2) is equivalent to being locally connected in codimension 1.

1. Main Results

Let $\Bbbk$ be a field and $R = \Bbbk[x_1, \cdots, x_n]$ a polynomial ring in n variables. The *standard grading* on R is the grading by $\mathbb{N}$, obtained by setting $\deg x_l = 1$ for all $1 \leq l \leq n$. We can also make R *multigraded*, *i.e.*, graded by $\mathbb{N}^n$, with $\deg x_l = \mathbf{e}_l, 1 \leq l \leq n$, $\mathbf{e}_l$ being the lth standard basis vector for $\mathbb{N}^n$. For any finitely generated R-module M, we say that M *satisfies Serre's property* (S_i) if for all $\mathfrak{p} \in \operatorname{Spec} R$, $\operatorname{depth} M_\mathfrak{p} \geq \min\{i, \dim M_\mathfrak{p}\}$ [1]. We adopt the convention that the zero module has property (S_i) for all i. The (S_i)*-locus of* M is the set $\mathcal{U}_{(S_i)}(M) := \{\mathfrak{p} \in \operatorname{Spec} R : M_\mathfrak{p} \text{ has property } (S_i)\}$. We know from [EGA, IV, 6.11.2] that this is an open subset of $\operatorname{Spec} R$. In Proposition 9, we describe an ideal that defines the complement of $\mathcal{U}_{(S_i)}(M)$, following which (Discussion 11) we show that if M is graded, either by $\mathbb{N}$ or by $\mathbb{N}^n$, then so are the minimal prime ideals in this closed set.

For any homogeneous ideal $I \subseteq R$, we say that I *satisfies property* $(N_{c,i})$ (after [EGHP05]) if all the minimal generators of I have degree c and a minimal graded free resolution of I is linear up to homological degree $i-1$. This definition is independent of the choice of the resolution, because I satisfies property $(N_{c,i})$ if and only if $\operatorname{Tor}_l^R(\Bbbk, I)_j = 0$ for all $0 \leq l \leq i-1$ and for all $j \neq l + c$. We now relate the properties (S_i) and $(N_{c,i})$:

Theorem 1. *Let $I \subseteq R$ be a square-free monomial ideal with $\operatorname{ht} I = c$. Then for $i > 1$, the following are equivalent:*

(a) *R/I satisfies property (S_i).*
(b) *The Alexander dual $I^\star$ satisfies $(N_{c,i})$.*

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[1] This definition follows [EGA, IV, 5.7.2] and [BH93, Section 2.1]. There is another definition of Serre's condition (S_i), used in [EG85, Section 0.B]: a module M is said to satisfy Serre's condition (S_i) if $\operatorname{depth} M_\mathfrak{p} \geq \min\{i, \dim R_\mathfrak{p}\}$, for all $\mathfrak{p} \in \operatorname{Spec} R$.

Remark 2. The Alexander dual of a square-free monomial ideal I, minimally generated by monomials $f_1, \cdots, f_s$, is the (square-free monomial) ideal $\cap_{i=1}^{s} \mathfrak{p}_{f_i}$, where for any square-free monomial $f = x_{l_1} \cdots x_{l_j}$, we set $\mathfrak{p}_f := (x_{l_1}, \cdots, x_{l_j})$. See [MS05] for more on Alexander duality, and, also for any unexplained terminology.

Remark 3. The motivation for Theorem 1 is the result of Eagon-Reiner [ER98] (see also [MS05, Theorem 5.56]) that R/I is Cohen-Macaulay if and only if $I^\star$ has a linear free resolution. We have that R/I is Cohen-Macaulay if and only if R/I satisfies property (S_i) for all i. By Theorem 1, this is equivalent to $I^\star$ having property $(N_{c,i})$ for all i.

Remark 4. Terai [Ter99] (see Proposition 13 below) gave a generalization of the Eagon-Reiner theorem; we require this in our proof of Theorem 1. For two other results generalizing the Eagon-Reiner theorem, see Herzog-Hibi [HH99, Theorem 2.1(a)] and Herzog-Hibi-Zheng [HHZ04, Theorem 1.2(c)].

