Borel-Weil-Bott theorem and geometry of Schubert varieties

Lectures by Shrawan Kumar during June, 2012
(MRC Meeting, Snowbird, Utah)

1 Introduction

We take the base field to be the field of complex numbers in these lectures. The varieties are, by definition, quasi-projective, reduced (but not necessarily irreducible) schemes.

Let G be a semisimple, simply-connected, complex algebraic group with a fixed Borel subgroup B, a maximal torus $H \subset B$, and associated Weyl group W. (Recall that a Borel subgroup is any maximal connected, solvable subgroup; any two of which are conjugate to each other.) For any $w \in W$, we have the Schubert variety $X_w := \overline{BwB/B} \subset G/B$. Also, let $X(H)$ be the group of characters of H and $X(H)_+$ the semigroup of dominant characters. For any $\lambda \in X(H)$, we have the homogeneous line bundle $\mathcal{L}(\lambda)$ on G/B (cf. Section 5) and its restriction (denoted by the same symbol) to any X_w.

The Lie algebras of G, B, and H are given by $\mathfrak{g}$, $\mathfrak{b}$, and $\mathfrak{h}$, respectively. For a fixed B, any subgroup $P \subset G$ containing B is called a *standard parabolic.*

The aim of these talks is to prove the following well-known results on the geometry and cohomology of Schubert varieties. Extension of these results to a connected reductive group is fairly straight forward.

(1) Borel-Weil theorem and its generalization to the Borel-Weil-Bott theorem.

(2) Any Schubert variety X_w is normal, and has rational singularities (in particular, is Cohen-Macaulay).

(3) For any $\lambda \in X(H)_+$, the linear system on X_w given by $\mathcal{L}(\lambda+\rho)$ embeds X_w as a projectively normal and projectively Cohen-Macaulay variety, where ρ is the half sum of positive roots.

(4) For any $\lambda \in X(H)_+$, we have

$$H^p(X_w, \mathcal{L}(\lambda)) = 0, \text{ for all } p > 0.$$

(5) For any $\lambda \in X(H)_+$ and $v \leq w \in W$, the canonical restriction map

$$H^0(X_w, \mathcal{L}(\lambda)) \to H^0(X_v, \mathcal{L}(\lambda))$$

is surjective.

(6) The Demazure character formula holds for the Demazure submodules (cf. Theorem 33 for the precise statement).

Proof of (1) is given in Sections (6)-(7).

One uniform and beautiful proof of the above results (2)-(6) was obtained via using the characteristic $p > 0$ methods (specifically the Frobenius splitting methods; cf. [BK, Chapters 2 and 3]).

Another uniform proof of the above results (including in the Kac-Moody setting) using only characteristic 0 methods was obtained by Kumar [K1]. There are various other proofs of these results including in characteristic $p > 0$ (see Remark 36). Kumar's proof of the above results relied on the following fundamental cohomology vanishing. (In fact, his result was more general and also worked in the Kac-Moody setting, but the following weaker version is enough for our applications in this note.)

For any sequence of simple reflections $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n})$ (called a word), let $Z_{\mathfrak{w}}$ be the associated Bott-Samelson-Demazure-Hansen variety, and for any $1 \leq j \leq n$, let $Z_{\mathfrak{w}(j)}$ be the divisor of $Z_{\mathfrak{w}}$ defined in Section 10. Also, for any $\lambda \in X(H)$, we have the line bundle $\mathcal{L}_{\mathfrak{w}}(\lambda)$ on $Z_{\mathfrak{w}}$ (cf. Section 10).

Theorem 1. *For any word $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n})$ and any $\lambda \in X(H)_+$,*

$$(a) \quad H^p\left(Z_{\mathfrak{w}}, \mathcal{O}_{Z_{\mathfrak{w}}}\left[-Z_{\mathfrak{w}(n)}\right] \otimes \mathcal{L}_{\mathfrak{w}}(\lambda)\right) = 0, \text{ for all } p > 0.$$

Also,

$$(b) \quad H^p(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) = 0, \text{ for all } p > 0.$$

Following Brion [B], we give a very short and simple proof of the above theorem using the Kawamata-Viehweg vanishing theorem (cf. Theorem 17). Once we have the above theorem, all the above stated results (2) - (6) follow by fairly standard arguments, which we give in Sections (11)- (12). *Thus, we have made this note self-contained.* We should mention that apart from the original proof of the above theorem (rather a generalization of it) due to Kumar (cf. [K1, Proposition 2.3]), there is another proof (of the generalization valid in characteristic $p > 0$ as well) due to Lauritzen-Thomsen using the Frobenius splitting methods (cf. [BK, Theorem 3.1.4]).

2 Representations of G

Let $R \subset \mathfrak{h}^*$ denote the set of roots of $\mathfrak{g}$. Recall,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha, \text{ where } \mathfrak{g}_\alpha := \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

Our choice of B gives rise to R^+, the set of positive roots, such that

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha.$$

We let $\{\alpha_1, \dots, \alpha_\ell\} \subset \mathfrak{h}^*$ be the simple roots and let $\{\alpha_1^\vee, \dots, \alpha_\ell^\vee\} \subset \mathfrak{h}$ be the simple coroots, where $\ell := \dim \mathfrak{h}$ (called the *rank* of $\mathfrak{g}$).

Elements of $X(H) := \mathrm{Hom}(H, \mathbb{C}^*)$ are called *integral weights*, and can be identified with

$$\mathfrak{h}^*_\mathbb{Z} = \{\lambda \in \mathfrak{h}^* : \lambda(\alpha_i^\vee) \in \mathbb{Z}, \forall\, i\},$$

by taking derivatives. The dominant integral weights $X(H)_+$ are those integral weights $\lambda \in X(H)$ such that $\lambda(\alpha_i^\vee) \geq 0$, for all i.

We let $V(\lambda)$ denote the irreducible G-module with highest weight $\lambda \in X(H)_+$. Then, $V(\lambda)$ has a unique B-stable line such that H acts on this line by λ. This gives a one-to-one correspondence between the set of isomorphism classes of irreducible finite dimensional algebraic representations of G and $X(H)_+$.

3 Tits system

Let $N = N_G(H)$ be the normalizer of H in G, and let $W = N/H$ be the Weyl group, which acts on H by conjugation. For each $i = 1, \dots, \ell$, consider the subalgebra

$$\mathfrak{sl}_2(i) := \mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{-\alpha_i} \oplus \mathbb{C}\,\alpha_i^\vee \subset \mathfrak{g}.$$

There is an isomorphism of Lie algebras $\mathfrak{sl}_2 \to \mathfrak{sl}_2(i)$, taking $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to $\mathfrak{g}_{\alpha_i}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ to $\mathfrak{g}_{-\alpha_i}$, and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ to $\alpha_i^\vee$. This isomorphism gives rise to a homomorphism $SL_2 \to G$. Let $\overline{s_i}$ denote the image of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in G. Then, $\overline{s_i} \in N$ and $S = \{s_i\}_{i=1}^\ell$ generates W as a group, where s_i denotes the image of $\overline{s_i}$ under $N \to N/H$. These $\{s_i\}$ are called *simple reflections*. For details about the Weyl group, see [Hu, §24,27].

The conjugation action of W on H gives rise to an action on $\mathfrak{h}$ via taking derivatives and also on $\mathfrak{h}^*$ by taking duals. Below are explicit formulae for these induced actions:

$$\begin{aligned} s_j : \mathfrak{h} \to \mathfrak{h} \quad &: \quad h \mapsto h - \alpha_j(h)\alpha_j^\vee \\ s_j : \mathfrak{h}^* \to \mathfrak{h}^* \quad &: \quad \beta \mapsto \beta - \beta(\alpha_j^\vee)\alpha_j. \end{aligned}$$

Theorem 2. *The quadruple (G, B, N, S) forms a Tits system (also called a BN-pair), i.e., the following are true:*

(a) $H = B \cap N$ and S generates W as a group;

(b) B and N generate G as a group;

(c) For every i, $s_iBs_i \not\subseteq B$;

(d) For every $1 \leq i \leq \ell$ and $w \in W$, $(Bs_iB)(BwB) \subset (Bs_iwB) \cup (BwB)$.

