SERRE DUALITY

MANOJ KUMMINI

We give a proof of the Serre duality theorem using duality for finite morphisms [Har77, Chapter III, Exercise 6.10].

1. Duality for finite morphisms

In this section, X and Y are noetherian schemes and $f : X \longrightarrow Y$ a finite morphism.

(a) [Har77, Chapter II, Exercise 5.17(e)]. If $\mathcal{M}$ is a quasi-coherent $f_*\mathscr{O}_X$-module, then define $\mathcal{M}^\dagger$ by

$$\mathcal{M}^\dagger(U) := \lim_{\substack{V \subseteq Y \text{ open} \\ U \subseteq f^{-1}(V)}} \mathcal{M}(V)$$

and an $\mathscr{O}_X$-module $\widetilde{\mathcal{M}}$ by

$$\widetilde{\mathcal{M}}(U) := \mathcal{M}^\dagger(U) \otimes_{(f_*\mathscr{O}_X)^\dagger(U)} \mathscr{O}_X(U)$$

(This is like defining $f^{-1}(-)$ and $f^*(-)$, which are from $\mathscr{O}_Y$-modules to $\mathscr{O}_X$-modules, but we want something from $f_*\mathscr{O}_X$-modules to $\mathscr{O}_X$-modules. Note that there is a natural map from $(f_*\mathscr{O}_X)^\dagger(U) \longrightarrow \mathscr{O}_X(U)$.) $\widetilde{\mathcal{M}}(U)$ is indeed quasi-coherent: For any open $V \subseteq Y$, $\mathcal{M}^\dagger(f^{-1}(V)) = \mathcal{M}(V)$, so $(f_*\mathscr{O}_X)^\dagger(f^{-1}(V)) = \mathscr{O}_X(f^{-1}(V))$ and, hence, $\widetilde{\mathcal{M}}(f^{-1}(V)) = \mathcal{M}(V)$. Moreover, if V is additionally affine, and $\mathcal{M}|_V$ is given by an $(f_*\mathscr{O}_Y)|_V$-module M, then $\widetilde{\mathcal{M}}|_{f^{-1}(V)}$ also is given by M, thought of as an $\mathscr{O}_X|_{f^{-1}(V)}$-module. Now apply these considerations on any affine open cover (V_i) of Y and the affine open cover $(f^{-1}(V_i))$ of X. It is also immediate that $f_*\widetilde{\mathcal{M}} = \mathcal{M}$ and that $\widetilde{(-)}$ is an exact functor from $f_*\mathscr{O}_X$-modules to $\mathscr{O}_X$-modules. (In the above argument, we have used only that f is affine.)

(b) [Har77, Chapter III, Exercise 6.10(a)]. Let $\mathcal{G}$ be a quasi-coherent $\mathscr{O}_Y$-module. Then $\mathcal{H}om_Y(f_*\mathscr{O}_X, \mathcal{G})$ is a quasi-coherent $f_*\mathscr{O}_X$-module. We denote the correponding $\mathscr{O}_X$-module, from (a) above, by $f^!\mathcal{G}$. Notice that $f^!$ is left-exact covariant functor from $f_*\mathscr{O}_X$-modules to $\mathscr{O}_X$-modules. It would be exact if $\mathcal{H}om_Y(f_*\mathscr{O}_X, -)$ is exact, which is the case if $f_*\mathscr{O}_X$ is locally free $\mathscr{O}_Y$-module.

(c) The natural map

$$f_*f^!\mathcal{G} = \mathcal{H}om_Y(f_*\mathscr{O}_X, \mathcal{G}) \longrightarrow \mathcal{H}om_Y(\mathscr{O}_Y, \mathcal{G}) = \mathcal{G}$$

which is dual to the natural map $\mathscr{O}_Y \longrightarrow f_*\mathscr{O}_X$ will be denoted $\operatorname{Tr}_{f,\mathcal{G}}$, and will be called the *trace map of f on $\mathcal{G}$*.

