WORKING NOTES

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ABSTRACT. These are (incomplete) notes from the workshop on *Representation Theory and syzygies*, held in CMI in Dec 2023.

1. PURE RESOLUTIONS (MK)

Throughout this lecture, $\Bbbk$ is a field, $S = \Bbbk[x_1, \dots, x_n]$ a polynomial ring in n variables over $\Bbbk$ with $\deg x_i = 1$ for each i. Write $\mathfrak{m} = (x_1, \dots, x_n)$. Every finitely generated graded R-module has a *minimal graded free resolution*, i.e., a complex

$$F_\bullet : \qquad 0 \longrightarrow F_n \xrightarrow{\partial_n} F_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \longrightarrow 0$$

such that

(a) For each $0 \le i \le n$, F_i is a graded free S-module and ∂_i preserves degrees.
(b) $\operatorname{coker} \partial_1 \simeq M$.
(c) For each $1 \le i \le n$, $\ker \partial_i = \operatorname{Im} \partial_{i+1}$.
(d) For each $1 \le i \le n$, $\operatorname{Im} \partial_i \subseteq \mathfrak{m} F_{i-1}$.

The first three conditions make $F_\bullet$ a *graded free resolution* of M and the last condition makes it *minimal*.

Let $F_\bullet$ be a minimal graded free resolution of M. Write

$$F_i = \oplus_j S(-j)^{\beta_{i,j}}$$

for non-negative integers $\beta_{i,j}$ (that depend on M). These are called the *graded Betti numbers* of M. They do not depend on the choice of $F_\bullet$ since

$$\beta_{i,j} = \operatorname{rk}_{\Bbbk} \operatorname{Tor}_i(M, \Bbbk)_j.$$

Definition 1.1. Let $p = \operatorname{pd} M$. Say that M has a *pure resolution* if for each $0 \le i \le p$, there exists a unique d_i such that $\beta_{i,j} \neq 0$ if and only if $j = d_i$. The (strictly increasing) sequence $(d_0, \dots, d_p)$ is called the *degree sequence* of M.

Example 1.2. $M = S/\mathfrak{m}^k$ for some $k \ge 1$. The output of a Macaulay2 output for $n = 6$ and $k = 4$. In general, the length of the resolution is n and the degree sequence is $(0, k, k+1, \dots, n+k-1)$ which we can prove using Propositions 1.3 and 1.4. (Note that $\operatorname{soc} S/\mathfrak{m}^k = \mathfrak{m}^{k-1}/\mathfrak{m}^k$, so $\beta_{n,j} \neq 0$ if and only if $j = n+k-1$.)

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i1 : S = kk[a..f]

o1 = S

o1 : PolynomialRing

i2 : betti res power(ideal vars S, 4)

               0   1   2   3   4   5  6
o2 = total: 1 126 504 840 720 315 56
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         0: 1   .   .   .   .   .   .  .
         1: .   .   .   .   .   .   .  .
         2: .   .   .   .   .   .   .  .
         3: . 126 504 840 720 315 56

o2 : BettiTally
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□

Proposition 1.3. *Write $c = \operatorname{codim} M := \operatorname{codim}_{\operatorname{Spec} S} \operatorname{Supp} M$ and $p = \operatorname{pd} M$. For $0 \leq i \leq p$, let $t_i = \min\{j \mid \beta_{i,j} \neq 0\}$ and $T_i = \max\{j \mid \beta_{i,j} \neq 0\}$. Then $t_0 < t_1 < \ldots < t_p$ and Then $T_0 < T_1 < \ldots < t_c$.*

Proof. Let $F_\bullet$ be a minimal graded free resolution of M. Use minimality to show that $t_0 < t_1 < \ldots < t_p$. Apply that result to $\operatorname{Hom}_S(F_\bullet, S)$ and use the fact that $\operatorname{Ext}^i_S(M, S) = 0$ for all $i < c$ to get the other assertion. □

Proposition 1.4. $\operatorname{Tor}^S_n(M, \Bbbk) \simeq (\operatorname{soc} M)(-n)$.