There are many consequences of this theorem. For example, (W, S) is a Coxeter group. In particular, there is a length function on W, denoted by $\ell : W \to \mathbb{Z}_+$. For any $w \in W$, $\ell(w)$ is defined to be the minimal $k \in \mathbb{Z}_+$ such that $w = s_{i_1} \ldots s_{i_k}$ with each $s_{i_j} \in S$. A decomposition $w = s_{i_1} \ldots s_{i_k}$ is called a *reduced decomposition* if $\ell(w) = k$.

We also have the *Bruhat-Chevalley ordering*: $v \leq w$ if v can be obtained by deleting some simple reflections from a reduced decomposition of w.

Axiom (d) above can be refined:

$$(Bs_iB)(BwB) \subset Bs_iwB \text{ if } s_iw > w. \qquad (d')$$

Thus, if we have a reduced decomposition $w = s_{i_1} \ldots s_{i_k}$, then

$$BwB = (Bs_{i_1}B) \ldots (Bs_{i_k}B), \qquad (1)$$

which can be obtained from (d') by inducting on $k = \ell(w)$.

We also have the Bruhat decomposition:

$$G = \bigsqcup_{w \in W} BwB.$$

Theorem 3. *The set of standard parabolics are in one-to-one correspondence with subsets of the set* $[\ell] = \{1, \ldots, \ell\}$*. Specifically, if* $I \subset [\ell]$*, let*

$$P_I = \bigsqcup_{w \in \langle s_i : i \in I \rangle} BwB,$$

where $\langle s_i : i \in I \rangle$ *denotes the subgroup of* W *generated by the enclosed elements. Then,* $I \mapsto P_I$ *is the bijection.*

Sketch of the proof. By (1) and (d), P_I is clearly a subgroup containing B. Conversely, if $P \supset B$, then, by the Bruhat decomposition,

$$P = \bigsqcup_{w \in S_P} BwB,$$

for some subset $S_P \subset W$. Let I be the following set:

$$\{i \in [\ell] : s_i \text{ occurs in a reduced decomposition of some } w \in S_P\}.$$

From the above (specifically Axiom (d) and (d')), one can prove $P_I = P$. □

4 A fibration

We begin with a technical theorem.

Theorem 4. *Let* F *be a closed, algebraic subgroup of* G *and* X *be an* F*-variety. Then,* $E = G \times_F X$ *is a* G*-variety, where*

$$G \times_F X := G \times X / \sim \qquad \text{with} \qquad (gf, x) \sim (g, fx)$$

for all $g \in G$*,* $f \in F$*, and* $x \in X$*. The equivalence class of* (g, x) *is denoted by* $[g, x]$*. Then,* G *acts on* E *by:*

$$g' \cdot [g, x] = [g'g, x].$$

In particular, $G \times_F \{pt\} = G/F$ *is a variety. Furthermore, the map* $\pi : E \to G/F$ *given by* $[g, x] \mapsto gF$ *is a* G*-equivariant isotrivial fibration with fiber* X*.*

The variety structure on G/F can be characterized by the following universal property: if Y is any variety, then $G/F \to Y$ is a morphism if and only if the composition $G \to G/F \to Y$ is a morphism.

Now, B is a closed subgroup. To see this, we only need to show that $\overline{B}$ is solvable (B being a maximal solvable subgroup, it will follow that $B = \overline{B}$). Since the commutator $G \times G \to G$ is a continuous map, we have that $[\overline{F}, \overline{F}] \subset \overline{[F, F]}$, for any $F \subset G$. Using this fact and induction, $D_n(\overline{F}) \subset \overline{D_n(F)}$ for all n, where $D_n(F)$ denotes the n–th term in the derived series of F. Since $D_n(B)$ is trivial for large n, $D_n(\overline{B})$ becomes trivial for large n, and $\overline{B}$ is solvable. Thus, G/B is a variety. We wish to give an explicit realization of this variety structure. In the process, we will show that G/B is a projective variety.

Take any regular $\lambda \in X(H)_+$, so that $\lambda(\alpha_i^\vee) > 0$ for all i. The representation $G \to \mathrm{Aut}(V(\lambda))$ gives rise to a map

$$\pi : G/B \to \mathbb{P}V(\lambda), \quad g \mapsto [g \cdot v],$$

since $[v]$ is fixed by B, where v is a highest weight vector of $V(\lambda)$.

Claim. *π is a morphism and injective.*

Proof. π is a morphism since the composition $G \to G/B \to \mathbb{P}V(\lambda)$ is a morphism. To prove injectivity, it suffices to show that the stabilizer of $[v]$ is exactly B. Let P be the stabilizer. Now, $B \subset P$, so P is parabolic and hence $P = P_I$ for some $I \subset [\ell]$. If $I = \varnothing$, then $P = B$. Towards a contradiction, assume $s_i \in P$. Then, s_i stabilizes λ, but

$$s_i(\lambda) = \lambda - \lambda(\alpha_i^\vee)\alpha_i \neq \lambda,$$

since λ is regular. □

We claim $X = \pi(G/B)$ is closed. We will need the following theorem:

Theorem 5 (Borel fixed–point theorem, see §21 in [Hu])**.** *Let Z be a projective variety with an action of a solvable group. Then, Z has a fixed point.*

Clearly, $\overline{X}$ is G-stable as a subspace of $\mathbb{P}V(\lambda)$. It follows that $\overline{X} \setminus X$ is G-stable. Thus, $\overline{X} \setminus X$ has a B-fixed point which contradicts the existence of a unique highest weight vector. Thus, $\overline{X} \setminus X = \varnothing$ and X is closed.

Lastly, to show X and G/B are isomorphic varieties, we use the following proposition from algebraic geometry:

Proposition 6 (Theorem A.11 in [K2]). *If $f : Y \to Z$ is a bijective morphism between irreducible varieties and Z is normal, then f is an isomorphism.*

Observe that X is smooth because it is a G-orbit (G takes smooth points to smooth points and any variety has at least one smooth point). In particular, X is normal and $\pi : G/B \to X$ is an isomorphism.

5 Line bundles on G/B

For any $\lambda \in X(H)$, we define a line bundle $\mathcal{L}(\lambda)$ on G/B. Recall that $B = H \ltimes U$, where $U = [B, B]$ is the unipotent radical. Extend $\lambda : H \to \mathbb{C}^*$ to $\lambda : B \to \mathbb{C}^*$ by letting λ map U to 1. Consider $\mathbb{C} = \mathbb{C}_\lambda$ as a B-module, where $b \cdot z = \lambda(b)z$. Then, $\mathcal{L}(\lambda)$ is the line bundle: $\pi : G \times_B \mathbb{C}_{-\lambda} \to G/B$. Note that λ is made negative in the definition of $\mathcal{L}(\lambda)$.

The space of global sections

$$H^0(G/B, \mathcal{L}(\lambda)) := \{\sigma : G/B \to G \times_B \mathbb{C}_{-\lambda} : \pi \circ \sigma = \mathrm{id}\}$$

is a G-module, where the G-action is given by

$$(g \cdot \sigma)(g'B) = g\sigma(g^{-1}g'B).$$

Also, this module is finite dimensional since G/B is projective and any cohomology of coherent sheaves on projective varieties is finite dimensional.

6 Borel–Weil theorem

Theorem 7 (Borel–Weil theorem). *If $\lambda \in X(H)_+$, then there is a G-module isomorphism*

$$H^0(G/B, \mathcal{L}(\lambda)) \simeq V(\lambda)^*.$$

Proof. If we pull back the line bundle $\mathcal{L} = \mathcal{L}(\lambda)$ (given by $\pi : G \times_B \mathbb{C}_{-\lambda} \to G/B$) under $G \to G/B$, we get the bundle $\hat{\mathcal{L}}$, which is $\hat{\pi} : G \times \mathbb{C}_{-\lambda} \to G$. We wish to compare sections of these two bundles.