Remark 5. We can extend the statement to include the case $i = 1$ by replacing the statement (a) by "R/I satisfies property (S_i) and I is unmixed" (*i.e.*, for all the associated primes $\mathfrak{p}$ of R/I, $\dim R/\mathfrak{p}$ is independent of $\mathfrak{p}$). Since R/I is reduced, it always satisfies property (S_1). Hence if I is unmixed, then $I^\star$ is generated by monomials of degree c; this is property $(N_{c,1})$ for $I^\star$. For larger i, the hypothesis that I is unmixed becomes superfluous: for any ideal I, not necessarily homogeneous, if R/I satisfies property (S_2), then I is unmixed [EGA, IV, 5.10.9].

For a commutative ring A, we say that $\operatorname{Spec} A$ is *connected in codimension* k, if for all ideals $\mathfrak{a} \subseteq A$ with $\operatorname{ht} \mathfrak{a} > k$, $\operatorname{Spec} A \setminus \{\mathfrak{p} \in \operatorname{Spec} A : \mathfrak{a} \subseteq \mathfrak{p}\}$ is connected, and that A is *locally connected in codimension* k if $A_\mathfrak{p}$ is connected in codimension k for all $\mathfrak{p} \in \operatorname{Spec} A$. It is known [Har62, Corollary 2.4] that for any ideal I, not necessarily homogeneous, if R/I satisfies property (S_2), then $\operatorname{Spec} R/I$ is locally connected in codimension 1. For square-free monomial ideals, we prove the converse, giving the following equivalence:

Theorem 6. *Let $R = \Bbbk[x_1, \cdots, x_n]$ be a polynomial ring in n variables and let $I \subseteq R$ be a square-free monomial ideal. Then $\operatorname{Spec} R/I$ is locally connected in codimension 1 if and only if R/I satisfies property (S_2).*

2. Free resolutions and the locus of non-(S_i) points

Many results in this section are part of folklore. We take R to be an arbitrary regular domain, and M a finitely generated R-module with a finite free resolution

$$\mathbb{F}_\bullet : \qquad 0 \longrightarrow F_p \xrightarrow{\phi_p} F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0 \ .$$

Let $c = \operatorname{codim} M$. For $1 \leq l \leq p$, set $r_l := \sum_{j=l}^{p} (-1)^{j-l} \operatorname{rk} F_j$ and $I_l := \sqrt{I_{r_l}(\phi_l)}$, where, for a map ϕ of free modules of finite rank, and a natural number t, $I_t(\phi)$ is ideal generated by the $t \times t$ minors of ϕ and $\sqrt{\ }$ denotes taking the radical of an ideal.

Remark 7. Since R is a domain, M has a well-defined rank. We apply [BE73, Lemma 1] to conclude that M is projective if and only if $I_1 = R$. We see immediately that the exact sequence $\big(0 \longrightarrow \operatorname{Im} \phi_l \longrightarrow F_{l-1} \longrightarrow \operatorname{coker} \phi_l \longrightarrow 0\big) \otimes_R R_\mathfrak{p}$ splits — we say that $\phi_l \otimes_R R_\mathfrak{p}$ *splits* if this happens — if and only if $I_l \nsubseteq \mathfrak{p}$. If $\phi_l \otimes_R R_\mathfrak{p}$ splits, then so does every $\phi_{l'} \otimes_R R_\mathfrak{p}$ for $l' \geq l$. Hence $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_p$.

Additionally, if R is local, with maximal ideal $\mathfrak{m}$, and M is not free, then $\operatorname{pd} M = \max\{l : 1 \le l \le p \text{ and } I_l \subseteq \mathfrak{m}\}$.

First we determine the Cohen-Macaulay locus of M, which is an open subset of $\operatorname{Spec} R$; see [EGA, IV, 6.11.3]. Let

$$J_{CM}(M) := \bigcap_{k=c+1}^{p} \left(I_k + \bigcap_{\substack{\mathfrak{q} \in \min M, \\ \operatorname{ht} \mathfrak{q} < k}} \mathfrak{q} \right), \tag{1}$$

taking $J_{CM}(M) = R$ if the intersection is empty.