(d) [Har77, Chapter III, Exercise 6.10(b)]. For every coherent $\mathcal{F}$ on X and quasi-coherent $\mathcal{G}$ on Y, there is an isomorphism

$$f_*\mathcal{H}om_X(\mathcal{F}, f^!\mathcal{G}) \xrightarrow{\simeq} \mathcal{H}om_Y(f_*\mathcal{F}, \mathcal{G}).$$

We see this as follows: For arbitrary sheaves $\mathcal{F}$ and $\mathcal{F}'$ on X, there is a morphism

$$f_*\mathcal{H}om_X(\mathcal{F}, \mathcal{F}') \longrightarrow \mathcal{H}om_Y(f_*\mathcal{F}, f_*\mathcal{F}').$$

Notes prepared for the workshop on Seshadri constants, February 2017 at CMI. Please send your comments by email.

Taking $\mathcal{F}' = f^!\mathcal{G}$ and using $\mathrm{Tr}_{f,\mathcal{G}}$ we get a morhpism

$$f_*\mathcal{H}om_X(\mathcal{F}, f^!\mathcal{G}) \xrightarrow{\simeq} \mathcal{H}om_Y(f_*\mathcal{F}, \mathcal{G}).$$

with $\mathcal{G}$ quasi-coherent. To prove this is an isomorphism when $\mathcal{F}$ is coherent, we may assume that Y and, hence, X are affine. When $\mathcal{F} = \mathscr{O}_X$, the asserted isomorphism follows from the definition of $f^!\mathcal{G}$. Therefore it is also true for the direct sum of finitely many copies of $\mathscr{O}_X$. For any coherent $\mathcal{F}$ on X, there is an exact sequence $\mathscr{O}_X^{b_1} \longrightarrow \mathscr{O}_X^{b_0} \longrightarrow \mathcal{F} \longrightarrow 0$, which gives the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow & f_*\mathcal{H}om_X(\mathcal{F}, f^!\mathcal{G}) & \longrightarrow & f_*\mathcal{H}om_X(\mathscr{O}_X^{b_0}, f^!\mathcal{G}) & \longrightarrow & f_*\mathcal{H}om_X(\mathscr{O}_X^{b_1}, f^!\mathcal{G}) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \mathcal{H}om_Y(f_*\mathcal{F}, \mathcal{G}) & \longrightarrow & \mathcal{H}om_Y(f_*\mathscr{O}_X^{b_0}, \mathcal{G}) & \longrightarrow & \mathcal{H}om_Y(f_*\mathscr{O}_X^{b_1}, \mathcal{G}) & \longrightarrow \end{array}$$

This gives the required isomorphism.

(e) [Har77, Chapter III, Exercise 6.10(c)]. This is similar to (d). If

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{E}_1 \longrightarrow \cdots \longrightarrow \mathcal{E}_i \longrightarrow \mathcal{F} \longrightarrow 0$$

is an exact sequence of quasi-coherent $\mathscr{O}_X$-modules representing an element in $\mathrm{Ext}^i_X(\mathcal{F}, \mathcal{F}')$ then its direct image is exact (since f is affine) and represents an element in $\mathrm{Ext}^i_Y(f_*\mathcal{F}, f_*\mathcal{F}')$. Now apply with $\mathcal{F}' = f^!\mathcal{G}$ and use $\mathrm{Tr}_{f,\mathcal{G}}$.

(f) It follows from (d) that $f^!\mathcal{I}$ is an injective $\mathscr{O}_X$-module for every quasi-coherent injective $\mathscr{O}_Y$-module $\mathcal{I}$. By applying $\Gamma(Y, -)$ to the isomorphism in (d), we get $\mathrm{Hom}_X(\mathcal{F}, f^!\mathcal{G}) \xrightarrow{\simeq} \mathrm{Hom}_Y(f_*\mathcal{F}, \mathcal{G})$. Therefore If $\mathcal{F}_1 \longrightarrow \mathcal{F}_2$ is an injective map of quasi-coherent $\mathscr{O}_X$-modules, then the map $\mathrm{Hom}_X(\mathcal{F}_2, f^!\mathcal{I}) \longrightarrow \mathrm{Hom}_X(\mathcal{F}_1, f^!\mathcal{I})$ is surjective when $\mathcal{I}$ is an injective $\mathscr{O}_Y$-module.