Proof. Let $K_\bullet$ be the (graded) Koszul complex on $x_1, \ldots, x_n$ Then

$$\operatorname{Tor}^S_n(M, \Bbbk) = \ker \left(M(-n) \xrightarrow{\begin{bmatrix} \pm x_1 \\ \pm x_2 \\ \vdots \\ \pm x_n \end{bmatrix}} M(-n+1)^n \right) = (\operatorname{soc} M)(-n).$$

□

Example 1.5. Let A be an $(n+1) \times n$ matrix of variables $x_{i,j}$ and $S = \Bbbk[x_{i,j}, 1 \leq i \leq n+1, 1 \leq j \leq n]$. Let $I = I_n(A)$ the ideal generated by the $n \times n$ minors of A. Then S/I has the minimal resolution

$$0 \longrightarrow S(-n-1)^n \xrightarrow{A} S(-n)^{n+1} \longrightarrow S \longrightarrow 0.$$

This is pure, with degree sequence $(0, n, n+1)$. □

Example 1.6. Consider the embedding of $\mathbb{P}^1 \longrightarrow \mathbb{P}^d$ using the complete linear series $|\mathscr{O}_{\mathbb{P}}(d)|$. Write X for the image. Consider the exact sequence

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathscr{O}_{\mathbb{P}^d} \longrightarrow \mathscr{O}_X \longrightarrow 0$$

and apply the functor $\Gamma_*(-) := \oplus_j \Gamma(\mathbb{P}^d, - \otimes \mathscr{O}_{\mathbb{P}^d}(j))$. Note that $\Gamma_*(\mathscr{O}_X) = \oplus_j \mathscr{O}_{\mathbb{P}^1}(jd) = \Bbbk[x^d, x^{d-1}y, \ldots, y^d] =: S_X$, where x, y are the homogeneous coordinates on $\mathbb{P}^1$. Write $\Gamma_* \mathscr{O}_{\mathbb{P}^d} = \Bbbk[z_0, \ldots, z_d] =: S$ Then we have an exact sequence

$$0 \longrightarrow I_X \longrightarrow S \longrightarrow S_X \longrightarrow 0.$$

Note that I_X is generated by elements of degree ≥ 2. On the other hand, since

$$\mathrm{H}^1(X, \mathscr{O}_X(m-1)) = 0 \text{ for all} m \geq 1$$

it follows that the Castelnuovo-Mumford regularity $\operatorname{reg} \mathscr{O}_X$ of $\mathscr{O}_X$ is at most 1. By a theorem of Eisenbud-Goto,

$$\operatorname{reg} \mathscr{O}_X = \max\{j - i \mid \beta_{i,j}(S_X) \neq 0\}$$

so $\beta_{i,j}(S_X) = 0$ for all $j > i + 1$. Hence the resolution of S_X is pure with degree sequence $(0, 2, 3, \ldots, d)$.

□

Example 1.7. Let Γ be an n-cycle, with $n \geq 4$. Think of Γ as a 1-dimensional simplicial complex. Let I be its Stanley-Reisner ideal inside $S := \Bbbk[x_1, \dots, x_n]$. I.e., if the vertices are labelled cyclically by $1, 2, \dots, n$, I is generated by

$$\{x_1x_3, x_1x_4, \dots, x_1x_{n-2}, x_2x_4, x_2x_5, \dots, x_2x_{n-1}, \cdots\}.$$

Then S/I is a two-dimensional Gorenstein ring. Its resolution is pure with degree sequence $(0, 2, 3, \dots, n-2, n)$. □

Theorem 1.8 (Herzog-Kuhl)**.** *Let M be a graded Cohen-Macaulay module with pure resolution with degree sequence $(d_0, \dots, d_c)$. Then there exists C such that*

$$\beta_{i,j} = \begin{cases} 0, & j \neq d_i \\ C \prod_{i' \neq i} \frac{1}{|d_{i'} - d_i|}, & \textit{otherwise.} \end{cases}$$