Proposition 8. *For all $\mathfrak{p} \in \operatorname{Spec} R$, $M_\mathfrak{p}$ is Cohen-Macaulay if and only if $J_{CM}(M) \not\subseteq \mathfrak{p}$.*

Proof. Let $l = \operatorname{codim} M_\mathfrak{p} + 1$. First, $\left(I_k + \bigcap_{\substack{\mathfrak{q} \in \min M, \\ \operatorname{ht} \mathfrak{q} < k}} \mathfrak{q} \right) \not\subseteq \mathfrak{p}$ for all $k < l$; otherwise, we would get an ideal $\mathfrak{q} \subseteq \mathfrak{p}$ with $\mathfrak{q} \in \min M$ and $\operatorname{ht} \mathfrak{q} < \operatorname{codim} M_\mathfrak{p}$, which is a contradiction. We now see that $M_\mathfrak{p}$ is Cohen-Macaulay if and only if $\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} = \operatorname{codim} M_\mathfrak{p}$, or, equivalently (by Remark 7), $I_l \not\subseteq \mathfrak{p}$, or, equivalently (by Remark 7, again), $\left(I_k + \bigcap_{\substack{\mathfrak{q} \in \min M, \\ \operatorname{ht} \mathfrak{q} < k}} \mathfrak{q} \right) \not\subseteq \mathfrak{p}$ for all $k \ge l$, or, equivalently (by above), $J_{CM}(M) \not\subseteq \mathfrak{p}$. □

In order to determine the (S_i)-locus of M, we first define $\Lambda_i = \Lambda_i(M)$ to be the set of all $\mathfrak{q} \in \operatorname{Spec} R$ such that $\mathfrak{q}$ is minimal over $I_l + J_{CM}(M)$ for some $l > \operatorname{ht} \mathfrak{q} - i$. Note that Λ_i is finite. Now let $J_{(S_i)}(M) = \bigcap_{\mathfrak{q} \in \Lambda_i} \mathfrak{q}$, taking $J_{(S_i)}(M) = R$ if $\Lambda_i = \emptyset$.

Proposition 9. *For all $\mathfrak{p} \in \operatorname{Spec} R$, $\mathfrak{p} \in \mathcal{U}_{(S_i)}(M)$ if and only if $J_{(S_i)}(M) \not\subseteq \mathfrak{p}$.*

Proof. Let $\mathfrak{p} \in \operatorname{Spec} R$ and $\Lambda_i \cap \mathfrak{p} := \{\mathfrak{q} \in \Lambda_i : \mathfrak{q} \subseteq \mathfrak{p}\}$. Since $J_{(S_i)}(M) \not\subseteq \mathfrak{p}$ if and only if $\Lambda_i \cap \mathfrak{p} = \emptyset$, we need to show that $\mathfrak{p} \in \mathcal{U}_{(S_i)}(M)$ if and only if $\Lambda_i \cap \mathfrak{p} = \emptyset$.

Let $\mathfrak{q} \in \Lambda_i \cap \mathfrak{p}$. Let $l > \operatorname{ht} \mathfrak{q} - i$ be such that $\mathfrak{q}$ is minimal over $I_l + J_{CM}(M)$. We apply Remark 7 to the regular local ring $(R_\mathfrak{q}, \mathfrak{q}R_\mathfrak{q})$ to conclude that $\operatorname{pd}_{R_\mathfrak{q}} M_\mathfrak{q} > \dim R_\mathfrak{q} - i$, and, by the Auslander-Buchsbaum formula, that $\operatorname{depth} R_\mathfrak{q} < i$. Since $J_{CM}(M) \subseteq \mathfrak{q}$, $M_\mathfrak{q}$ is not Cohen-Macaulay. Hence $M_\mathfrak{q}$ does not have property (S_i), so $\mathfrak{p} \notin \mathcal{U}_{(S_i)}(M)$.

