F-RATIONALITY

MANOJ KUMMINI

Introduction

These are notes from my lectures at the workshop *Commutative Algebra and Algebraic Geometry in Positive Characteristics* held at IIT Bombay in December 2018. The goal is to give a proof of a theorem of K. Smith which asserts that F-rational rings have pseudo-rational singularities [Smi97].

Notation. By a ring we mean a commutative ring with multiplicative identity. Ring homomorphisms are assumed to take the multiplicative identity to the multiplicative identity.

$\Bbbk$: field

R, S: rings.

1. Double-complex spectral sequences

In this lecture, we list some results, mostly without proofs, about double-complex spectral sequences. References are [CE99, Chapter XV], [Eis95, Appendix A3], and [Wei94, Chapter 5].

Let $\mathcal{A}$ be an abelian category and $C^{\bullet,\bullet}$ a first-quadrant double complex in $\mathcal{A}$, i.e., a double complex with $C^{i,j} = 0$ if $i < 0$ or $j < 0$. Write $F^\bullet = \operatorname{Tot}(C^{\bullet,\bullet})$. We wish to understand $\mathrm{H}^*(F^\bullet)$. To this end, we take a filtration $F^\bullet \supseteq F^\bullet_1 \supseteq F^\bullet_2 \supseteq \cdots$. Fix $n \geq 0$. Write $M_p = \operatorname{Im}(\mathrm{H}^n({}'F^\bullet_p) \longrightarrow \mathrm{H}^n(F^\bullet))$. Since H^n is a functor from the category of complexes over $\mathcal{A}$ to $\mathcal{A}$, we get an induced filtration $\mathrm{H}^n(F^\bullet) \supseteq M_1 \supseteq M_2 \cdots$ on $\mathrm{H}^n(F^\bullet)$. Using a spectral sequence, we start from

$$\mathrm{H}^*\left(\bigoplus_p \left(F^\bullet_p / F^\bullet_{p+1}\right)\right)$$

and obtain the associated graded object

$$\bigoplus_p M_p / M_{p+1}$$

of the filtration of $\mathrm{H}^n(F^\bullet)$.

Filtration by columns. For $p \geq 0$, define

$${}'C^{i,j}_p = \begin{cases} C^{i,j}, & \text{if } i \geq p; \\ 0, & \text{otherwise,} \end{cases}$$

for every j. Write ${}'F^\bullet_p = \operatorname{Tot}({}'C^{\bullet,\bullet}_p)$. This gives a filtration $F^\bullet = {}'F^\bullet_0 \supseteq {}'F^\bullet_1 \supseteq {}'F^\bullet_2 \supseteq \cdots$ with

$${}'F^\bullet_p / {}'F^\bullet_{p+1} = C^{p,\bullet}.$$

Version of December 15, 2018.

Set ${}'E_0^{i,j} = C^{i,j}$ for every i, j. Think of ${}'E_0^{\bullet,\bullet}$ as the collection of complexes $C^{i,\bullet}$ $i \geq 0$, with the horizontal arrows as maps of these complexes. Now ${}'E_1^{\bullet,\bullet}$ is the homology of ${}'E_0^{\bullet,\bullet}$; more precisely, the maps in ${}'E_0^{\bullet,\bullet}$ are of the form

$$\begin{array}{c} C_{i,j+1} \\ \uparrow \\ C_{i,j} \\ \uparrow \\ C_{i,j-1} \end{array}$$

so ${}'E_1^{i,j} = \mathrm{H}^j(C^{i,\bullet})$. The horizontal maps of $C^{\bullet,\bullet}$, which are thought of as maps of complexes $C^{i,\bullet} \longrightarrow C^{i+1,\bullet}$, give maps

$${}'E_1^{i-1,j} \longrightarrow {}'E_1^{i,j} \longrightarrow {}'E_1^{i+1,j}.$$

We define ${}'E_2^{\bullet,\bullet}$ as the homology of ${}'E_1^{\bullet,\bullet}$. One can show that there are maps

$$\begin{array}{ccccc} {}'E_2^{i-2,j+1} & & & & \\ & \searrow & & & \\ & & {}'E_2^{i,j} & & \\ & & & \searrow & \\ & & & & {}'E_2^{i+2,j-1} \end{array}$$

and that these form a complex. Define ${}'E_3^{\bullet,\bullet}$ as the homology of ${}'E_2^{\bullet,\bullet}$; there are maps

$${}'E_3^{i-3,j+2} \longrightarrow {}'E_3^{i,j} \longrightarrow {}'E_3^{i+3,j-2}.$$

Inductively define ${}'E_r^{\bullet,\bullet}$ as the homology of ${}'E_{r-1}^{\bullet,\bullet}$; the maps are

$${}'E_r^{i-r,j+r-1} \longrightarrow {}'E_r^{i,j} \longrightarrow {}'E_r^{i+r,j-r+1}.$$

Note for each $s \geq r \geq 1$, and and each i, j, ${}'E_s^{i,j}$ is a subquotient of ${}'E_r^{i,j}$ and that ${}'E_0^{i,j}$ is a subquotient of $C^{i,j}$. Hence, for each i, j, there exists r such that for every $s \geq r$, the map coming into ${}'E_s^{i,j}$ is from the second quadrant and the map leaving from ${}'E_s^{i,j}$ is to the fourth quadrant; therefore these maps are zero, which gives that ${}'E_s^{i,j} = {}'E_r^{i,j}$; define

$${}'E_\infty^{i,j} = {}'E_r^{i,j}$$

for this r.