Proof. The Hilbert series $H_M(t)$ of M is given by

$$\frac{\sum_{i=0}^{c} (-1)^i \beta_{i,d_i} t^{d_i}}{(1-t)^n}.$$

Hence the numerator $\sum_{i=0}^{c} (-1)^i \beta_{i,d_i} t^{d_i}$ has a zero of order c at $t = 1$. Evaluating it at $t = 1$, we see that

$$\sum_{i=0}^{c} (-1)^i \beta_{i,d_i} = 0.$$

Differentiating it once and evaluating at $t = 1$, we get

$$\sum_{i=0}^{c} (-1)^i d_i \beta_{i,d_i} = 0. \tag{1.9}$$

Differentiating it once more, evaluating at $t = 1$ and using (1.9), we get

$$\sum_{i=0}^{c} (-1)^i d_i^2 \beta_{i,d_i} = 0.$$

Thus, we get the following c linear relations among the $c+1$ numbers b_{i,d_i}, $0 \leq i \leq c$:

$$\sum_{i=0}^{c} (-1)^i d_i^k \beta_{i,d_i} = 0, \ 0 \leq k < c.$$

The $c \times (c+1)$ matrix

$$D := \left[(-1)^i d_i^k \right]_{k,i}$$

has rank c since each $c \times c$ submatrix is a Vandermonde matrix. Hence up to multiplication by a constant, the vector $(\beta_{i,d_i})_i$ is uniquely determined. Let δ_i, $1 \leq i \leq c+1$ be the maximal minors of D. Then, using Cramer's rule, we see that the vector $(\beta_{i,d_i})_i$ is a multiple of the vector $(\delta_1, \dots, \delta_{c+1})$. Note that

$$\delta_i = \prod_{\substack{t<s \\ t \neq i \neq s}} (d_t - d_s)$$

Dividing by $\prod_{t<s} (d_t - d_s)$, we get the proposition. □

The significance of Cohen-Macaulay modules of pure resolution stems from the following conjectures (since then proved) of Boij-Soderberg [BS08].

Conjecture 1.10 ([BS08])**.** Let M be a graded Cohen-Macaulay S-module. Then the Betti table of M (as an element of $\prod_j \mathbb{Q}^{n+1}$) is a positive rational combination of the Betti tables with pure resolution. In other words, the extremal rays of the cone generated by the Betti tables of Cohen-Macaulay modules are those corresponding to pure resolutions.

Conjecture 1.11 ([BS08])**.** For every degree sequence $(d_0, \dots, d_c)$, there exists a Cohen-Macaulay module with pure resolution of degree sequence $(d_0, \dots, d_c)$.

Remark 1.12. Conjecture 1.11 was proved by Eisenbud-Floystad-Weyman [EFW11] in characteristic zero, by constructing resolutions using Pieri rules. Then both conjectures were proved in a characteristic-free way by Eisenbud-Schreyer [ES09]; they discovered an analogous picture for vector bundles on projective spaces. Later Boij-Soderberg extended Conjecture 1.10 to the non-Cohen-Macaulay situation, with suitable changes in the statement [BS12]. □

2. Determinantal varieties

Let V and W $\Bbbk$-vector spaces with $\dim V = n$, $\dim W = m$. Let $0 \le t \le \min\{m, n\}$. Write $\mathbb{A} = \mathbb{A}^{mn} = \operatorname{Hom}_{\Bbbk}(V, W) = V^* \otimes_{\Bbbk} W$. Let $X_t = \{\phi \in \mathbb{A} \mid \operatorname{rk} \phi \le t\}$. Let $y_{i,j}, 1 \le i \le m, 1 \le j \le n$ be indeterminates over $\Bbbk$ and $S = [\{y_{i,j} \mid 1 \le i \le m, 1 \le j \le n\}]$. Write Y for the $m \times n$ matrix $[y_{i,j}]$. By $I_{t+1}(Y)$ we mean the S-ideal generated by the $(t \times t)$-minors of Y.

We list some properties of X_t and $I_{t+1}(Y)$, mostly from Bruns-Vetter.

(a) X_t is irreducible. For all $\phi \in \mathbb{A}$, $\phi \in X_t$ if and only if $f(\phi) = 0$ for all $f \in I_{t+1}(Y)$.
(b) $I_{t+1}(Y)$ is a prime ideal. Hence X_t is scheme-theoretically defined by $I_{t+1}(Y)$.
(c) $S/I_{t+1}(Y)$ is a normal and Cohen-Macaulay ring.
(d) If $\operatorname{char} \Bbbk = 0$ X_t has rational singularities. I.e., for every proper and birational map $\mu : Z \longrightarrow X_t$ with Z non-singular (we say that μ or Z is *desingularization* or *resolution of singularities*), the map $\mathscr{O}_{X_t} \longrightarrow \mu_* \mathscr{O}_Z$ is an isomorphism $\mathbf{R}^i \mu_* \mathscr{O}_Z = 0$ for each $i \neq 0$.

