PROBLEM SET FOR WORKSHOP OF REPRESENTATION THEORY AND SYZYGIES

1. Pieri resolutions

Sometimes complexes can be constructed (and the maps determined) using Pieri rules. See [SW11, Section 2]. Computational help: `PieriMaps` package of `Macaulay2`. There is a recent paper [HMS21] that has a lot of examples.

2. More examples of the Kempf-Lascoux-Weyman technique

(1) Eisenbud-Schreyer [ES09]. Characteristic-free way of construction of pure resolutions. Understand the proof of Theorem 5.1. If you understand this, understand the proof of the similar result Theorem 6.4 for constructing vector bundles.
(2) Kummini-Lakshmibai-Sastry-Seshadri[KLSS15]. Let $w \in S_n$ and $1 \leq d \leq n-1$. Let O^- be the opposite big cell in $\mathrm{Grass}(d,n)$. Let X_w be the Schubert variety in $\mathrm{Grass}(d,n)$ given by w. Let $Y_w = X_w \cap O^-$. This paper calculates the free resolution of the ideal of Y_w in the ring of regular functions on O^-. Theorem 3.7, Corollary 3.9 and Theorem 4.2 establish the desingularization and the direct images. The necessary cohomology calculations are in Section 5. There are some examples in Section 6.
(3) Rank varieties [Wey03, Chapter 7]. Section 7.1 has the definition, and proof of the fact that rank varieties have rational singularities and a description of the defining ideals.
(4) Nilpotent orbit closures [Wey03, Chapter 8]. Section 8.1. For background and some preliminary results, see [ES89].

3. Syzygies of embeddings

Let $\Bbbk$ be an algebraically closed field. Let X be a projective variety over $\Bbbk$ and $\mathcal{L}$ a very ample line bundle on X. Let $\phi : X \longrightarrow \mathbb{P}^n_{\Bbbk}$ (where $n = \mathrm{rk}_{\Bbbk} \mathrm{H}^0(X, \mathcal{L}) - 1$) be the embedding given by $\mathcal{L}$. There are two rings one can associate to X. Consider the coordinate ring $S = \Bbbk[x_0, \dots, x_n]$ of $\mathbb{P}^n_{\Bbbk}$. The *ideal* of X is $\{f \in S \mid f(p) = 0 \text{ for all } p \in X\}$. The *coordinate ring* of X is $S_X := S/I_X$. There is also another ring: $\tilde{S}_X := \oplus_{m \in \mathbb{N}} \mathrm{H}^0(X, \mathcal{L}^m)$. There is a natural map $S_X \longrightarrow \tilde{S}_X$, and, in nice situations, this map is an isomorphism. E.g., X is a Grassmannian and ϕ is the Plucker embedding.

Question 3.1. What is a natural generating set of I_X? What is a minimal free resolution of I_X?

An answer is unknown, in general, even for Grassmannians.

(1) [GKR07] Sections 1 and 2.

4. Computational techniques

Pick your favourite computer algebra system that has the ability to do Schur functors. Re-engineer the code and understand how these calculations are implemented.

References

[ES89] D. Eisenbud and D. Saltman. Rank varieties of matrices. In *Commutative algebra (Berkeley, CA, 1987)*, volume 15 of *Math. Sci. Res. Inst. Publ.*, pages 173–212. Springer, New York, 1989. 1

[ES09] D. Eisenbud and F.-O. Schreyer. Betti numbers of graded modules and cohomology of vector bundles. *J. Amer. Math. Soc.*, 22(3):859–888, 2009. 1

17/12/2023 10:10.

[GKR07] A. L. Gorodentsev, A. S. Khoroshkin, and A. N. Rudakov. On syzygies of highest weight orbits. In *Moscow Seminar on Mathematical Physics. II*, volume 221 of *Amer. Math. Soc. Transl. Ser. 2*, pages 79–120. Amer. Math. Soc., Providence, RI, 2007. https://arxiv.org/abs/math/0602316. 1

[HMS21] M. Hunziker, J. A. Miller, and M. Sepanski. Explicit Pieri inclusions. *Electron. J. Combin.*, 28(3):Paper No. 3.49, 51, 2021. 1

[KLSS15] M. Kummini, V. Lakshmibai, P. Sastry, and C. S. Seshadri. Free resolutions of some Schubert singularities. *Pacific J. Math.*, 279(1-2):299–328, 2015. arXiv:1504.04415v1 [math.AG]. 1

[SW11] S. V. Sam and J. Weyman. Pieri resolutions for classical groups. *J. Algebra*, 329:222–259, 2011. 1

[Wey03] J. Weyman. *Cohomology of vector bundles and syzygies*, volume 149 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2003. 1