Sections of $\hat{\mathcal{L}}$ are of the form $\sigma(g) = (g, f(g))$, for some map $f : G \to \mathbb{C}_{-\lambda}$, so we can identify $H^0(G, \hat{\mathcal{L}})$ with $k[G] \otimes \mathbb{C}_{-\lambda}$. There is a B-action on $k[G]$ given by $(b \cdot f)(g) = f(gb)$. Acting diagonally, we get an action on $k[G] \otimes \mathbb{C}_{-\lambda}$.

Since $k[G]\otimes\mathbb{C}_{-\lambda}$ is naturally isomorphic to $k[G]$ (make the second coordinate 1), we get a new B-action on $k[G]$ given by

$$(b \cdot f)(g) = \lambda(b)^{-1} f(gb). \tag{2}$$

Use this action to make $H^0(G, \hat{\mathcal{L}})$ a B-module.

Sections of $\mathcal{L}$ are of the form $\sigma(gB) = [g, f(g)]$, for some map $f : G \to \mathbb{C}_{-\lambda}$. In order to insure that σ is well-defined, we require that for any $b \in B$:

$$[g, f(g)] = [gb, f(gb)] = [g, b \cdot f(gb)] = [g, \lambda(b)^{-1} f(gb)].$$

Therefore, f must have the property that $f(g) = \lambda(b)^{-1} f(gb)$ for all $b \in B$. It follows that

$$\left[H^0(G, \hat{\mathcal{L}})\right]^B = H^0(G/B, \mathcal{L}).$$

Now, it suffices to show $\left[H^0(G, \hat{\mathcal{L}})\right]^B \simeq V(\lambda)^*$.

Consider the following two $(G \times G)$-modules. First, $k[G]$ has a $(G \times G)$-action given by $((g_1, g_2) \cdot f)(g) = f(g_1^{-1} g g_2)$. Second, acting coordinate-wise, we have:

$$\mathcal{M} := \bigoplus_{\mu \in X(H)_+} V(\mu)^* \otimes V(\mu).$$

It follows from the Peter–Weyl theorem and Tanaka–Krein duality that these are isomorphic as $(G\times G)$-modules. The explicit isomorphism is $\Phi = \sum_\mu \Phi_\mu : \mathcal{M} \to k[G]$, where $\Phi_\mu : V(\mu)^* \otimes V(\mu) \to k[G]$ is given by

$$\Phi_\mu(f \otimes v)(g) = f(gv).$$

Furthermore, $k[G] \otimes \mathbb{C}_{-\lambda}$ has a $(G \times B)$-action given diagonally, where G is forgotten when $G \times B$ acts the second coordinate $\mathbb{C}_{-\lambda}$, and the action of $G \times B$ on $k[G]$ is the restriction of the $G \times G$ action given above. Since $H^0(G, \hat{\mathcal{L}}) \simeq k[G] \otimes \mathbb{C}_{-\lambda}$ as (left) G-modules, where G acts on $k[G]$ via $(g \cdot f)(x) = f(g^{-1}x)$, for $g, x \in G$ and $f \in k[G]$. Since the action of G on $k[G] \otimes \mathbb{C}_{-\lambda}$ commutes with the B-action given by equation (2), we get an induced G-action on the space of B-invariants:

$$\begin{aligned}
\left[H^0(G,\hat{\mathcal{L}})\right]^B &\simeq [k[G]\otimes \mathbb{C}_{-\lambda}]^B \\
&\simeq \bigoplus_{\mu\in X(H)_+} [V(\mu)^*\otimes V(\mu)\otimes \mathbb{C}_{-\lambda}]^B \\
&\simeq \bigoplus_{\mu\in X(H)_+} V(\mu)^*\otimes [V(\mu)\otimes \mathbb{C}_{-\lambda}]^B \\
&\simeq \bigoplus_{\mu\in X(H)_+} V(\mu)^*\otimes [\mathbb{C}_{\mu}\otimes \mathbb{C}_{-\lambda}]^H \\
&\simeq V(\lambda)^*,
\end{aligned}$$

since $\mathbb{C}_\mu \otimes \mathbb{C}_{-\lambda}$ will only have H-invariants if $\mu = \lambda$. □

It follows from the next section that the higher cohomology vanishes; that is, for $\lambda \in X(H)_+$ and $i \geq 1$, $H^i(G/B, \mathcal{L}(\lambda)) = 0$.

7 Borel–Weil–Bott theorem

Let ρ be half the sum of the positive roots. Since G is simply-connected, $\rho \in X(H)_+$. Also, ρ has the property that $\rho(\alpha_i^\vee) = 1$ for all i. We will need a shifted action of the Weyl group on $\mathfrak{h}^*$ given by:

$$w \star \lambda = w(\lambda + \rho) - \rho.$$

Theorem 8 (Borel–Weil–Bott)**.** *If $\lambda \in X(H)_+$ and $w \in W$, then*

$$H^p(G/B, \mathcal{L}(w\star\lambda)) = \begin{cases} V(\lambda)^* & \text{if } p = \ell(w) \\ 0 & \text{if } p \neq \ell(w) \end{cases}.$$

Before we prove this theorem, we need to establish a number of results. For any i, let P_i denote the minimal parabolic subgroup $P_i = B \sqcup Bs_iB$. In what follows, if M is a B-module, the notation $H^p(G/B, M)$ is the p-th sheaf cohomology for the sheaf of sections of the bundle $G \times_B M \to G/B$.

Lemma 9. *If M is a P_i-module, then $H^p(G/B, M \otimes \mathbb{C}_\mu) = 0$, for all $p \geq 0$ and any $\mu \in X(H)$ such that $\mu(\alpha_i^\vee) = 1$.*

Proof. Apply the Leray–Serre spectral sequence to the fibration $G/B \to G/P_i$ with fiber P_i/B and the vector bundle on G/B corresponding to the B-module $M \otimes \mathbb{C}_\mu$. Thus,

$$E_2^{p,q} = H^p(G/P_i, H^q(P_i/B, M \otimes \mathbb{C}_\mu)) \Longrightarrow H^*(G/B, M \otimes \mathbb{C}_\mu).$$

If we can show $E_2^{p,q} = 0$, then we are done.

It suffices to show $H^q(P_i/B, M \otimes \mathbb{C}_\mu)$ vanishes for all $q \geq 0$. By the next exercise, we have

$$H^q(P_i/B, M \otimes \mathbb{C}_\mu) \simeq M \otimes H^q(P_i/B, \mathbb{C}_\mu),$$

since M is a P_i-module by assumption. Since $P_i/B \simeq SL_2(i)/B(i) \simeq \mathbb{P}^1$, where $SL_2(i)$ is the subgroup of P_i with Lie algebra $\mathfrak{sl}_2(i)$ and $B(i)$ is the standard Borel subgroup of $SL_2(i)$, we have that

$$H^q(P_i/B, \mathbb{C}_\mu) \simeq H^q(\mathbb{P}^1, \mathcal{O}(-\mu(\alpha_i^\vee))) = H^q(\mathbb{P}^1, \mathcal{O}(-1)),$$

which is known to be zero (for example, [H, Ch. III, Theorem 5.1]). □

Exercise 10. For any closed subgroup $F \subset G$, if M is a G-module, then $G \times_F M \to G/F$ is a trivial vector bundle.