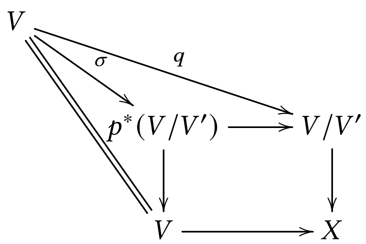
A desingularization of X_t is as follows: Write $G = \operatorname{Grass}(t, W)$, the Grassmannian of t-dimensional subspaces of W. Let $Z = \{(\phi, W') \in \mathbb{A} \times G \mid \operatorname{Im} \phi \subseteq W'\}$. Then Z has the structure of a sub-bundle of the trivial bundle $\mathbb{A} \times G$ (over G). Hence Z is non-singular. The image of Z under the projection $\mathbb{A} \times G \longrightarrow \mathbb{A}$ is X_t. Write q for this map. It is proper, since it is the restriction of the projection map whose fibres are proper. Moreover, over the open set $X_t \smallsetminus X_{t-1}$, q is a bijective. In characteristic zero, this is enough to show that q is birational. (q is birational in prime characteristic, too.)

3. Kozsul complex

Some information on Koszul complexes that is relevant for these lectures is in my notes (`syzSeminar.pdf`, Appendix A.2). Here I explain one point that is relevant, but is not included in those notes.

Let V be a vector space and V' a subspace. Without loss of generality, $\{v_1, \dots, v_d\}$ is a basis of V' and $\{v_1, \dots, v_n\}$ is a basis of V. Let $x_1, \dots, x_n$ be the basis of V^*, dual to $\{v_1, \dots, v_n\}$. Then V' is defined (as an algebraic subset of V) by the equations $x_{d+1} = \cdots = x_n = 0$. If we write $q : V \longrightarrow V/V'$ for the natural map, then $\{x_{d+1}, \dots, x_n\}$ is a basis of $\operatorname{Im}\left((V/V')^* \xrightarrow{q^*} V^*\right)$.

Now suppose we have a vector bundle V on a variety X and a sub-bundle V'. (We think of these as spaces.) Write $p : V \longrightarrow X$ be the structure morphism. Then the total space of V' is the zero-locus of a section σ of the bundle $p^*(V/V')$. This is the bundle $V \times_X (V/V')$. We get σ uniquely as follows.



4. Free resolutions of determinantal varieties

This is a summary of the arguments in [Wey03, Chapters 5, 6]. The desingularization describe above gives the following diagram

$$\begin{array}{ccccc} Z & \hookrightarrow & \mathbb{A}\times G & \longrightarrow & G \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \\ X_t & \hookrightarrow & \mathbb{A} & & \end{array}$$

Write $\mathcal{R}$ and Q for the tautological sub-bundle and quotient bundle of G, respectively. Denote the map $Z \longrightarrow G$ by p. Z is the total space of the vector bundle $V^* \otimes_{\Bbbk} \mathcal{R}$ where $\mathcal{R}$ is the tautological sub-bundle of G. (I.e., for each $W' \in G$, $p^{-1}(W')$ is the space of maps $V \longrightarrow W'$.) Hence there is a section σ of $V^* \otimes_{\Bbbk} Q$) whose zero locus is Z. Since $\operatorname{codim}_{\mathbb{A}\times G}(Z) = n(m-t) = \operatorname{rk} p^*(V^* \otimes_{\Bbbk} Q)$, it follows that the Koszul complex

$$K_\bullet \;:\; 0 \longrightarrow \wedge^{n(m-t)} p^*(V^* \otimes_{\Bbbk} Q)^* \longrightarrow \cdots \longrightarrow \wedge^2 p^*(V^* \otimes_{\Bbbk} Q)^* \longrightarrow p^*(V^* \otimes_{\Bbbk} Q)^* \longrightarrow 0$$

given by σ is a locally free resolution of $\mathcal{O}_Z$ as an $\mathcal{O}_{\mathbb{A}\times G}$-module.

We then find a suitable double complex $J^{\bullet,\bullet}$ such that

(a) it is a q_*-acyclic resolution of $K_\bullet$;
(b) $q_*J^{\bullet,\bullet}$ is a double complex of *free* graded R-modules.