1.1. **Theorem**. *For the filtration on $\mathrm{H}^n(F^\bullet)$ induced by the filtration of $\{{}'F_p^\bullet\}_p$ of $F^\bullet$, the associated graded object of $\mathrm{H}^n(F^\bullet)$ has ${}'E_\infty^{i,n-i}$ as its ith component.*

Filtration by rows. For $q \geq 0$, define

$${}''C_q^{i,j} = \begin{cases} C^{i,j}, & \text{if } j \geq q; \\ 0, & \text{otherwise}, \end{cases}$$

for every i. Write ${}''F_q^\bullet = \mathrm{Tot}({}''C_q^{\bullet,\bullet})$. This gives a filtration $F^\bullet = {}''F_0^\bullet \supseteq {}''F_1^\bullet \supseteq {}''F_2^\bullet \supseteq \cdots$ with

$${}''F_q^\bullet / {}''F_{q+1}^\bullet = C^{\bullet,q}.$$

Set $''E_0^{i,j} = C^{i,j}$ for every i, j. Think of $''E_0^{\bullet,\bullet}$ as the collection of complexes $C^{\bullet,j}$ $j \geq 0$, with the vertical arrows as maps of these complexes. Now $''E_1^{\bullet,\bullet}$ is the homology of $''E_0^{\bullet,\bullet}$; more precisely, the maps in $''E_0^{\bullet,\bullet}$ are of the form

$$C_{i-1,j} \longrightarrow C_{i,j} \longrightarrow C_{i+1,j}$$

so $''E_1^{i,j} = \mathrm{H}^i(C^{\bullet,j})$. The vertical maps of $C^{\bullet,\bullet}$, which are thought of as maps of complexes $C^{\bullet,j} \longrightarrow C^{\bullet,j+1}$, give maps

$$\begin{array}{c} ''E_1^{i,j+1} \\ \uparrow \\ ''E_1^{i,j} \\ \uparrow \\ ''E_1^{i,j-1} \end{array}$$

We define $''E_2^{\bullet,\bullet}$ as the homology of $''E_1^{\bullet,\bullet}$. One can show that there are maps

$$\begin{array}{ccccc} ''E_2^{i-2,j+1} & & & & \\ & \nwarrow & & & \\ & & ''E_2^{i,j} & & \\ & & & \nwarrow & \\ & & & & ''E_2^{i+2,j-1} \end{array}$$

and that these form a complex. Inductively define $''E_r^{\bullet,\bullet}$ as the homology of $''E_{r-1}^{\bullet,\bullet}$; the maps are

$$''E_r^{i+r,j-r+1} \longrightarrow {}'E_r^{i,j} \longrightarrow {}''E_r^{i-r,j+r-1}.$$

As with the filtration by columns, for each i, j, there exists r such that for every $s \geq r$, $''E_s^{i,j} = {}''E_r^{i,j}$; define

$$''E_\infty^{i,j} = {}''E_r^{i,j}$$

for this r.

1.2. **Theorem**. *For the filtration on $\mathrm{H}^*(F^\bullet)$ induced by the filtration of $\{''F_q^\bullet\}_q$ of $F^\bullet$, the associated graded object of $\mathrm{H}^n(F^\bullet)$ has $''E_\infty^{n-i,i}$ as its ith component.*

Terminology. We often refer to $'E_r^{\bullet,\bullet}$ and $''E_r^{\bullet,\bullet}$ as the *rth page* of the spectral sequence. We also say that the spectral sequences $'E_r^{\bullet,\bullet}$ and $''E_r^{\bullet,\bullet}$ *converge* to $\mathrm{H}^*(F^\bullet)$. We denote this by

$$'E_r^{i,j} \Rightarrow \mathrm{H}^{i+j}(F^\bullet) \text{ and } ''E_r^{i,j} \Rightarrow \mathrm{H}^{i+j}(F^\bullet)$$

Edge maps. Fix $n \geq 0$ and consider the filtration on $\mathrm{H}^n(F^\bullet)$ induced by the filtration of $\{'F^\bullet_p\}_p$ of $F^\bullet$. Since this is a decreasing filtration, we see that $'E^{n,0}_\infty$ is a submodule of $\mathrm{H}^n(F^\bullet)$. For $r \geq 2$, there is a surjective morphism $'E^{n,0}_r \longrightarrow {'E^{n,0}_\infty}$. The composite map $'E^{n,0}_r \longrightarrow {'E^{n,0}_\infty} \longrightarrow \mathrm{H}^n(F^\bullet)$ is called an *edge homomorphism*. Similarly, we get an *edge homomorphism* $''E^{0,n}_r \longrightarrow {''E^{0,n}_\infty} \longrightarrow \mathrm{H}^n(F^\bullet)$

Grothendieck spectral sequence. We give an application of the double complex spectral sequence to obtain a relation between the derived functors of a composite of two functors.

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories such that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{C}$ be left-exact covariant additive functors such that F takes injectives in $\mathcal{A}$ to G-acyclic objects in $\mathcal{B}$, i.e., objects Y of $\mathcal{B}$ such that $R^iGY = 0$ for every $i > 0$.

1.3. **Theorem.** *With notation as above, there is a spectral sequence*

$$E^{i,j}_2 = \mathrm{R}^j G(\mathrm{R}^i F(X)) \Rightarrow \mathrm{R}^{i+j}(GF)(X)$$

for every object X of $\mathcal{A}$.

Proof. Let X be an object of $\mathcal{A}$. Let $I^\bullet$ be an injective resolution of X. Let $J^{\bullet,\bullet}$ be a Cartan-Eilenberg injective resolution (double complex) of $F(I^\bullet)$. (See [CE99, Chapter XVII] and [Wei94, Section 5.7] for the construction of Cartan-Eilenberg resolutions.) Let $C^{\bullet,\bullet} = G(J^{\bullet,\bullet})$. Then

$$'E^{i,j}_1 = \mathrm{H}^j(G(J^{i,\bullet})) = \mathrm{R}^j G(F(I^i)) = \begin{cases} (GF)(I^i), & \text{if } j = 0; \\ 0, & \text{otherwise,} \end{cases}$$

by the hypothesis on F. Hence the $'E_1$ page is the complex $(GF)(I^\bullet)$, from which we conclude that

$$'E^{i,j}_\infty = \begin{cases} \mathrm{R}^i(GF)(X), & \text{if } j = 0, \\ 0, & \text{otherwise,} \end{cases}$$

In particular, for every n, the associated graded object of $\mathrm{H}^n(\mathrm{Tot}(C^{\bullet,\bullet}))$ has only one potentially non-zero term $\mathrm{R}^n(GF)(X)$; it follows that $\mathrm{H}^n(\mathrm{Tot}(C^{\bullet,\bullet})) = \mathrm{R}^n(GF)(X)$.