Proposition 11. *If for some i, $\mu \in X(H)$ has the property that $\mu(\alpha_i^\vee) \geq -1$, then for all $p \geq 0$,*

$$H^p(G/B, \mathcal{L}(\mu)) \simeq H^{p+1}(G/B, \mathcal{L}(s_i \star \mu)).$$

Proof. First, consider the case where $\mu(\alpha_i^\vee) \geq 0$. Let $X_i := P_i/B \simeq \mathbb{P}^1$ and $\mathcal{H} := H^0(X_i, \mathcal{L}(\mu + \rho))$. It can easily be seen (by using the definition of the action of P_i on $\mathcal{H}$) that the action of the unipotent radical U_i of P_i is trivial on $\mathcal{H}$. Moreover, P_i/U_i is isomorphic with the subgroup $\widehat{SL_2(i)}$ of G generated by $SL_2(i)$ and H. Thus, by the Borel-Weil theorem for $G = \widehat{SL_2(i)}$, we get $\mathcal{H} \simeq V_i(\mu+\rho)^*$, as $\widehat{SL_2(i)}$-modules, where $V_i(\mu+\rho)$ is the irreducible $\widehat{SL_2(i)}$-module with highest weight $\mu + \rho$. (Even though we stated the Borel-Weil theorem for semisimple, simply-connected groups, the same proof gives the result for any connected, reductive group.) Thus, we have the weight space decomposition (as H-modules):

$$\mathcal{H} \simeq V_i(\mu + \rho)^* = \bigoplus_{j=0}^{(\mu+\rho)(\alpha_i^\vee)} \mathbb{C}_{-(\mu+\rho)+j\alpha_i}.$$

There is a short exact sequence of B-modules:

$$0 \longrightarrow K \longrightarrow \mathcal{H} \longrightarrow \mathbb{C}_{-(\mu+\rho)} \longrightarrow 0,$$

where K, by definition, is the kernel of the projection. Tensoring with $\mathbb{C}_\rho$, we get the following exact sequence of B-modules:

$$0 \longrightarrow K \otimes \mathbb{C}_\rho \longrightarrow \mathcal{H} \otimes \mathbb{C}_\rho \longrightarrow \mathbb{C}_{-\mu} \longrightarrow 0.$$

Passing to the long exact cohomology sequence, we get:

$$\cdots \to H^p\left(G/B, \mathcal{H} \otimes \mathbb{C}_\rho\right) \to H^p(G/B, \mathbb{C}_{-\mu}) \to \\ H^{p+1}(G/B, K \otimes \mathbb{C}_\rho) \to H^{p+1}\left(G/B, \mathcal{H} \otimes \mathbb{C}_\rho\right) \to \cdots.$$

By the previous lemma, $H^p(G/B, \mathcal{H} \otimes \mathbb{C}_\rho) = 0$ for all p. Thus,

$$H^p(G/B, \mathcal{L}(\mu)) = H^p(G/B, \mathbb{C}_{-\mu}) \simeq H^{p+1}(G/B, K \otimes \mathbb{C}_\rho). \tag{3}$$

Consider another short exact sequence of B-modules:

$$0 \longrightarrow \mathbb{C}_{-s_i(\mu+\rho)} \longrightarrow K \longrightarrow M \longrightarrow 0,$$

where M is just the cokernal of the inclusion. In particular, as H-modules,

$$M = \bigoplus_{j=1}^{(\mu+\rho)(\alpha_i^\vee)-1} \mathbb{C}_{-(\mu+\rho)+j\alpha_i},$$

so it may be regarded as a P_i-module. Then, as B-modules, we can tensor with $\mathbb{C}_\rho$ to arrive at the following exact sequence:

$$0 \longrightarrow \mathbb{C}_{-s_i \star \mu} \longrightarrow K \otimes \mathbb{C}_\rho \longrightarrow M \otimes \mathbb{C}_\rho \longrightarrow 0.$$

Again, passing to the long exact sequence, we see:

$$\cdots \to H^p(G/B, M \otimes \mathbb{C}_\rho) \to H^{p+1}(G/B, \mathbb{C}_{-s_i \star \mu}) \to \\ H^{p+1}(G/B, K \otimes \mathbb{C}_\rho) \to H^{p+1}(G/B, M \otimes \mathbb{C}_\rho) \to \cdots.$$

By the previous lemma, $H^p(G/B, M \otimes \mathbb{C}_\rho) = 0$ for all p. Thus,

$$H^{p+1}(G/B, \mathcal{L}(s_i \star \mu)) = H^{p+1}(G/B, \mathbb{C}_{-s_i \star \mu}) \simeq H^{p+1}(G/B, K \otimes \mathbb{C}_\rho). \tag{4}$$

Combining equations (3) and (4), we get the proposition in the case where $\mu(\alpha_i^\vee) \geq 0$.

For the case that $\mu(\alpha_i^\vee) = -1$, we have that $s_i \star \mu = \mu$, so the statement reduces to proving that $H^p(G/B, \mathcal{L}(\mu)) = 0$, for all p. In this case, $K = 0$. From the isomorphism $\mathcal{H} \otimes \mathbb{C}_\rho \simeq \mathbb{C}_{-\mu}$, we conclude $H^p(G/B, \mathcal{L}(\mu)) \simeq H^p(G/B, \mathcal{H} \otimes \mathbb{C}_\rho)$ which vanishes by the previous lemma. □

Corollary 12. *If $\mu \in X(H)_+$ and $w \in W$, then for all $p \in \mathbb{Z}$, as G-modules:*

$$H^p(G/B, \mathcal{L}(\mu)) \simeq H^{p+\ell(w)}(G/B, \mathcal{L}(w \star \mu)).$$

Proof. We induct on $\ell(w)$. Assume the above for all $v \in W$ such that $\ell(v) < \ell(w)$, and write $w = s_i v$ for some $v < w$. Then,

$$H^p(G/B, \mathcal{L}(\mu)) \simeq H^{p+\ell(v)}(G/B, \mathcal{L}(v \star \mu)).$$

Now $(v \star \mu)(\alpha_i^\vee) = (\mu + \rho)(v^{-1}\alpha_i^\vee) - 1 \geq -1$, since $v^{-1}\alpha_i^\vee$ is a positive coroot and $\mu + \rho$ is dominant. So, applying Proposition 11, we get:

$$H^p(G/B, \mathcal{L}(\mu)) \simeq H^{p+\ell(v)+1}(G/B, \mathcal{L}(s_i \star (v \star \mu))) = H^{p+\ell(w)}(G/B, \mathcal{L}(w \star \mu)),$$

which is our desired result. □

We are now ready to prove the Borel–Weil–Bott theorem.

Proof of the Borel–Weil–Bott theorem. From the above corollary,

$$H^p(G/B, \mathcal{L}(w \star \lambda)) \simeq H^{p-\ell(w)}(G/B, \mathcal{L}(\lambda)).$$

We claim that $H^j(G/B, \mathcal{L}(\lambda)) = 0$ if $j \neq 0$. Indeed, if $j < 0$, this is true. Let w_0 denote the unique longest word in the Weyl group, so that $\ell(w_0) = \dim(G/B)$. If $j > 0$, then by Corollary 12,

$$H^j(G/B, \mathcal{L}(\lambda)) \simeq H^{j+\dim(G/B)}(G/B, \mathcal{L}(w_0 \star \lambda)) = 0.$$

This implies

$$H^p(G/B, \mathcal{L}(w \star \lambda)) = \begin{cases} H^0(G/B, \mathcal{L}(\lambda)) & \text{if } p = \ell(w) \\ 0 & \text{if } p \neq \ell(w) \end{cases},$$

which is our desired result, by the Borel–Weil theorem. □

Exercise 13. Show that for any μ not contained in $W\star(X(H)_+)$, $H^p(G/B, \mathcal{L}(\mu)) = 0$, for all $p \geq 0$.