Conversely, if $\mathfrak{p} \notin \mathcal{U}_{(S_i)}(M)$, then there exists $\mathfrak{q} \subseteq \mathfrak{p}$ be such that $\operatorname{depth} M_\mathfrak{q} < \min\{i, \dim M_\mathfrak{q}\}$. Then $M_\mathfrak{q}$ is not Cohen-Macaulay, *i.e.*, $J_{CM}(M) \subseteq \mathfrak{q}$, and $\operatorname{pd}_{R_\mathfrak{q}} M_\mathfrak{q} > \dim R_\mathfrak{q} - i$. By Remark 7, there exists $l > \operatorname{ht} \mathfrak{q} - i$ such that $I_l \subseteq \mathfrak{q}$. Let $\mathfrak{q}'$ be minimal such that $I_l + J_{CM}(M) \subseteq \mathfrak{q}' \subseteq \mathfrak{q}$. Since $\mathfrak{q}'$ is minimal over $I_l + J_{CM}(M)$ and $l > \operatorname{ht} \mathfrak{q}' - i$, we see that $\mathfrak{q}' \in \Lambda_i \cap \mathfrak{p}$. □

Remark 10. Suppose that $\operatorname{ht} \mathfrak{p} = c$ for all $\mathfrak{p} \in \min M$, *i.e.*, that $\operatorname{Ann} M$ is unmixed. Then $J_{CM}(M) = I_{c+1} + \sqrt{\operatorname{Ann} M}$. If $M = R/I$ for some radical ideal I, then $r_1 = 1$ and $I_1 = I$, so we get $J_{CM}(R/I) = I_{c+1}$. Hence Λ_i consists of those primes $\mathfrak{q}$ minimal over I_l for some $l \ge c+1$ with $\operatorname{ht} \mathfrak{q} < l - i$.

Discussion 11. Let $R = \Bbbk[x_1, \cdots, x_n]$, taken with standard grading, and M a finitely generated graded R-module. Let $\mathbb{F}_\bullet$ be a *graded* free resolution of M, with maps of degree 0. Then the $I_{r_l}(\phi_l)$ are homogeneous: to show this, it is enough to show that if F and G are graded free modules of same finite rank and $\phi : F \to G$ is a map of degree 0, then $\det \phi$ is homogeneous. Indeed, giving bases $f_1, \cdots, f_r$ for F and $g_1, \cdots, g_r$ for G, we can write $\phi = \left[a_{ij}\right]$. If $a_{ij} \neq 0$, then $\deg a_{ij} = \deg g_j - \deg f_i$. Since $\det \phi = \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{r\sigma(r)}$ (where, S_r is a permutation group of r elements, and $\operatorname{sgn}(\sigma)$ is the sign of a permutation σ), it suffices to show that $\deg a_{1\sigma(1)} \cdots a_{r\sigma(r)}$ is independent of σ, whenever $a_{i\sigma(i)} \neq 0$ for all $1 \leq i \leq r$. This is true, since if $a_{i\sigma(i)} \neq 0$ for all $1 \leq i \leq r$, then $\deg a_{1\sigma(1)} \cdots a_{r\sigma(r)} = \sum_{i=1}^{r} \left(\deg g_{\sigma(i)} - \deg f_i\right) = \sum_{i=1}^{r} \left(\deg g_i - \deg f_i\right)$, which is independent of σ. Radicals of homogeneous ideals are homogeneous. Minimal prime ideals of M are homogeneous. Therefore the ideals $J_{CM}(M)$ and $J_{(S_i)}(M)$ are homogeneous. Minimal prime ideals of homogeneous ideals are homogeneous, so the Cohen-Macaulay and (S_i)-loci of M are determined by homogeneous prime ideals. Hence to determine whether M has property (S_i), (or, is Cohen-Macaulay), it suffices to check this at homogeneous prime ideals. We remark here that the above argument carries over *mutatis mutandis* to the situation of multigrading, for instance, when $M = R/I$ for a monomial ideal I.