In the spectral sequence associated to filtration by rows of $C^{\bullet,\bullet}$, we have

$$''E^{i,j}_1 = \mathrm{H}^i\, G(J^{\bullet,j})$$

One can check, using the definition and properties of Cartan-Eilenberg resolutions that

$$\mathrm{H}^i\, G(J^{\bullet,j}) = G(\text{an injective resolution of } \mathrm{H}^i(F(I^\bullet))).$$

Hence

$$''E^{i,j}_2 = \mathrm{R}^j G(\mathrm{R}^i F(X))$$

Set $E_2 = {''E_2}$. $\square$

The edge homomorphisms of the above spectral sequence are $\mathrm{R}^n G(F(X)) \longrightarrow \mathrm{R}^n(GF)(X)$.

2. Pseudo-rational rings

In this lecture, we look at pseudo-rational rings [LT81]. We begin with some remarks on local cohomology.

Cohomology with supports. Let X be a topological space, Z a (locally) closed subset of X and $\mathcal{F}$ a sheaf of abelian groups on X. We denote the category of abelian groups by **Ab** and, for a topological space Y, the category of sheaves of abelian groups on Y by $\mathbf{Ab}_Y$.

Write $U = X \setminus Z$. Define

$$\Gamma_Z(X, F) := \ker\left(\Gamma(X, F) \longrightarrow \Gamma(U, F)\right).$$

This is a functor from $\mathbf{Ab}_X$ to **Ab**. It is left exact (Exercise 5.5). Define *cohomology groups with support in* Z, denoted $\mathrm{H}^*_Z(X)$, to be its right-derived functors.

2.1. **Proposition.** *Suppose that* $X = \operatorname{Spec} R$*, that* Z *is defined by a finitely generated* R*-ideal* I *and that* $\mathcal{F}$ *is the sheaf defined by an* R*-module* M*. Then*

$$\mathrm{H}^i_Z(X, \mathcal{F}) = \mathrm{H}^i_I(M)$$

for every i*.*

For a proof, see [Har67, Proposition 2.2] or [ILL+07, Theorem 12.47].

2.2. **Proposition.** *Let* $f : X' \longrightarrow X$ *be a continuous map,* Z *a closed subset of* X*,* $Z' := f^{-1}(Z)$ *and* $\mathcal{F}$ *a sheaf of abelian groups on* X*. Then we have a spectral sequence*

$$E_2^{i,j} = \mathrm{H}^j_Z(X, \mathrm{R}^i f_* \mathcal{F})$$

converging to $\mathrm{H}^{i+j}_{Z'}(X', \mathcal{F})$*. The edge homomorphisms of this page are the maps* $\mathrm{H}^n_Z(X, f_*\mathcal{F}) \longrightarrow \mathrm{H}^n_{Z'}(X', \mathcal{F})$*.*

Proof. Use Theorem 1.3 with $\mathcal{A} = \mathbf{Ab}_{X'}$, $\mathcal{B} = \mathbf{Ab}_X$, $\mathcal{C} = \mathbf{Ab}$, $F = f_*$ and $G = \Gamma_Z(X, -)$. Note that f_* takes injectives in $\mathbf{Ab}_{X'}$ to injectives in $\mathbf{Ab}_X$, which are acyclic for $\Gamma_Z(X, -)$. See [Har67, Proposition 5.5] for details. The assertion about edge homomorphisms follows from the definition. □

Pseudo-rational rings.

2.3. **Definition.** Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay, normal, analytically unramified local ring. Then R is said to be *pseudo-rational* if the edge homomorphism

$$\mathrm{H}^d_{\mathfrak{m}}(R) \xrightarrow{\delta_f} \mathrm{H}^d_{f^{-1}(\{\mathfrak{m}\})}(Z, \mathcal{O}_Z)$$

is injective, for every proper birational map $f : Z \longrightarrow \operatorname{Spec} R$ with Z normal.

2.4. **Example.** Regular local rings are pseudo-rational [LT81, Section 4].

2.5. **Example.** Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay, normal local ring that is essentially of finite type over a field of characteristic zero. Suppose that R has *rational singularities*, i.e., there exists a proper birational morphism $h : Z \longrightarrow \operatorname{Spec} R$ such that Z is nonsingular (such a morphism is called a *desingularization*) and $\mathrm{R}^i h_* \mathcal{O}_Z = 0$ for every $i > 0$. In fact, if this holds for one desingularization, it holds for every desingularization. Let $f : W \longrightarrow \operatorname{Spec} R$ be a proper birational morphism with W normal. Let $g : Z \longrightarrow W$ be a desingularization. Then $h = fg$ is a desingularization of $\operatorname{Spec} R$. Then the edge homomorphism δ_f is injective. (Exercise 5.8).