8 Schubert varieties

For any $w \in W$, let $X_w := \overline{BwB/B} \subset G/B$ denote the corresponding *Schubert variety*. This variety is projective and irreducible of dimension $\ell(w)$. By the Bruhat decomposition, we have the following decomposition of X_w:

$$X_w = \bigsqcup_{v \leq w} BvB/B.$$

9 Bott–Samelson–Demazure–Hansen variety

Let $\mathfrak{W}$ be the set of all ordered sequences $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n})$, $n \geq 0$, of simple reflections, called *words*. For any such word, define the *Bott–Samelson–Demazure–Hansen variety* (for short *BSDH variety*) as follows: if $n = 0$ (thus, $\mathfrak{w}$ is the empty sequence), $Z_{\mathfrak{w}}$ is a point. For $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n})$, with $n \geq 1$, define

$$Z_{\mathfrak{w}} = P_{i_1} \times \cdots \times P_{i_n}/B^n,$$

where the product group B^n acts on $P_{\mathfrak{w}} := P_{i_1} \times \cdots \times P_{i_n}$ from the right via:

$$(p_1, \dots, p_n) \cdot (b_1, \dots, b_n) = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{n-1}^{-1} p_n b_n).$$

We denote the $B^{\times n}$ orbit of $(p_1, \dots, p_n)$ by $[p_1, \dots, p_n]$. This action is free and proper. The group P_{i_1} (in particular, B) acts on $Z_{\mathfrak{w}}$ via its left multiplication on the first factor.

Lemma 14. *$Z_{\mathfrak{w}}$ is a smooth projective variety.*

Sketch of the proof. Induct on the length of $\mathfrak{w}$, where length refers to the number of terms in the sequence. Let $\mathfrak{v}$ be the last $n-1$ terms in the sequence $\mathfrak{w}$, so that $\mathfrak{w} = (s_{i_1}) \cup \mathfrak{v}$, where order is preserved when taking the union.

Let

$$\pi : Z_{\mathfrak{w}} \simeq P_{i_1} \times_B Z_{\mathfrak{v}} \longrightarrow Z_{(s_{i_1})} = P_{i_1}/B \simeq \mathbb{P}^1$$

be the map $[p_1, \dots, p_n] \mapsto p_1 B$. This map has fiber $Z_{\mathfrak{v}}$ and since it is a fibration, we get that $Z_{\mathfrak{w}}$ is smooth. Furthermore, $Z_{\mathfrak{w}}$ is complete since $\mathbb{P}^1$ is complete and the fibers of π are complete by induction.

Furthermore, it is a trivial fibration restricted to $\mathbb{P}^1 \smallsetminus \{x\}$, for any $x \in \mathbb{P}^1$. Hence, projectivity follows from the Chevalley–Kleiman criterion asserting that a smooth complete variety is projective if and only if any finite set of points is contained in an affine open subset. $\square$

There is a map $\xi : \mathfrak{W} \to W$ given by $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n}) \mapsto s_{i_1} \cdots s_{i_n}$. For any $\mathfrak{w} \in \mathfrak{W}$, we say $\mathfrak{w}$ is reduced if $s_{i_1} \cdots s_{i_n}$ is a reduced decomposition of $\xi(w)$.

For $\mathfrak{w} \in \mathfrak{W}$, consider the map $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \to G/B$ given by $[p_1, \ldots, p_n] \mapsto p_1 \cdots p_n B$.

Lemma 15. *If $\mathfrak{w}$ is reduced, then $\theta_{\mathfrak{w}}(Z_{\mathfrak{w}}) = X_{\xi(\mathfrak{w})}$. Moreover, $\theta_{\mathfrak{w}}$ is a desingularization of $X_{\xi(\mathfrak{w})}$; that is, it is birational and proper.*

If $\mathfrak{w}$ is not reduced, then $\theta_{\mathfrak{w}}(Z_{\mathfrak{w}})$ is NOT equal to $X_{\xi(\mathfrak{w})}$ in general.

Sketch of the proof. The open subset of $Z_{\mathfrak{w}}$ given by

$$(Bs_{i_1}B) \times \cdots \times (Bs_{i_n}B)/B^n$$

maps isomorphically to the open cell BwB/B by (1) of Section (3). $\square$

10 A fundamental cohomology vanishing theorem

For any $1 \le j \le n$, define $\mathfrak{w}(j) = (s_{i_1}, \ldots, \widehat{s_{i_j}}, \ldots, s_{i_n})$. The variety $Z_{\mathfrak{w}(j)}$ embeds into $Z_{\mathfrak{w}}$ via:

$$[p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n] \mapsto [p_1, \ldots, p_{j-1}, 1, p_{j+1}, \ldots, p_n].$$

Denote also by $Z_{\mathfrak{w}(j)}$ the image of this map. These are divisors in $Z_{\mathfrak{w}}$.

For $\lambda \in X(H)$, let $\mathcal{L}_{\mathfrak{w}}(\lambda) = \theta_{\mathfrak{w}}^*(\mathcal{L}(\lambda))$ be the pull-back of $\mathcal{L}(\lambda)$ under the map $\theta_{\mathfrak{w}}$.

The following result is fundamental to the notes. This is a special case of a more general result proved by Kumar (cf. [K1, Proposition 2.3] or [K2, Theorem 8.1.8]).

Theorem 16. *For any word $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$ and any $\lambda \in X(H)_+$,*

$$(a) \quad H^p\left(Z_{\mathfrak{w}}, \mathcal{O}_{Z_{\mathfrak{w}}}\left[-Z_{\mathfrak{w}(n)}\right] \otimes \mathcal{L}_{\mathfrak{w}}(\lambda)\right) = 0, \;\; \text{for all} \, p > 0.$$

Also,

$$(b) \quad H^p(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) = 0, \;\; \text{for all} \, p > 0.$$

To prove the above theorem, we make use of the following Kawamata-Viehweg vanishing theorem (cf. [D, Theorem 7.21]).

Theorem 17. *Let X be a smooth projective variety. Let D be a nef and big $\mathbb{Q}$-divisor on X and $\Delta = \sum a_i X_i$ a $\mathbb{Q}$-divisor on X with simple normal crossings such that $0 \leq a_i < 1$. Assume further that $D + \Delta$ is an integral divisor. Then,*

$$H^p(X, K_X + D + \Delta) = 0, \ \ \textit{for all} \, p > 0,$$

where K_X is a canonical divisor of X.

We recall the following from [BK, Proposition 2.2.2]. Various properties of $Z_{\mathfrak{w}}$ are summarized in [BK, §2.2.1]; in particular, it is smooth.

Proposition 18. *The canonical line bundle of $Z_{\mathfrak{w}}$ is given by:*

$$\omega_{Z_{\mathfrak{w}}} \simeq \mathcal{O}_{Z_{\mathfrak{w}}} \left[-\sum_{j=1}^{n} Z_{\mathfrak{w}(j)} \right] \otimes \mathcal{L}_{\mathfrak{w}}(-\rho),$$

where ρ is the usual half sum of positive roots.

Now, we are ready to prove Theorem 16 (cf. [B, Proof of Theorem 2.3.1]).

Proof of Theorem 16. We first prove (a). Consider the projection

$$\pi : Z_{\mathfrak{w}} \to Z_{\mathfrak{w}(n)}, \ [p_1, \ldots, p_n] \mapsto [p_1, \ldots, p_{n-1}].$$

By [BK, Exercise 3.1.E.3(f)], there exist positive integers $a_1, \ldots, a_{n-1}$ such that $\mathcal{O}_{Z_{\mathfrak{v}}} \left[\sum_{j=1}^{n-1} a_j Z_{\mathfrak{v}(j)} \right]$ is an ample line bundle on $Z_{\mathfrak{v}}$, where $\mathfrak{v} := \mathfrak{w}(n)$. Moreover, $\mathcal{L}_{\mathfrak{w}}(\rho)$ has degree 1 along the fibers of π. Thus, there exists a large enough positive integer a such that

$$\mathcal{O}_{Z_{\mathfrak{w}}} \left[\sum_{j=1}^{n-1} a a_j Z_{\mathfrak{w}(j)} \right] \otimes \mathcal{L}_{\mathfrak{w}}(\rho)$$

is an ample line bundle on $Z_{\mathfrak{w}}$.