(g) [Har77, Chapter III, Exercise 6.10(d)]. Let $\mathcal{I}^\bullet$ be an injective resolution of $\mathcal{G}$ by quasi-coherent injectives. (Note: Noetherian schemes have enough quasi-coherent injectives.) Since $f_*\mathscr{O}_X$ is locally free, we see from (b) that $0 \longrightarrow f^!\mathcal{G} \longrightarrow f^!\mathcal{I}^\bullet$ is an exact sequence, and, hence from (f) that $f^!\mathcal{I}^\bullet$ is an injective resolution of $f^!\mathcal{G}$ as an $\mathscr{O}_X$-module. Now,

$$\begin{aligned} \mathrm{Ext}^i_X(\mathcal{F}, f^!G) &= \mathrm{H}^i(\mathrm{Hom}_X(\mathcal{F}, f^!\mathcal{I}^\bullet)) \\ &= \mathrm{H}^i(\Gamma(X, \mathcal{H}om_X(\mathcal{F}, f^!\mathcal{I}^\bullet))) \\ &\simeq \mathrm{H}^i(\Gamma(Y, \mathcal{H}om_Y(f_*\mathcal{F}, \mathcal{I}^\bullet))) \\ &= \mathrm{H}^i(\mathrm{Hom}_Y(f_*\mathcal{F}, \mathcal{I}^\bullet)) \\ &= \mathrm{Ext}^i_Y(f_*\mathcal{F}, G). \end{aligned}$$

(h) We now explain these statements in the case of affine schemes: $Y = \mathrm{Spec}\, R$, $X = \mathrm{Spec}\, S$, and f corresponds a ring map $\phi : R \longrightarrow S$. Let M be an S-module, thought of as an R-module through ϕ. From (a), we see that the corresponding sheaf on $X = \mathrm{Spec}\, S$ is given by M itself. Therefore, in (b), we see that for any R-module N, $f^!N = \mathrm{Hom}_R(S, N)$, considered as a natural S-module. The trace map is the composite of $\mathrm{Hom}_R(S, N) \longrightarrow \mathrm{Hom}_R(R, N) \longrightarrow N$, $\alpha \mapsto \alpha \circ \phi \mapsto (\alpha \circ \phi)(1_R) = \alpha(1_S)$, *evaluation at* 1. The duality of (d) is $\mathrm{Hom}_S(M, \mathrm{Hom}_R(S, N)) \simeq \mathrm{Hom}_R(M, N)$. In (g), S is a projective (and finitely generated) R-module, M a finitely generated S-module and N an R-module. Let $F_\bullet$ be a free resolution of M as an S-module. It is also a projective resolution of M as an R-module. Hence

$$\begin{aligned} \mathrm{Ext}^i_S(M, f^!N) &= \mathrm{H}^i(\mathrm{Hom}_S(F_\bullet, \mathrm{Hom}_R(S, N))) \\ &\simeq \mathrm{H}^i(\mathrm{Hom}_R(F_\bullet, N)) \\ &= \mathrm{Ext}^i_R(M, N). \end{aligned}$$

2. Finite morphisms to projective spaces

(a) **Noether normalization**: Let $\Bbbk$ denote a field and X an n-dimensional projective $\Bbbk$-scheme. Embed $X \subseteq \mathbb{P}^N_{\Bbbk}$. Let $S = \Bbbk[x_0, \ldots, x_N]$ be a homogeneous coordinate ring of $\mathbb{P}^N_{\Bbbk}$ and I an S-ideal such that $X = \operatorname{Proj}(S/I)$. (For example, if $\mathcal{I}$ is the ideal sheaf of X, then we can take $I = \bigoplus_{k\in\mathbb{Z}} \Gamma(X, \mathcal{I}(k))$.) Let $y_0, \ldots, y_n \in (S/I)_1$ be such that $\Bbbk[y_0, \ldots, y_n] \subseteq S/I$ is a *homogeneous* Noether normalization of S/I. In particular $y_0, \ldots, y_n$ are algebraically independent over $\Bbbk$, S/I is finite over the subring, and the ideal $(y_0, \ldots, y_n)(S/I)$ is primary to irrelevant ideal $(x_0, \ldots, x_N)(S/I)$. Then we have a finite morphism $X \longrightarrow \mathbb{P}^n_{\Bbbk} = \operatorname{Proj} \Bbbk[y_0, \ldots, y_n]$. This is surjective, since $\dim X = n$ and $\mathbb{P}^n_{\Bbbk}$ is irreducible.