2.6. **Example.** Let $\Bbbk$ be a field of characteristic different from 3, $S = \Bbbk[x, y, z]$ and $R = S/(x^3 + y^3 + z^3)$. Write $\mathfrak{m}$ for the homogeneous maximal ideal of R. After replacing $\Bbbk$ by an algebraic closure and using the jacobian criterion [Eis95, 16.19] we see that the singular locus of $\operatorname{Spec} R$ is $\{\mathfrak{m}\}$, which has codimension two. Since it is Cohen-Macaulay, it satisfies the Serre condition (S_2). Hence R is a normal domain. Let A be the Rees algebra $R[\mathfrak{m}t]$ and $X = \operatorname{Proj} A$. Write f for the natural map $X \longrightarrow \operatorname{Spec} R$. We now make several observations and conclude that R is not pseudo-rational.

(1) X is nonsingular: X has an affine open covering

$$\operatorname{Spec}\left(\left(R\left[\frac{\mathfrak{m}t}{xt}\right]\right)_0\right) \cup \operatorname{Spec}\left(\left(R\left[\frac{\mathfrak{m}t}{yt}\right]\right)_0\right).$$

(Observe that $zt \in \sqrt{(xt, yt)}$.) Note that

$$\left(R\left[\frac{\mathfrak{m}t}{xt}\right]\right)_0 \simeq R\left[\frac{y}{x}, \frac{z}{x}\right].$$

Write $u = \frac{y}{x}$ and $y = \frac{z}{x}$ to see that

$$R\left[\frac{y}{x}, \frac{z}{x}\right] \simeq \mathbb{k}[x, u, v]/(1 + u^3 + v^3)$$

which is non-singular; similarly for the other open set.

(2) $\mathrm{H}^2(X, \mathcal{F}) = 0$ for every coherent sheaf $\mathcal{F}$ on X, since X has an affine cover with two open sets.

(3) The map f is birational: for, let $0 \neq a \in \mathfrak{m}$. Then $A_a \simeq R_a[t]$, so $f^{-1}(\operatorname{Spec} R_a) \simeq \operatorname{Proj}(R_a \otimes_R A) \simeq \operatorname{Proj}(R_a[t]) \simeq \operatorname{Spec} R_a$. Write $U = \operatorname{Spec} R \smallsetminus \{\mathfrak{m}\}$ and $V = f^{-1}(U)$. Then $f|_V : V \longrightarrow U$ is an isomorphism, since U has an affine covering by $\operatorname{Spec} R_a, a \in \mathfrak{m}, a \neq 0$.

(4) $\operatorname{Supp}\left(\mathrm{H}^1(X, \mathcal{F})\right) \subseteq \{\mathfrak{m}\}$ for every coherent sheaf $\mathcal{F}$ on X. This follows from applying the flat-base change theorem for cohomology [Har77, III.9.3] for the flat (in fact open) morphism $U \longrightarrow \operatorname{Spec} R$, and noting that all higher direct images vanish for the isomorphism $V \longrightarrow U$.

(5) Let $E = \operatorname{Proj}(R/\mathfrak{m} \otimes_R A)$, the scheme-theoretic pre-image of $\operatorname{Spec}(R/\mathfrak{m}) \subseteq \operatorname{Spec} R$. Note that $R/\mathfrak{m} \otimes_R A \simeq \mathbb{k}[x, y, z]/(x^3 + y^3 + z^3)$, so $E \simeq \operatorname{Proj} R$. Note that we have an exact sequence

$$0 \longrightarrow \mathfrak{m}\mathcal{O}_X \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_E \longrightarrow 0.$$

(6) $\mathrm{H}^1(E, \mathcal{O}_E) \neq 0$: Since $E \simeq \operatorname{Proj} R$, it suffices [ILL+07, 13.21] to show that

$$\mathrm{H}^2_{\mathfrak{m}}(R)_0 \neq 0.$$

Note that we have an exact sequence

$$0 \longrightarrow \mathrm{H}^2_{\mathfrak{m}}(R) \longrightarrow \mathrm{H}^3_{\mathfrak{m}}(S)(-3) \longrightarrow \mathrm{H}^3_{\mathfrak{m}}(S) \longrightarrow 0$$

A description of $\mathrm{H}^3_{\mathfrak{m}}(S)$ as a graded S-module is given in [ILL+07, Example 7.16], whence we conclude that

$$\mathrm{H}^2_{\mathfrak{m}}(R)_0 \simeq \mathrm{H}^3_{\mathfrak{m}}(S)_{-3} \simeq \mathbb{k}.$$

(7) $\mathrm{H}^0_{\mathfrak{m}}(\mathrm{H}^1(X, \mathcal{O}_X)) = \mathrm{H}^1(X, \mathcal{O}_X) \neq 0$, since $\mathrm{H}^1(X, \mathcal{O}_X)$ is a finite-length non-zero module.

(8) The 'exact sequence of low-degree terms' (Exercise 5.1) for the spectral sequence of Proposition 2.2

$$\mathrm{H}^j_{\mathfrak{m}}(\mathrm{R}^i f_* \mathcal{O}_X) \Rightarrow \mathrm{H}^{i+j}_E(\mathcal{O}_X)$$

is

$$0 \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(R) \xrightarrow{\text{edge}} \mathrm{H}^1_E(\mathcal{O}_X) \longrightarrow \mathrm{H}^1(X, \mathcal{O}_X) \longrightarrow \mathrm{H}^2_{\mathfrak{m}}(R) \xrightarrow{\text{edge}} \mathrm{H}^2_E(\mathcal{O}_X) \longrightarrow$$

(9) $\mathrm{H}^1_E(\mathcal{O}_X) = 0$ [Lip78, Theorem 2.4, p. 177].

Hence the edge map $\mathrm{H}^2_{\mathfrak{m}}(R) \longrightarrow \mathrm{H}^2_E(\mathscr{O}_X)$ is non-zero, and $\operatorname{Spec} R$ is not pseudo-rational. What we essentially used is the fact that

$$\mathrm{H}^2_{\mathfrak{m}}(R)_j \neq 0$$

for some $j \geq 0$. See Exercise 5.11 in this context. □

3. F-RATIONALITY

Tight closure. For this lecture and the next, p is a prime number and R is a noetherian ring of characteristic p. Let I be an R-ideal. By q, we mean a power of p. By $I^{[q]}$, we mean the ideal generated by $\{x^q \mid x \in I\}$. By R^o, we mean the set $R \setminus \cup_{\mathfrak{p}\in \operatorname{Min}(R)} \mathfrak{p}$.