Take any integer N bigger than each aa_j and take

$$D = \sum_{j=1}^{n-1} \frac{aa_j}{N} Z_{\mathfrak{w}(j)} + \frac{1}{N} \hat{\mathcal{L}}_{\mathfrak{w}}(\rho) + \hat{\mathcal{L}}_{\mathfrak{w}}(\lambda) + (1 - \frac{1}{N}) \hat{\mathcal{L}}_{\mathfrak{w}}(\rho), \ \Delta = \sum_{j=1}^{n-1} (1 - \frac{aa_j}{N}) Z_{\mathfrak{w}(j)},$$

where $\hat{\mathcal{L}}_{\mathfrak{w}}(\lambda)$ denotes a divisor on $Z_{\mathfrak{w}}$ representing the line bundle $\mathcal{L}_{\mathfrak{w}}(\lambda)$.

Since $\sum_{j=1}^{n-1} aa_j Z_{\mathfrak{w}(j)} + \hat{\mathcal{L}}_{\mathfrak{w}}(\rho)$ is an ample divisor and $\mathcal{L}_{\mathfrak{w}}(\lambda)$ is globally generated for any dominant weight λ, we get that ND is an ample divisor (cf. [H, Exercise 7.5(a), Chap. II]). In particular, D is a nef and big $\mathbb{Q}$-divisor. Moreover, by [BK, §2.2.1], $\{Z_{\mathfrak{w}(j)}\}_{1\le j\le n}$ are nonsingular prime divisors with simple normal crossings in $Z_{\mathfrak{w}}$. Hence, Δ satisfies the assumptions of Theorem 17.

Finally, by Proposition 18,

$$K_{Z_{\mathfrak{w}}} + D + \Delta = -Z_{\mathfrak{w}(n)} + \hat{\mathcal{L}}_{\mathfrak{w}}(\lambda).$$

Thus, by Theorem 17, the (a)-part of Theorem 16 follows.

To prove the (b)-part, take an ample line bundle $\mathcal{O}_{Z_{\mathfrak{w}}}[\sum_{j=1}^{n} b_j Z_{\mathfrak{w}(j)}]$ for some $b_j > 0$. Now, take N larger than each b_j and define

$$D = \sum_{j=1}^{n} \frac{b_j}{N} Z_{\mathfrak{w}(j)} + \hat{\mathcal{L}}_{\mathfrak{w}}(\rho+\lambda), \;\; \Delta = \sum_{j=1}^{n} (1-\frac{b_j}{N}) Z_{\mathfrak{w}(j)}.$$

Again use Theorem 17 to conclude the (b)-part. □

As an immediate corollary of Theorem 16 (a), we get the following:

Corollary 19. *For any word $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$, and any $\lambda \in X(H)_+$, the canonical restriction map*

$$H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) \to H^0(Z_{\mathfrak{w}(n)}, \mathcal{L}_{\mathfrak{w}(n)}(\lambda))$$

is surjective.

11 Geometry of Schubert varieties

In this section we show that Schubert varieties are normal and have rational singularities (in particular, they are Cohen-Macaulay).

We recall the Zariski's Main Theorem, see, e.g., [H, Chap. III, Corollary 11.4 and its proof].

Theorem 20. *If $f : X \to Y$ is a birational projective morphism between irreducible varieties and X is smooth, then Y is normal if and only if $f_*\mathcal{O}_X = \mathcal{O}_Y$.*

Lemma 21. *If $f : X \to Y$ is a surjective morphism between projective varieties and $\mathcal{L}$ is an ample line bundle on Y such that $H^0(Y, \mathcal{L}^{\otimes d}) \to H^0(X, (f^*\mathcal{L})^{\otimes d})$ is an isomorphism for all large d, then $f_*\mathcal{O}_X = \mathcal{O}_Y$.*

For a proof see [K2, Lemma A.32].

For any $w \in W$, choose a reduced decomposition $w = s_{i_1} \cdots s_{i_n}$, with each $s_{i_j} \in S$, and take $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$. Such a $\mathfrak{w}$ is called a *reduced word.* Then, $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \to X_w$ is a desingularization (cf. Lemma 15).

By the last two results, in order to show that X_w is normal, it suffices to prove the following theorem:

Theorem 22. *If $\lambda \in X(H)_+$ and $w \in W$, then $H^0(X_w, \mathcal{L}(\lambda)) \to H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda))$ is an isomorphism.*

Before we give the proof, we recall the following Projection formula (cf. [H, Exercise 8.3 of Chap. III]:

Lemma 23. *If $f : X \to Y$ is any morphism of varieties, η is a vector bundle on Y, $\mathcal{S}$ is a quasi-coherent sheaf on X, then for all i:*

$$R^i f_*(\mathcal{S} \otimes f^*\eta) \simeq (R^i f_*\mathcal{S}) \otimes \eta.$$

Proof of Theorem 22. This map is clearly injective since $Z_{\mathfrak{w}} \twoheadrightarrow X_w$. Choose a reduced decomposition of the longest element $w_0 \in W$, $w_0 = s_{i_1} \cdots s_{i_N}$, each $s_{i_j} \in S$, $N = \dim(G/B) = |R^+|$, and let $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_N})$. For $0 \leq j \leq N$, let $w_j = s_{i_1} \cdots s_{i_j}$ and $\mathfrak{w}_j = (s_{i_1}, \ldots, s_{i_j})$. Consider the following diagram:

$$\begin{array}{ccc}
Z_{\mathfrak{w}_N} & \xrightarrow{\theta_{\mathfrak{w}_N}} & X_{w_N} = G/B \\
\uparrow & & \uparrow \\
Z_{\mathfrak{w}_{N-1}} & \xrightarrow{\theta_{\mathfrak{w}_{N-1}}} & X_{w_{N-1}} \\
\uparrow & & \uparrow \\
Z_{\mathfrak{w}_{N-2}} & \xrightarrow{\theta_{\mathfrak{w}_{N-2}}} & X_{w_{N-2}} \\
\uparrow & & \uparrow \\
\vdots & & \vdots
\end{array}$$

In this diagram, the horizontal arrows are surjective and the vertical arrows (which are the canonical inclusions) are injective. Passing to global sections, we get:

$$\begin{array}{ccc}
H^0(Z_{\mathfrak{w}_N}, \mathcal{L}_{\mathfrak{w}_N}(\lambda)) & \longleftarrow & H^0(X_{w_N}, \mathcal{L}(\lambda)) \\
\big\downarrow & & \big\downarrow \\
H^0(Z_{\mathfrak{w}_{N-1}}, \mathcal{L}_{\mathfrak{w}_{N-1}}(\lambda)) & \longleftarrow & H^0(X_{w_{N-1}}, \mathcal{L}(\lambda)) \\
\big\downarrow & & \big\downarrow \\
H^0(Z_{\mathfrak{w}_{N-2}}, \mathcal{L}_{\mathfrak{w}_{N-2}}(\lambda)) & \longleftarrow & H^0(X_{w_{N-2}}, \mathcal{L}(\lambda)) \\
\big\downarrow & & \big\downarrow \\
\vdots & & \vdots
\end{array}$$

In this diagram, the horizontal arrows are of course injective and the vertical arrows on the left are surjective by Corollary 19. Furthermore, by Lemma 23 (with $\mathcal{S} = \mathcal{O}_{Z_{\mathfrak{w}_N}}$ and $\eta = \mathcal{L}(\lambda)$) and Theorem 20, the top horizontal arrow is an isomorphism. Then, by a standard diagram chase, all of the horizontal arrows are isomorphisms.