3. Property $(N_{c,i})$ for Alexander Duals

To every square-free monomial ideal J in R, we can associate a simplicial complex Δ, called the *Stanley-Reisner complex* of J. See [MS05, Chapter 1]. For any monomial ideal J, R/J inherits the multigrading of R. For any multigraded R-module M, we define multigraded Betti numbers $\beta_{l,\sigma}(M) := \dim_{\Bbbk} \operatorname{Tor}_l^R(\Bbbk, M)_\sigma$, where $1 \leq l \leq n$ and $\sigma \subseteq \mathbb{N}^n$ is a *multidegree*. When σ is square-free, *i.e.*, when the every entry in σ is 0 or 1, then we identify σ with the subset $\{x_i : \sigma_i \neq 0\}$, and, by abuse of notation, say that $\sigma \subseteq \{x_1, \cdots, x_n\}$. For a simplicial complex Δ and square-free multidegree $\sigma \subseteq \{x_1, \cdots, x_n\}$, we define $\Delta|_\sigma := \{F \in \Delta : F \subseteq \sigma\}$.

Proposition 12 (Hochster, [MS05, Corollary 5.12])**.** *Let J be a square-free monomial ideal and Δ its Stanley-Reisner complex. Non-zero multigraded Betti numbers of R/I occur at square-free multidegrees. Moreover, for a square-free multidegree $\sigma \subseteq \{x_1, \cdots, x_n\}$,*

$$\beta_{i,\sigma}(J) = \beta_{i-1,\sigma}(R/J) = \dim_{\Bbbk} \widetilde{\mathrm{H}}_{|\sigma|-i-2}(\Delta|_\sigma; \Bbbk).$$

An immediate corollary to Hochster's formula is that $\operatorname{depth} R/J = 1$ if and only if Δ is not connected: indeed, the Auslander-Buchsbaum formula implies that $\operatorname{depth} R/J = 1$ if and only if $\operatorname{Tor}_{n-1}^R(\Bbbk, R/J) \neq 0$. Since $\operatorname{Tor}_i^R(\Bbbk, R/J)_\sigma = 0$ if $|\sigma| \leq i$, Hochster's formula gives the equivalence with $\operatorname{Tor}_{n-1}^R(\Bbbk, R/J)_{\{x_1, \cdots, x_n\}} \neq 0$, and, again, with $\widetilde{\mathrm{H}}_0(\Delta; \Bbbk) \neq 0$, which is equivalent to Δ being disconnected.

Proposition 13 (Terai [Ter99]; [MS05, Theorem 5.59])**.** *For any square-free monomial ideal J, $\operatorname{pd} R/J = \operatorname{reg} J^\star$.*

Lemma 14. *With notation as above,*

(a) *For all $1 \leq l \leq n$, $(I : x_l)^\star = (I^\star \cap \Bbbk[x_1, \cdots, \widehat{x_l}, \cdots, x_n])R$.*
(b) *If R/I satisfies (S_i), then, for all $1 \leq l \leq n$, $R/(I : x_l)$ satisfies (S_i).*

Proof. (a): Associated primes of $(I : x_l)$ are exactly those of I not containing x_l. Hence while computing the dual, we take the generators not involving x_l.

(b): It suffices to show that $J_{(S_i)}(R/(I : x_l)) = R$. By way of contradiction, if $J_{(S_i)}(R/(I : x_l)) \neq R$, then let $\mathfrak{p}$ be a minimal prime ideal over $J_{(S_i)}(R/(I : x_l))$;

hence $(R/(I : x_l))_{\mathfrak{p}}$ does not have property (S_i). Since no monomial minimal generator of $(I : x_l)$ is divisible by x_l, $\mathfrak{p}$ is a monomial ideal not containing x_l; see Discussion 11. Therefore $(R/(I : x_l))_{\mathfrak{p}} \simeq (R/I)_{\mathfrak{p}}$, which has property (S_i), a contradiction. □

We are now ready to prove Theorem 1.