3.1. **Definition**. The *tight closure* of I, denoted I^*, is the set

$$\{x \in R \mid \text{there exists } c \in R^o \text{ such that } cx^q \in I^{[q]} \text{ for all } q \gg 0\}.$$

We say that I is *tightly closed* if $I = I^*$.

Some facts:
(1) I^* is an ideal containing I; $(I^*)^* = I^*$.
(2) $x \in I^*$ if and only if $x \in (IR/\mathfrak{p})^*$ for every $\mathfrak{p} \in \operatorname{Min}(R)$.
(3) Every ideal in a regular local ring is tightly closed.

F-**rational rings**. Let $x_1, \ldots, x_n \in R$. We say that $(x_1, \ldots, x_n)$ is a *parameter ideal* if the images of $x_1, \ldots, x_n$ in $R_{\mathfrak{p}}$ form part of a system of parameters for $R_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of R containing $x_1, \ldots, x_n$. We say that R is F*-rational* if every parameter ideal is tightly closed.

Some facts:
(1) Every F-rational ring is normal.
(2) Every ideal in a Gorenstein F-rational ring is tightly closed.
(3) If R is a quotient of a Cohen-Macaulay ring and is F-rational, then R is Cohen-Macaulay, and localizations of R are F-rational.
(4) Let R be a local ring that is a quotient of a Cohen-Macaulay ring. Then R is F-rational if and only if R is equi-dimensional and there exists a system of parameters that generates a tightly closed ideal.
(5) Let R be a local ring and $\widehat{R}$ its completion. If $\widehat{R}$ is F-rational, then R is F-rational. The converse is true if R is excellent (e.g., essentially of finite type over a field).

Frobenius action on local cohomology. The Frobenius map $F : R \longrightarrow R, r \mapsto r^p$ commutes with localization. Let $I = (x_1, \ldots, x_n)$; then F commutes with the maps in $\check{\mathrm{C}}(x_1, \ldots, x_n; R)$, so it induces a map on $\mathrm{H}^i_I(R)$ for every i. On $\mathrm{H}^n_I(R)$, this map is

$$\left[\frac{z}{x_1^t x_2^t \cdots x_n^t}\right] \mapsto \left[\frac{z^p}{x_1^{tp} x_2^{tp} \cdots x_n^{tp}}\right].$$

$\check{\mathrm{C}}^\bullet(x_1, \ldots, x_n; R)$ is also the limit of the Koszul complexes $K^\bullet(x_1^t, \ldots, x_n^t; R)$ [ILL+07, Chapter 7]. We have

$$\varinjlim_t \left(\frac{R}{(x_1^t, \ldots, x_n^t)} \xrightarrow{x_1 x_2 \cdots x_n} \frac{R}{(x_1^{t+1}, \ldots, x_n^{t+1})} \right) = \mathrm{H}^i_I(R).$$

If $x_1, \ldots, x_n$ is an R-regular then the maps in the above system are injective, so

$$\frac{R}{(x_1^t, \ldots, x_n^t)} \hookrightarrow \mathrm{H}_I^i(R).$$

Under this map the element

$$\left[\frac{z}{x_1^t x_2^t \cdots x_n^t}\right]$$

corresponds to $z \mod (x_1^t, \ldots, x_n^t)$. For a proof, see [LT81, p. 104–105].

3.2. **Definition**. A submodule M of $\mathrm{H}_I^i(R)$ is said to be *F-stable* if $F(M) \subseteq M$.

3.3. **Example**. Let $\eta \in \mathrm{H}_I^n(R)$. Then the R-submodule of $\mathrm{H}_I^n(R)$ generated by $F^e(\eta), e \geq 1$ is F-stable. In the proof of the theorem below, we will denote it by M_η. □

3.4. **Theorem** ([Smi97, Theorem 2.6]). *Let $(R, \mathfrak{m})$ be a d-dimension excellent Cohen-Macaulay local ring of characteristic p. Then R is F-rational if and only if $\mathrm{H}_\mathfrak{m}^d(R)$ has no proper non-zero F-stable submodules.*

A special case of this was proved by R. Fedder and K. i. Watanabe: assuming that R is an isolated singularity and that $\mathrm{H}_\mathfrak{m}^i(R)$ has finite length for every $i < d$; see [FW89, Theorem 2.8].

Proof. ‘Only if’: Since R is excellent and F-rational, $\widehat{R}$ is Cohen-Macaulay and F-rational. Since $\mathrm{H}_\mathfrak{m}^d(R)$ is both an R-module and an $\widehat{R}$-module (compatibly), we may assume that R is complete.

By way of contradiction suppose that $0 \neq M \subsetneq \mathrm{H}_\mathfrak{m}^d(R)$ is an F-stable R-submodule of $\mathrm{H}_\mathfrak{m}^d(R)$. Let $C = \mathrm{H}_\mathfrak{m}^d(R)/M$. Taking Matlis duals, we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^\vee & \longrightarrow & \left(\mathrm{H}_\mathfrak{m}^d(R)\right)^\vee & \longrightarrow & M^\vee & \longrightarrow & 0 \\ & & & & \| & & & & \\ & & & & \omega_R & & & & \end{array}$$

where ω_R is a canonical module of R. The isomorphism $\left(\mathrm{H}_\mathfrak{m}^d(R)\right)^\vee \simeq \omega_R$ is local duality [ILL+07, Theorem 11.44]. Note that $M \neq 0 \neq C$, so $C^\vee \neq 0 \neq M^\vee$. Since ω_R is a torsion-free R-module of rank 1, $K \otimes_R M^\vee = 0$. Since $M^\vee$ is a finitely generated R-module, there exists $0 \neq c \in R$ such that $cM^\vee = 0$, so $cM = 0$. Let