Since $w_0 = w(w^{-1}w_0)$ and $\ell(w^{-1}w_0) = \ell(w_0) - \ell(w)$, a reduced decomposition of w_0 can always be obtained so that the first $\ell(w)$ terms of the decomposition give the word $\mathfrak{w}$. This completes the proof. □

Thus, using Theorems 20, 22 and Lemma 21, we get the following:

Corollary 24. *Any Schubert variety X_w is normal.*

Corollary 25. *For any $v \leq w$ and $\lambda \in X(H)_+$, the restriction map*

$$H^0(X_w, \mathcal{L}(\lambda)) \to H^0(X_v, \mathcal{L}(\lambda))$$

is surjective.

Proof. By the above proof, $H^0(G/B, \mathcal{L}(\lambda)) \to H^0(X_v, \mathcal{L}(\lambda))$ is surjective and hence so is $H^0(X_w, \mathcal{L}(\lambda)) \to H^0(X_v, \mathcal{L}(\lambda))$. □

An irreducible projective variety Y has *rational singularities* if for some desingularization $f : X \to Y$ we have that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and $R^i f_*\mathcal{O}_X = 0$ for all $i > 0$. This definition does not depend on the choice of a desingularization. (In characteristic $p > 0$, we also need to assume that $R^i f_*\omega_X = 0$, for the canonical bundle ω_X.) To prove that X_w has rational singularities, we use the following theorem of Kempf (cf. [K2, Lemma A.31]:

Theorem 26. *Let $f : X \to Y$ be a morphism of projective varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Y$. Assume there exists an ample line bundle $\mathcal{L}$ on Y such that $H^i(X, (f^*\mathcal{L})^{\otimes d}) = 0$ for all $i > 0$ and all large d. Then, $R^i f_*\mathcal{O}_X = 0$ for $i > 0$.*

Corollary 27. *Any Schubert variety X_w has rational singularities.*

Proof. In view of Corollary 24, it suffices to prove $H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(d\lambda)) = 0$ for all large d, for all $i > 0$, and some regular $\lambda \in X(H)_+$, which follows from Theorem 16. □

We recall the following general theorem (cf. [K2, Lemma A.38]:

Theorem 28. *Any projective variety Y which has rational singularities is Cohen-Macaulay.*

In fact, in this case, for any ample line bundle $\mathcal{L}$ on Y,

$$H^p(Y, \mathcal{L}^{-n}) = 0, \quad \text{for all } p < \text{ dim} Y \text{ and } n > 0.$$

Thus, we get:

Corollary 29. *Any Schubert variety X_w is Cohen-Macaulay.*

Another consequence of having rational singularities (which we will use in the next section) is given in the following two results.

Proposition 30. *Let Y be a projective variety with rational singularities. Then, for any desingularization $f : X \to Y$ and any vector bundle η on Y, $H^i(Y, \eta) \to H^i(X, f^*\eta)$ is an isomorphism for $i \geq 0$.*

Proof. Applying the Leray-Serre spectral sequence, we have

$$E_2^{p,q} = H^p(Y, R^q f_* f^*\eta) \Longrightarrow H^*(X, f^*\eta).$$

By the projection formula (with $\mathcal{S} = \mathcal{O}_X$),

$$R^q f_*(\mathcal{O}_X \otimes f^*\eta) \simeq (R^q f_*\mathcal{O}_X) \otimes \eta.$$

Since Y has rational singularities, $R^q f_*\mathcal{O}_X = 0$ for $q > 0$. Therefore, $E_2^{p,q} = 0$ for $q > 0$, and hence $H^p(Y, \eta) \simeq E_2^{p,0}$ for all p, and the result follows. □

Corollary 31. *For any* $\lambda \in X(H)$ *and* $i \geq 0$,

$$H^i(X_w, \mathcal{L}(\lambda)) \simeq H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)),$$

for any reduced word $\mathfrak{w}$ *with* $\xi(\mathfrak{w}) = w$, *where* $\xi(s_{i_1}, \ldots, s_{i_n}) := s_{i_1} \ldots s_{i_n}$.

In particular, for any $\lambda \in X(H)_+$, $H^i(X_w, \mathcal{L}(\lambda)) = 0$ *if* $i > 0$.

Proof. By Corollary 27 and Proposition 30, $H^i(X_w, \mathcal{L}(\lambda)) \simeq H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda))$, which vanishes by Theorem 16(b) for $\lambda \in X(H)_+$ and $i > 0$. □

As a consequence of the above corollary and Theorem 28, we get the following:

Corollary 32. *For any* $\lambda \in X(H)_+$, *the linear system on* X_w *given by* $\mathcal{L}(\lambda+\rho)$ *embeds* X_w *as a projectively normal and projectively Cohen-Macaulay variety.*

Proof. To prove the projective normality of X_w, by using its normality (cf. Corollary 24) and [H, Exercise 5.14(d) of Chap. II and Theorem 5.1 of Chap. III], it suffices to show that the canonical multiplication map (for $\lambda, \lambda' \in X(H)_+$)

$$H^0(X_w, \mathcal{L}(\lambda)) \otimes H^0(X_w, \mathcal{L}(\lambda')) \to H^0(X_w, \mathcal{L}(\lambda + \lambda'))$$

is surjective. By Corollary 25, to prove the above surjectivity, it suffices to show that

$$H^0(G/B, \mathcal{L}(\lambda)) \otimes H^0(G/B, \mathcal{L}(\lambda')) \to H^0(G/B, \mathcal{L}(\lambda + \lambda'))$$

is surjective. But, the above map is a G-module map (under the diagonal action of G on the domain) and $H^0(G/B, \mathcal{L}(\lambda + \lambda'))$ is an irreducible G-module and hence it is surjective. This proves the projective normality of X_w.

We now prove that X_w is projectively Cohen-Macaulay: Since X_w is projectively normal, in view of [E, Exercise 18.16], it suffices to show that

$$H^p(X_w, \mathcal{L}(n(\lambda + \rho))) = 0, \text{ for all } n \in \mathbb{Z} \text{ and } 0 < p < \dim X_w.$$

For any $p > 0$ and $n \geq 0$, this vanishing follows from Corollary 31. For any $p < \dim X_w$ and $n < 0$, the vanishing follows from Theorem 28 and Corollary 27. □

12 Demazure character formula

Let $w \in W$ and $\lambda \in X(H)_+$. The Demazure module $V_w(\lambda) \subset V(\lambda)$ is the B-submodule defined by $V_w(\lambda) = \mathcal{U}(\mathfrak{b}) \cdot V(\lambda)_{w\lambda}$, where $\mathcal{U}(\mathfrak{b})$ is the enveloping algebra of $\mathfrak{b}$ and $V(\lambda)_{w\lambda}$ is the weight space of $V(\lambda)$ with weight $w\lambda$. Observe that $V(\lambda)_{w\lambda}$ is one-dimensional. The formal character of $V_w(\lambda)$ is defined by

$$\text{ch } V_w(\lambda) = \sum_{\mu \in X(H)} \dim(V_w(\lambda)_\mu)\, e^\mu.$$

If $w = w_0$, then $V_w(\lambda) = V(\lambda)$. Therefore, ch $V_{w_0}(\lambda)$ is given by the Weyl character formula.

For an arbitrary $\mathfrak{w} \in \mathfrak{W}$, we need to introduce the Demazure operators $D_\mathfrak{w}$. For each simple reflection s_i, let $D_{s_i} : \mathbb{Z}[X(H)] \to \mathbb{Z}[X(H)]$ be the $\mathbb{Z}$-linear map given by:

$$D_{s_i}(e^\mu) = \frac{e^\mu - e^{s_i\mu - \alpha_i}}{1 - e^{-\alpha_i}}.$$

Given $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n}) \in \mathfrak{W}$, define $D_\mathfrak{w} : \mathbb{Z}[X(H)] \to \mathbb{Z}[X(H)]$ by

$$D_\mathfrak{w} = D_{s_{i_1}} \circ \cdots \circ D_{s_{i_n}}.$$

In what follows, we will also need $* : \mathbb{Z}[X(H)] \to \mathbb{Z}[X(H)]$ given by

$$* \, e^\mu = e^{-\mu},$$

and extended $\mathbb{Z}$-linearly.