Theorem 1. *Then for $i > 1$, the following are equivalent:*
(a) *R/I satisfies property (S_i).*
(b) *The Alexander dual $I^\star$ satisfies $(N_{c,i})$.*

Proof. We prove both the directions by induction on n. Let $n = 3$. For any non-zero ideal $I \subseteq R = \Bbbk[x_1, x_2, x_3]$, if R/I satisfies (S_2) (equivalently, since $\dim R/I \leq 2$, (S_i) for all $i \geq 2$), then R/I is Cohen-Macaulay, and, hence $\operatorname{pd} R/I = \operatorname{ht} I$. By Proposition 13, we see that $\operatorname{reg} I^\star = \operatorname{ht} I$; however, since $I^\star$ is generated by monomials of degree $\operatorname{ht} I$, $I^\star$ has a linear resolution; in particular, $I^\star$ has property $(N_{c,2})$. Conversely, if $I^\star$ has property $(N_{c,2})$, and $c = 1$, then R/I is a complete intersection, and Cohen-Macaulay. If $c = 2$, then $\dim R/I = 1$. One-dimensional reduced Noetherian local rings are Cohen-Macaulay.

(a) $\Longrightarrow$ (b): By way of contradiction, assume that $I^\star$ does not have the property $(N_{c,i})$. By induction, assume that n is the least integer for which there is such a counter-example. By Lemma 14(a), $(I : x_l)^\star$ satisfies $(N_{c,i})$ for all $1 \leq l \leq n$. Now, since I does not have $(N_{c,i})$, there is a (square-free) multidegree σ and $j \leq i - 1$ such that $|\sigma| > j + c$ and $\beta_{j,\sigma}(I^\star) \neq 0$. We now claim that $\sigma = \{x_1, \cdots, x_n\}$: for, if, say, $x_1 \notin \sigma$, then let Δ be the Stanley-Reisner complex of $I^\star$, and $\widetilde{\Delta}$ of $(I^\star \cap \Bbbk[x_2, \cdots, x_n])R$. Then, by applying Hochster's formula, we have

$$\begin{aligned}\beta_{j,\sigma}(I^\star) &= \dim_{\Bbbk} \widetilde{\mathrm{H}}_{|\sigma|-j-2}(\Delta|_\sigma; \Bbbk) = \dim_{\Bbbk} \widetilde{\mathrm{H}}_{|\sigma|-j-2}(\widetilde{\Delta}|_\sigma; \Bbbk) \\ &= \beta_{j,\sigma}((I^\star \cap \Bbbk[x_2, \cdots, x_n])R) \\ &= \beta_{j,\sigma}((I : x_1)^\star)\end{aligned}$$

contradicting the fact that $(I : x_1)^\star$ satisfies $(N_{c,i})$. Hence $\sigma = \{x_1, \cdots, x_n\}$, and, therefore, $j < n - c = \dim R/I$. By choice, $j < i$. Moreover, $\operatorname{reg} I^\star \geq n - j$. By Proposition 13, $\operatorname{pd} R/I \geq n - j$, and, therefore $\operatorname{depth} R/I \leq j$, contradicting the hypothesis that R/I satisfies (S_i).

(b) $\Longrightarrow$ (a) : By way of contradiction, assume that R/I does not satisfy (S_i). We may again assume that n is the least number of variables where such a counter-example exists. Since $I^\star$ satisfies $(N_{c,i})$, $(I : x_l)^\star$ has $(N_{c,i})$ for all $1 \leq l \leq n$. By choice of n, $R/(I : x_l)$ satisfies (S_i) for all $1 \leq l \leq n$.

Now let $\mathfrak{p} \in \operatorname{Spec} R$ be such that $\operatorname{depth}(R/I)_{\mathfrak{p}} < \min\{i, \dim(R/I)_{\mathfrak{p}}\}$. If $x_l \notin \mathfrak{p}$, then, $(R/I)_{\mathfrak{p}} \simeq (R/(I : x_l))_{\mathfrak{p}}$. Hence $\operatorname{depth}(R/I)_{\mathfrak{p}} \geq \min\{i, \dim(R/I)_{\mathfrak{p}}\}$. Therefore $\mathfrak{p} = \mathfrak{m}$. Hence $\operatorname{depth} R/I < \min\{i, \dim R/I\}$. By Auslander-Buchsbaum formula, $\operatorname{pd} R/I > n - i$. Again, by the result of Terai, $\operatorname{reg} I^\star > n - i$, *i.e.*, there exists j and a multidegree σ such that $\beta_{j,\sigma}(I^\star) \neq 0$ and $|\sigma| - j > n - i$. By Hochster's theorem, non-zero Betti numbers are in square-free multidegrees, so, $|\sigma| \leq n$. Hence $j < i$, contradicting the hypothesis that $I^\star$ has $(N_{c,i})$. □