$$\eta := \left[\frac{z}{x_1^t x_2^t \cdots x_d^t}\right] \in M$$

be a non-zero element. Hence

$$cF^e(\eta) = \left[\frac{cz^q}{x_1^{tq} x_2^{tq} \cdots x_d^{tq}}\right] = 0.$$

for every q. This means that $cz^q \in (x_1^{tq}, x_2^{tq}, \ldots, x_d^{tq})$ for every q, i.e., $z \in (x_1^t, x_2^t, \ldots, x_d^t)^* = (x_1^t, x_2^t, \ldots, x_d^t)$, so $\eta = 0$, a contradiction.

‘If’: To make the argument simple, we will assume that R is a domain. By way of contradiction, assume that R is not F-rational. Let $x_1, \ldots, x_d$ be a system of parameters and $z \in (x_1, \ldots, x_d)^* \smallsetminus (x_1, \ldots, x_d)$. Write

$$\eta = \left[\frac{z}{x_1 x_2 \cdots x_d}\right] \in \mathrm{H}^d_{\mathfrak{m}}(R).$$

Note that $\eta \neq 0$, so $M_\eta \neq 0$. Let $0 \neq c \in R$ be such that $cz^q \in (x_1^q, x_2^q, \ldots, x_d^q)$ for every $q \geq 1$. Then $c\eta^q = 0$ for every $q \geq 1$, so $cM_\eta = 0$. Note that $M_\eta \neq \mathrm{H}^d_{\mathfrak{m}}(R)$ since the annihilator of $\mathrm{H}^d_{\mathfrak{m}}(R)$ is 0. This contradicts the hypothesis. □

3.5. **Example**. Suppose that R is positively graded with R_0. Write $\mathfrak{m}$ for the homogeneous maximal ideal. Then

$$\bigoplus_{j \geq 0} \left(\mathrm{H}^d_{\mathfrak{m}}(R)\right)_j$$

is a proper F-stable submodule of $\mathrm{H}^d_{\mathfrak{m}}(R)$. Using this, we see that the ring

$$\mathbb{k}[x, y, z]/(x^3 + y^3 + z^3)$$

is not F-rational for any field $\mathbb{k}$. □

3.6. **Example**. Let $R = \mathbb{F}_2[x, y, z]/(x^2+y^3+z^5)$. This is a Cohen-Macaulay normal domain. It is a graded ring if we set $\deg x = 15$, $\deg y = 10$ and $\deg z = 6$. Let $S = \mathbb{F}_2[x, y, z]$. Then

$$\left[\mathrm{H}^3_{(x,y,z)}(S)\right]_{-31} \neq 0 \text{ and } \left[\mathrm{H}^3_{(x,y,z)}(S)\right]_j = 0 \text{ for every } j \geq -30.$$

Hence

$$\left[\mathrm{H}^3_{(x,y,z)}(R)\right]_{-1} \neq 0 \text{ and } \left[\mathrm{H}^3_{(x,y,z)}(R)\right]_j = 0 \text{ for every } j \geq 0.$$

However, R is not F-rational, as we see now. Since $x \notin (y, z)$,

$$0 \neq \left[\frac{x}{yz}\right] \in \mathrm{H}^2_{(x,y,z)}(R).$$

On the other hand, $x^2 \in (y^2, z^2)$, so

$$\left[\frac{x^2}{y^2 z^2}\right] = F\left(\left[\frac{x}{yz}\right]\right) = 0.$$

Hence F has a non-zero kernel. It is easy to check that kernel of F is an F-stable submodule of $\mathrm{H}^2_{(x,y,z)}(R)$. □

4. F-rationality implies pseudo-rationality

5. Exercises

5.1. Derive the 'exact sequence of low-degree terms' for the $''E_2$ page:

$$0 \longrightarrow {}''E_2^{0,1} \xrightarrow{\text{edge}} \mathrm{H}^1(F^\bullet) \longrightarrow {}''E_2^{1,0} \xrightarrow{d_2^{1,0}} {}''E_2^{0,2} \xrightarrow{\text{edge}} \mathrm{H}^2(F^\bullet) \longrightarrow$$

5.2. Place an exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

on the horizontal axis and take a Cartan-Eilenberg injective resolution. Let F be a left-exact covariant functor. Show that the $'E_3$ page is zero and that the maps on the $'E_1$ and $'E_2$ pages give the familiar exact sequence in R^iF.

5.3. Show that $\operatorname{Tor}_*^R(M, N) \simeq \operatorname{Tor}_*^R(N, M)$ by looking at the third quadrant double complex

$$C^{-i,-j} = F_i \otimes G_j$$

where $F_\bullet$ and $G_\bullet$ are projective resolutions of M and N respectively.

5.4. The following is an example of a step in the construction of pure resolutions by Eisenbud and Schreyer.

Let $X = \mathbb{P}^1_{\Bbbk} \times \mathbb{P}^1_{\Bbbk}$, where $\Bbbk$ is a field. Give homogeneous coordinates u, v and x, y respectively. Let $S = \Bbbk[u, v, x, y]$, with $\deg u = \deg v = (1, 0)$ and $\deg x = \deg y = (0, 1)$.

(1) The Koszul complex on S with respect to $ux, uy + vx, vy$ gives an exact sequence

$$K_\bullet : \qquad 0 \longrightarrow \mathscr{O}_X(-3, -3) \longrightarrow \mathscr{O}_X(-2, -2)^{\oplus 3} \longrightarrow \mathscr{O}_X(-1, -1)^{\oplus 3} \longrightarrow \mathscr{O}_X \longrightarrow 0.$$

(Hint: X can be thought of as the set of bigraded ideals not containing the *irrelevant* ideal $(u, v) \cap (x, y)$. The two projection maps from X are given by contraction to $\Bbbk[u, v]$ and $\Bbbk[x, y]$.)