Write $G_\bullet$ for the total complex of $q_*J^{\bullet,\bullet}$. It is a complex of *free* graded R-modules and is quasi-isomorphic to $\mathbf{R}q_*\mathcal{O}_Z$.

Notational convention: $K_\bullet$ is on the negative horizontal axis; $J^{\bullet,\bullet}$ is in the second quadrant. Hence $G_i = \oplus_j q_* J^{-j-i,j}$. (We freely switch between homological indexing, with subscripts that decrease along the arrows and cohomological indexing, with superscripts that increase along the arrows.)

Then there is a subcomplex $F_\bullet$ of $G_\bullet$ such that the inclusion map is a quasi-isomorphism and the maps in $F_\bullet$ are minimal. Moreover, for all $i \in \mathbb{Z}$,

$$F_i = \oplus_j \mathrm{H}^j(G, \wedge^{i+j} V \otimes_{\Bbbk} Q^*) \otimes_{\Bbbk} R(-i-j).$$

The groups $\mathrm{H}^j(G, \wedge^{i+j} V \otimes_{\Bbbk} Q^*)$ can be computed using the Pieri formula for exterior powers

$$\wedge^{i+j} V \otimes_{\Bbbk} Q^* = \bigoplus_{\lambda \vdash i+j} L_\lambda V \otimes L_{\lambda'} Q^*$$

followed by the Borel-Weil-Bott theorem to compute

$$\mathrm{H}^j(G, L_\lambda V \otimes_{\Bbbk} L_{\lambda'} Q^*) = L_\lambda V \otimes_{\Bbbk} \mathrm{H}^j(G, L_{\lambda'} Q^*).$$

Remark 4.1. Since X_t has rational singularities, $G_\bullet$ is quasi-isomorphic to $\mathcal{O}_{X_t}$. Note that since $\mathbf{R}^i q_* \mathcal{O}_Z = 0$ for all $i > 0$, the complex

$$\longrightarrow G_{-1} \longrightarrow G_{-2} \longrightarrow$$

(i.e., to the right of G_0 has no homology. This complex is bounded on the right, so we see that for each $i < 0$, the module of cycles and the module of boundaries at i are free and that

$$\operatorname{rk} G_i = \operatorname{rk} \operatorname{Im}(G_{i+1} \longrightarrow G_i) + \operatorname{rk} \ker(G_i \longrightarrow G_{i-1}).$$

In particular $F_i = 0$ for each $i < 0$. Similar argument also shows that $F_0 = R$. Hence

$$\cdots \longrightarrow F_2 \longrightarrow F_0 \longrightarrow F_0 \longrightarrow 0$$

is a minimal graded free resolution of $\mathcal{O}_{X_t}$.

Remark 4.2. In fact, the prior knowledge that X_t has rational singularities is not necessary. One can compute the cohomology groups and conclude that $F_i = 0$ for each $i < 0$ and that $F_0 = R$. Hence $\mathbf{R}^i q_* \mathcal{O}_Z = 0$ for all $i > 0$ and $q_*\mathcal{O}_Z$ is a cyclic $\mathcal{O}_{X_t}$-module. Since $q_*\mathcal{O}_Z$ gives the normalization of X_t, it follows that X_t is normal and hence the map $\mathcal{O}_{X_t} \longrightarrow q_*\mathcal{O}_Z$ is an isomorphism. Hence X_t has rational singularities.

References

[BS08] M. Boij and J. Söderberg. Graded Betti numbers of Cohen-Macaulay modules and the multiplicity conjecture. *J. Lond. Math. Soc. (2)*, 78(1):85–106, 2008. 3

[BS12] M. Boij and J. Söderberg. Betti numbers of graded modules and the multiplicity conjecture in the non-Cohen-Macaulay case. *Algebra Number Theory*, 6(3):437–454, 2012. 4

[EFW11] D. Eisenbud, G. Fløystad, and J. Weyman. The existence of equivariant pure free resolutions. *Ann. Inst. Fourier (Grenoble)*, 61(3):905–926, 2011. 4

[ES09] D. Eisenbud and F.-O. Schreyer. Betti numbers of graded modules and cohomology of vector bundles. *J. Amer. Math. Soc.*, 22(3):859–888, 2009. 4

[Wey03] J. Weyman. *Cohomology of vector bundles and syzygies*, volume 149 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2003. 5

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