Before we proceed, we observe that if $\dim R/I \geq 2$ and R/I is connected in codimension 1, then Stanley-Reisner complex Δ of I is connected; in fact, it is *strongly connected*, *i.e.*, for any two faces F and F' of Δ of maximal dimension, we can find a sequence $F_0 = F, F_1, \cdots, F_r = F'$ of faces of maximal dimension such that for all $1 \leq i \leq n-1$, $F_i \cap F_{i-1}$ is a face of codimension 1 in F_i and F_{i-1}. To prove this, it suffices, using the correspondence between faces of Δ and prime ideals containing I [MS05, Theorem 1.7], to show that for any $\mathfrak{p}, \mathfrak{p}' \in \operatorname{Ass} R/I$, there is a sequence $\mathfrak{p}_0 = \mathfrak{p}, \mathfrak{p}_1, \cdots, \mathfrak{p}_r = \mathfrak{p}'$ of associated primes of R/I such that for all $1 \leq i \leq n-1$, $\operatorname{ht}(\mathfrak{p}_i + \mathfrak{p}_{i+1}) = \operatorname{ht} \mathfrak{p}_i + 1 = \operatorname{ht} \mathfrak{p}_{i+1} + 1$. This follows from setting $d = 2$ in [EGA, IV, 5.10.8]. Finally, since R/I is connected in codimension 1, it is equidimensional; this is the content of the proof of [EGA, IV, 5.10.9]. Hence every vertex of Δ is in some face of maximal dimension, so Δ is connected.

Theorem 6. *Let $R = \Bbbk[x_1, \cdots, x_n]$ be a polynomial ring in n variables and let $I \subseteq R$ be a square-free monomial ideal. Then $\operatorname{Spec} R/I$ is locally connected in codimension 1 if and only if R/I satisfies property (S_2).*

Proof. We will show that if $\operatorname{Spec} R/I$ is locally connected in codimension 1, then R/I has property (S_i); the other implication is already known [Har62, Corollary 2.4]. If $c \geq n-1$, then it is clear that R/I is locally connected in codimension 1 and that R/I has property (S_2). Therefore we will assume that $c \leq n-2$.

We proceed by induction on n. Let $n = 3$. It is easy to verify that any unmixed monomial ideal in three variables in locally connected in codimension 1. Since $c = 1$, R/I is a complete intersection and, hence has property (S_2). Now assume that $n > 3$.

We first observe that for all $1 \leq l \leq n$, $\operatorname{Spec} R/(I : x_l)$ is locally connected in codimension 1, because, as topological spaces, $\operatorname{Spec} R/(I : x_l)$ is homeomorphic to $\operatorname{Spec}(R/I)_{x_l}$, which is locally connected in codimension 1, $(R/I)_{x_l}$ being a localization of R/I. Since x_l does not divide any minimal generator of $(I : x_l)$, $(I : x_l)$ is extended from the subring $\Bbbk[x_1, \cdots, \widehat{x_l}, \cdots, x_n] \subseteq R$. By induction $R/(I : x_l)$ has property (S_2). Now let $\mathfrak{p} \in \operatorname{Spec} R, \mathfrak{p} \neq \mathfrak{m}$. We can then pick $x_l \notin \mathfrak{p}$. Since $(R/I)_\mathfrak{p} \simeq (R/(I : x_l))_\mathfrak{p}$, we see that $\operatorname{depth}(R/I)_\mathfrak{p} \geq \min\{2, \dim(R/I)_\mathfrak{p}\}$. It remains to show that $\operatorname{depth} R/I \geq 2$, *i.e.*, that the Stanley-Reisner complex Δ of I is connected, which follows from the preceding discussion. □

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