(b) Let $f : X \longrightarrow Y$ be a finite surjective morphism of non-singular noetherian schemes. Then $f_*\mathscr{O}_X$ is a locally free $\mathscr{O}_Y$-module. Since the question is local on Y, we may assume that $Y = \operatorname{Spec} R$ for a regular local ring R and that $X = \operatorname{Spec} S$ for some regular ring S that is finite over R. We need to show that S is a free R-module, which is equivalent to S being a flat R-module, which is equivalent to the vanishing of $\operatorname{Tor}_1^R(R/\mathfrak{m}, S)$, where $\mathfrak{m}$ is the maximal ideal of R. Write $n = \dim R = \dim S$. Let $r_1, \ldots, r_n$ be minimal generators for $\mathfrak{m}$. Then $\operatorname{ht}(r_1, \ldots, r_n)S = n$, since the map $R \longrightarrow S$ (and hence $R/\mathfrak{m} \longrightarrow S/\mathfrak{m}S$) is finite. Since S is a Cohen-Macaulay ring, (the images in S of) $r_1, \ldots, r_n$ form a regular sequence, so $\operatorname{Tor}_1^R(R/\mathfrak{m}, S)$, which is equal to the first Koszul homology of the sequence $r_1, \ldots, r_n$ in S, is zero.

(c) In (b), we have not used the hypothesis that X is non-singular very strongly; that X is Cohen-Macaulay (i.e., all the local rings $\mathscr{O}_{X,x}$ are Cohen-Macaulay) would do.

3. Serre duality

In this section X and Y denote n-dimensional projective varieties over a field $\Bbbk$ and $P = \mathbb{P}^n_{\Bbbk}$.

(a) Suppose that $f : X \longrightarrow Y$ is a finite surjective morphism. Suppose that (ω_Y, t_Y) is a dualizing sheaf for Y, i.e., ω_Y is a coherent sheaf on Y, $t_Y : \mathrm{H}^n(Y, \omega_Y) \longrightarrow \Bbbk$ is $\Bbbk$-linear and the composite map

$$\operatorname{Hom}_Y(\mathcal{F}, \omega_Y) \times \mathrm{H}^n(Y, \mathcal{F}) \longrightarrow \mathrm{H}^n(Y, \omega_Y) \xrightarrow{t_Y} \Bbbk,$$

where the first map is $(\phi, c) \mapsto \mathrm{H}^n(\phi)(c)$, is a perfect pairing. Define $\omega_X = f^!\omega_Y$ and t_X to be the composite of

$$\mathrm{H}^n(X, \omega_X) = \mathrm{H}^n(Y, f_*\omega_X) \xrightarrow{\mathrm{H}^n(\operatorname{Tr}_{f,\omega_Y})} \mathrm{H}^n(Y, \omega_Y) \xrightarrow{t_Y} \Bbbk.$$

Then (ω_X, t_X) is a dualizing sheaf for X. (It is straightforward to check that the conditions in the definition given above are satisfied.)

(b) Let $f : X \longrightarrow P$ be a Noether normalization (§2,(a)). Let $x_0, \ldots, x_n$ be homogeneous coordinates for P. Then $\mathrm{H}^n(P, \mathscr{O}_P(-n-1))$ can identified with the a one-dimensional vector-space with basis $\frac{1}{x_0 x_1 \ldots x_n}$. Let t_P be the basis dual to this. Then $(\mathscr{O}_P(-n-1), t_P)$ is a dualizing sheaf for P. Hence X has a dualizing sheaf (ω_X, t_X).

(c) Additionally if X is non-singular (or, merely, Cohen-Macaulay), then we have isomorphisms

$$\begin{aligned}\operatorname{Ext}^i_X(\mathcal{F}, \omega_X) &\simeq \operatorname{Ext}^i_P(f_*\mathcal{F}, \omega_Y)\\ &\simeq \mathrm{H}^{n-i}(P, f_*\mathcal{F})^\vee\\ &\simeq \mathrm{H}^{n-i}(X, \mathcal{F})^\vee\end{aligned}$$

for coherent sheaves $\mathcal{F}$ on X. (One must first check that the statement is true for P; see [Har77, Chapter III, Theorem 7.1]

Acknowledgments

I happily thank Pramath Sastry whose comments clarified the exposition considerably.

REFERENCES

[Har77] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52. 1, 2, 3

CHENNAI MATHEMATICAL INSTITUTE, SIRUSERI, TAMILNADU 603103. INDIA
E-mail address: mkummini@cmi.ac.in