(2) Let $\pi : X \longrightarrow \mathbb{P}^1_{\Bbbk}$ be the projection to the first factor. Let $I^{\bullet,\bullet}$ be a Cartan-Eilenberg injective resolution of $K_\bullet$. Let $C^{\bullet,\bullet} = \pi_*(I^{\bullet,\bullet})$. (This is a 'first-quadrant' double complex.) Use the projection formula to see that

$$'E_1^{i,j} = \begin{cases} \mathscr{O}_{\mathbb{P}^1_{\Bbbk}}(-3)^{\oplus 2}, & \text{if } i = -3 \text{ and } j = 1; \\ \mathscr{O}_{\mathbb{P}^1_{\Bbbk}}(-2)^{\oplus 3}, & \text{if } i = -2 \text{ and } j = 1; \\ \mathscr{O}_{\mathbb{P}^1_{\Bbbk}}, & \text{if } i = 0 \text{ and } j = 0; \\ 0, & \text{otherwise.} \end{cases}$$

(3) Use the $''E$ spectral sequence to conclude that $'E_\infty^{i,j} = 0$ for every i, j.

(4) Conclude that the non-zero terms of the $'E_1$ page give an exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^1_{\Bbbk}}(-3)^{\oplus 2} \longrightarrow \mathscr{O}_{\mathbb{P}^1_{\Bbbk}}(-2)^{\oplus 3} \longrightarrow \mathscr{O}_{\mathbb{P}^1_{\Bbbk}} \longrightarrow 0.$$

(Getting a pure resolution over $\Bbbk[u, v]$ from the above exact sequence requires a little more work, which we omit in this exercise.)

(5) Using the same strategy, construct an exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^1_{\Bbbk}}(-3)^{\oplus a} \longrightarrow \mathscr{O}_{\mathbb{P}^1_{\Bbbk}}(-1)^{\oplus b} \longrightarrow \mathscr{O}_{\mathbb{P}^1_{\Bbbk}}(1)^{\oplus c} \longrightarrow 0.$$

5.5. Do a 'diagram-chasing' in the commutative diagram below

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma_Z(X, \mathcal{F}_1) & \longrightarrow & \Gamma_Z(X, \mathcal{F}_2) & \longrightarrow & \Gamma_Z(X, \mathcal{F}_3) & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(X, \mathcal{F}_1) & \longrightarrow & \Gamma(X, \mathcal{F}_2) & \longrightarrow & \Gamma(X, \mathcal{F}_3) & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(U, \mathcal{F}_1) & \longrightarrow & \Gamma(U, \mathcal{F}_2) & \longrightarrow & \Gamma(U, \mathcal{F}_3) & \longrightarrow &
\end{array}$$

to conclude that $\Gamma_Z(X, -)$ is left-exact.

5.6. Let $(R, \mathfrak{m})$ be a two-dimensional analytically unramified normal domain and $f : W \longrightarrow \operatorname{Spec} R$ a proper birational morphism with W normal.

(1) Show that $\operatorname{Supp}(\mathrm{H}^1(W, \mathscr{O}_W)) \subseteq \{\mathfrak{m}\}$. (Hint: Localize in the base and use flat base-change for cohomology and the fact that over a DVR, every proper birational map is an isomorphism.)

(2) R is pseudo-rational if and only if $\mathrm{H}^1(Z, \mathscr{O}_Z) = 0$ for every Z that has a proper birational map to $\operatorname{Spec} R$. (Use: $\mathrm{H}_{f^{-1}(\{\mathfrak{m}\})}(\mathscr{O}_Z) = 0$ [Lip78, Theorem 2.4, p. 177].)

5.7. Let $(R, \mathfrak{m})$ be a normal local ring. Let $W \xrightarrow{g} Z \xrightarrow{f} \operatorname{Spec} R$ be a proper birational morphisms with Z and W normal. Write $h = fg$. Then the edge map

$$\mathrm{H}^d_{\mathfrak{m}}(R) \xrightarrow{\delta_h} \mathrm{H}^d_{h^{-1}(\{\mathfrak{m}\})}(W, \mathscr{O}_W)$$

factors as

$$\mathrm{H}^d_{\mathfrak{m}}(R) \xrightarrow{\delta_f} \mathrm{H}^d_{f^{-1}(\{\mathfrak{m}\})}(Z, \mathscr{O}_Z) \longrightarrow \mathrm{H}^d_{h^{-1}(\{\mathfrak{m}\})}(W, \mathscr{O}_W).$$

5.8. Show that rational singularities (in characteristic zero) are pseudo-rational. You need to use the fact that if R has rational singularities, then $\mathrm{R}^i f_* \mathscr{O}_Z = 0$ for every desingularization $f : Z \longrightarrow \operatorname{Spec} R$ and every $i > 0$.

5.9. Let R be a Cohen-Macaulay ring of characteristic zero. Show that R has rational singularities if and only if there exists a proper birational morphism $f : Z \longrightarrow \operatorname{Spec} R$ such that Z has rational singularities and $\mathrm{R}^i f_* \mathscr{O}_Z = 0$ for every $i > 0$.

5.10. Let $(R, \mathfrak{m})$ be a noetherian ring. Let $X = \{\mathfrak{p} \in \operatorname{Spec} R \mid \dim R/\mathfrak{p} = \dim R\}$. Let $a \in \mathfrak{m} \smallsetminus \cup_{\mathfrak{p} \in X} \mathfrak{p}$. If $R/(a)$ is regular, then so is R. Show that the hypothesis on a is necessary.