Theorem 33. *For any reduced word $\mathfrak{w}$ and $\lambda \in X(H)_+$,*

$$\text{ch}\, V_{\xi(\mathfrak{w})}(\lambda) = D_\mathfrak{w}(e^\lambda).$$

Proof. The first step is to show $V_w(\lambda)^* \simeq H^0(X_w, \mathcal{L}(\lambda))$, for any $w \in W$.

By the Borel–Weil theorem, $V(\lambda)^* \simeq H^0(G/B, \mathcal{L}(\lambda))$. The isomorphism $\phi : V(\lambda)^* \to H^0(G/B, \mathcal{L}(\lambda))$ is explicitly given by $\phi(f)(gB) = [g, f(gv_\lambda)]$, where v_λ is a highest weight vector in $V(\lambda)$.

By Corollary 25, the restriction $H^0(G/B, \mathcal{L}(\lambda)) \to H^0(X_w, \mathcal{L}(\lambda))$ is surjective. Let ϕ_w denote the composition

$$V(\lambda)^* \xrightarrow{\phi} H^0(G/B, \mathcal{L}(\lambda)) \to H^0(X_w, \mathcal{L}(\lambda)).$$

We compute the kernal of ϕ_w; i.e., find all $f \in V(\lambda)^*$ such that $\phi_w(f)$ is the zero section. It suffices to check that $\phi_w(f) = 0$ on BwB/B, since BwB/B is a dense open subset of X_w. For $f \in V(\lambda)^*$,

$$\begin{aligned}\phi_w(f) = 0 &\iff f(BwB \cdot v_\lambda) = 0\\ &\iff f(B \cdot v_{w\lambda}) = 0\\ &\iff f \text{ vanishes on } V_w(\lambda).\end{aligned}$$

Thus, $\ker \phi_w = \{f \in V(\lambda)^* : f|_{V_w(\lambda)} = 0\}$; that is, we have the following exact sequence:

$$0 \longrightarrow \left(\frac{V(\lambda)}{V_w(\lambda)}\right)^* \longrightarrow V(\lambda)^* \longrightarrow H^0(X_w, \mathcal{L}(\lambda)) \longrightarrow 0.$$

Therefore, $H^0(X_w, \mathcal{L}(\lambda))^* \simeq V_w(\lambda)$, which completes the first step.

Now, take a reduced decomposition of $w = s_{i_1} \cdots s_{i_n}$ and let $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$. The map $Z_\mathfrak{w} \to X_w$ is B-equivariant and by Corollary 31, $H^i(Z_\mathfrak{w}, \mathcal{L}_\mathfrak{w}(\lambda)) \simeq H^i(X_w, \mathcal{L}(\lambda))$ for all i as B-modules (for any $\lambda \in X(H)$). Therefore, their characters coincide; that is,

$$\text{ch } H^i(Z_\mathfrak{w}, \mathcal{L}_\mathfrak{w}(\lambda)) = \text{ch } H^i(X_w, \mathcal{L}(\lambda)).$$

Consider the Euler–Poincaré characteristic:

$$\chi_H(Z_\mathfrak{w}, \mathcal{L}_\mathfrak{w}(\lambda)) := \sum_i (-1)^i \operatorname{ch} H^i(Z_\mathfrak{w}, \mathcal{L}_\mathfrak{w}(\lambda)) \in \mathbb{Z}[X(H)].$$

Since $\operatorname{ch} H^0(X_w, \mathcal{L}(\lambda)) = \chi_H(Z_\mathfrak{w}, \mathcal{L}_\mathfrak{w}(\lambda))$ for $\lambda \in X(H)_+$, it suffices to show:

$$\chi_H(Z_\mathfrak{w}, \mathcal{L}_\mathfrak{w}(\lambda)) = * D_\mathfrak{w}(e^\lambda).$$

In fact, we will prove a stronger result which is given as the next proposition. □

Proposition 34. *For a B-module M, let $G \times^B M \to G/B$ be the associated vector bundle. Denote its pull-back to $Z_\mathfrak{w}$ (for any word $\mathfrak{w}$) under the morphism $\theta_\mathfrak{w} : Z_\mathfrak{w} \to G/B$ by $\theta_\mathfrak{w}^* M$. Then,*

$$\chi_H(Z_\mathfrak{w}, \theta_\mathfrak{w}^* M) = * D_\mathfrak{w}(* \text{ch } M).$$

Proof. We induct on the length n of $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$. The Leray spectral sequence for the fibration $\pi : Z_{\mathfrak{w}} \to Z_{\mathfrak{w}(n)}$, with fibers $\mathbb{P}^1 \simeq P_{i_n}/B$, takes the form

$$E_2^{p,q} = H^p\left(Z_{\mathfrak{w}(n)}, \theta^*_{\mathfrak{w}(n)}(H^q(P_{i_n}/B, \theta^*_{(s_{i_n})}M))\right),$$

and converges to $H^{p+q}(Z_{\mathfrak{w}}, \theta^*_{\mathfrak{w}}M)$. From this we see that

$$\chi_H(Z_{\mathfrak{w}}, \theta^*_{\mathfrak{w}}M) = \chi_H(Z_{\mathfrak{w}(n)}, \theta^*_{\mathfrak{w}(n)}(\chi_H(P_{i_n}/B, \theta^*_{(s_{i_n})}M))).$$

It is easy to see that $\chi_H(P_{i_n}/B, \theta^*_{(s_{i_n})}\mathbb{C}_\mu) = *\,D_{s_{i_n}}(e^{-\mu})$, where $\mathbb{C}_\mu$ denotes the one-dimensional B-module with character μ and hence (by Lie's theorem)

$$\chi_H(P_{i_n}/B, \theta^*_{s_{i_n}}M) = *\,D_{s_{i_n}}(*\operatorname{ch} M). \tag{5}$$

By induction on n,

$$\begin{aligned} \chi_H(Z_{\mathfrak{w}}, \theta^*_{\mathfrak{w}}M) &= *\,D_{\mathfrak{w}(n)}\left(*\,\chi_H(P_{i_n}/B, \theta^*_{(s_{i_n})}M)\right) \\ &= *\,D_{\mathfrak{w}(n)}(*\,*\,D_{(s_{i_n})}(*\operatorname{ch} M)), \text{ by the above equality (5)} \\ &= *\,D_{\mathfrak{w}}(*\operatorname{ch} M). \end{aligned}$$

□

Combining Proposition 34 for $M = \mathbb{C}_\lambda$ and Corollary 31, we get the following:

Corollary 35. *For any reduced word $\mathfrak{w}$, the operator $D_{\mathfrak{w}}$ depends only upon $\xi(\mathfrak{w})$.*

Remark 36. *We have not given any historical comments. The interested reader can find them in [K2, § 8.C] and [BK, §§2.C and 3.C].*

References

[B] Brion, M., Lectures on the geometry of flag varieties, In: 'Topics in Cohomological Studies of Algebraic Varieties, Springer, 2005.

[BK] Brion, M. and Kumar, S., *Frobenius Splitting Methods in Geometry and Representation Theory.* Boston, Birkhäuser, 2005.

[D] Debarre, O., *Higher-Dimensional Algebraic Geometry.* Springer, 2001.

[E] Eisenbud, D., *Commutative Algebra with a View Toward Algebraic Geometry.* New York, Springer–Verlag, 1995.

[H] Hartshorne, R., *Algebraic Geometry.* New York, Springer–Verlag, 1977.

[Hu] Humphreys, James. *Linear Algebraic Groups.* New York: Springer–Verlag, 1975.

[K1] Kumar, S., Demazure character formula in arbitrary Kac-Moody setting, *Invent. Math.* **89** (1987), 395–423.

[K2] Kumar, S., *Kac-Moody Groups, their Flag Varieties and Representation Theory.* Boston, Birkhäuser, 2002.

Address: Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599-3250, USA. (shrawan@email.unc.edu)