5.11. Let R be a two-dimensional standard graded normal domain, with $R_0 = \Bbbk$, with homogeneous maximal ideal $\mathfrak{m}$. Assume that $\operatorname{Spec} R \smallsetminus \{\mathfrak{m}\}$ has pseudo-rational singularities. Show that R has pseudo-rational singularities if and only if $\mathrm{H}^2_{\mathfrak{m}}(R)_j = 0$ for every $j \geq 0$ as follows:

(1) Let $X = \operatorname{Proj} R[\mathfrak{m}t]$. Then X has pseudo-rational singularities, and there is a proper birational map $f : X \longrightarrow \operatorname{Spec} R$.

(2) Let $h : W \longrightarrow \operatorname{Spec} R$ with W normal. Let W' be the blow-up of W along the ideal sheaf $\mathfrak{m}\mathscr{O}_W$, and h' the composite map $W' \longrightarrow W \longrightarrow \operatorname{Spec} R$. It suffices to show that the edge map $\delta_{h'}$ is injective. Hence replacing W by W', we may assume that h factors as $W \xrightarrow{g} X \xrightarrow{f} \operatorname{Spec} R$.

(3) $\mathrm{R}^1 g_* \mathscr{O}_W = 0$.

(4) Let E be the divisor of X defined by $\mathfrak{m}\mathscr{O}_X$. Write $\widetilde{E} = h^{-1}(\{\mathfrak{m}\})$. The map

$$\mathrm{H}^2_E(\mathscr{O}_X) \longrightarrow \mathrm{H}^2_{\widetilde{E}}(\mathscr{O}_W)$$

is an isomorphism.

(5) The map

$$\mathrm{H}^2_{\mathfrak{m}}(R) \longrightarrow \mathrm{H}^2_{\widetilde{E}}(\mathscr{O}_W)$$

is injective if and only if the map

$$\mathrm{H}^2_{\mathfrak{m}}(R) \longrightarrow \mathrm{H}^2_E(\mathscr{O}_X)$$

is injective, which holds if and only if $\mathrm{H}^1(X, \mathscr{O}_X) = 0$ which holds if and only if $\mathrm{H}^1(E, \mathscr{O}_E(j)) = 0$ for every $j \geq 0$ which holds if and only if $\mathrm{H}^2_{\mathfrak{m}}(R)_j = 0$ for every $j \geq 0$. You will need to use two facts: $E \simeq \operatorname{Proj} R$ and that $\mathfrak{m}^j \mathscr{O}_X \otimes_X \mathscr{O}_E \simeq \mathscr{O}_E(j)$.

5.12. Let $R = \Bbbk[x^2, x^3]$ where $\Bbbk$ is a field of characteristic $p > 0$. Show that $x^3 \in (x^2)^* \smallsetminus (x^2)$.

5.13. Let $R = \Bbbk[x, y, z]/(x^3 + y^3 + z^3)$ where $\Bbbk$ is a field of characteristic $p > 0$, $p \neq 3$. Show that $z^2 \in (x, y)^* \smallsetminus (x, y)$.

5.14. Let R be a noetherian ring, and I an R-ideal. Show that if I is tightly closed, then $(I : J)$ is tightly closed for every ideal J.

5.15. Show that an intersection of tightly closed ideals is tightly closed.

5.16. Let $(R, \mathfrak{m})$ be a Gorenstein ring, I an unmixed R-ideal, and $x_1, \ldots, x_c \in I$ a maximal regular sequence. Write $J = (x_1, \ldots, x_c)$. Show that $(J : (J : I)) = I$.

5.17. Show that every ideal in a Gorenstein F-rational ring is tightly closed.

5.18. Let R be a local ring and $\widehat{R}$ its completion. If $\widehat{R}$ is F-rational, then R is F-rational.

5.19. Let $(R, \mathfrak{m})$ be a two-dimensional pseudo-rational ring. Following [LT81, Section 5], prove the (special case of) Brian con-Skoda theorem:

$$\overline{I^{n+2}} \subseteq I^n$$

for every $n \geq 1$, as follows:

(1) We may assume that $R/\mathfrak{m}$ is an infinite field [LT81, Example (c), p. 103].

(2) I has a *reduction* generated by two elements, i.e., there exists $J = (x, y) \subseteq I$ such that $I^{n+1} = JI^n$ for every $n \gg 0$. (Hint: take a Noether normalization of $R[It]/\mathfrak{m}R[It]$.)

(3) The ideal generated by xt, yt in $A := \overline{R[It]}$ is primary to the irrelevant ideal.

(4) Let $X = \operatorname{Proj} A$. The Koszul complex $K_\bullet(xt, yt; A)$ gives an exact sequence

$$0 \longrightarrow \mathscr{O}_X(n) \longrightarrow \mathscr{O}_X(n+1) \longrightarrow \mathscr{O}_X(n+2) \longrightarrow 0$$

for every $n \in \mathbb{Z}$. (Here, by $\mathscr{O}_X(1)$ we mean the invertible sheaf $I\mathscr{O}_X$.)

(5) For $n \geq 0$, this gives an exact sequence

$$0 \longrightarrow \overline{I^n} \longrightarrow \overline{I^{n+1}} \xrightarrow{[x\ y]} \overline{I^{n+2}} \longrightarrow 0.$$

(Use: $\overline{I^n} = \mathrm{H}^0(X, \mathscr{O}_X(n))$ for every $n \geq 1$, $I^0 := R = \mathrm{H}^0(X, \mathscr{O}_X)$.)

(6) $\overline{I^{n+2}} = I\overline{I^{n+1}}$ for every $n \geq 0$.

(7) $\overline{I^{n+2}} \subseteq I^n$ for every $n \geq 1$.

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Chennai Mathematical Institute, Siruseri, Tamilnadu 603103. India

E-mail address: mkummini@cmi.ac.in