Big Cohen-Macaulay algebras

Part 1: Applications

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https://www.cmi.ac.in/~mkummini/bigcm.pdf

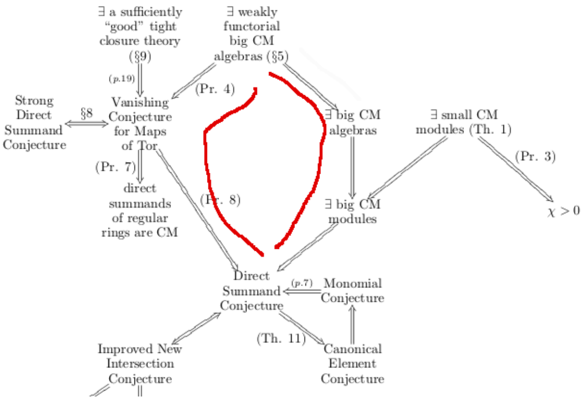
These are expository lectures on the “big Cohen-Macaulay algebras” conjecture (Hochster) and its proof in the prime characteristic case.

This lecture: the conjecture and some applications.

Next lecture: proof in the prime characteristic case (Huneke-Lyubeznik).

Background

1 Homological Conjectures: a diagram



1. Hochster, Topics in the homological theory of ..., CBMS notes, AMS.
2. Hochster, Current state of the homological conjectures, Univ. Utah. www.math.utah.edu/vigre/minicourses/algebra/hochster.pdf

Throughout this talk R is a noetherian ring.

(But not R-algebras, necessarily.)

Definition

Let R be a local ring. An R-algebra S is said to be a *Cohen-Macaulay R-algebra* if a system of parameters of R is a S-regular sequence.

big Cohen-Macaulay R-algebra: to emphasise that it is not necessarily finitely generated as an R-module.

Definition

Let R be a local ring and S a Cohen-Macaulay R-algebra. Say that S is *balanced* if every system of parameters of R is S-regular.

Absolute Integral Closure

Definition

Let R be a domain. The *absolute integral closure* R^+ of R is the integral closure of R in an algebraic closure of its fraction field.

Theorem ([HH92, Theorem 1.1])

Let R be an excellent local domain of characteristic $p > 0$. Then R^+ is a balanced (big) Cohen-Macaulay R-algebra.

Theorem ([HL07, Corollary 2.3])

Let R be a local domain of characteristic $p > 0$, that is a homomorphic image of a Gorenstein local ring. Then R^+ is a balanced (big) Cohen-Macaulay R-algebra.

Weak functoriality

Let $R \to S$ be be a local map of excellent local domains (or local domains that are homomorphic images of Gorenstein rings) of characteristic $p > 0$. Then there exists a commutative diagram

$$\begin{array}{ccc} R^{+} & \longrightarrow & S^{+} \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

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We can consider $R \hookrightarrow S$ and $R \twoheadrightarrow S$ separately.

Weak functoriality

Injective case: $R \subseteq S$.

$K \subseteq L$: respective fraction fields.

$$\begin{array}{ccc}
\overline{K} & \subseteq & \overline{L} \\
\uparrow & & \uparrow \\
R^{+} & \subseteq & S^{+} \\
\uparrow & & \uparrow \\
R & \subseteq & S
\end{array}$$

Weak functoriality

Surjective case: $R \twoheadrightarrow S$.

In general: for a domain A, A^+ is characterised by

1. A^+ is a domain and contains A as a subring;
2. A^+ is integral over A;
3. every monic $f(T) \in A^+[T]$ splits into monic linear factors over A^+.

Write $S = R/\mathfrak{p}$.

Let $\mathfrak{q} \subseteq R^+$ be a prime ideal lying over $\mathfrak{p}$.

Then

$$S^+ \simeq R^+/\mathfrak{q}$$

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The above results do not hold verbatim in characteristic 0 in dim ≥ 3.

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x, y, z system of parameters.

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Suppose $a \in (x, y)R^+$. Write $a = s_1 x + s_2 y$ for some $s_1, s_2 \in R^+$.

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Let $S \subseteq R^+$ be a finite extension of R with $s_1, s_2 \in S$.
$K \subseteq L$ fraction fields of R and S.
Then there exists a non-zero integer m (invertible in R) such that

$$ma = \mathrm{Trace}_{L/K}(a) = x\mathrm{Trace}_{L/K}(s_1) + y\mathrm{Trace}_{L/K}(s_2) \in (x, y)R.$$

Hence x, y, z cannot be R^+-regular unless R is Cohen-Macaulay.

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Nonetheless, we have the following:

Theorem ([HH92, Theorem 8.1])

Let $(R, \mathfrak{m})$ *be an equi-characteristic local domain. Then there exists a local (not necessarily noetherian) ring* $(S, \mathfrak{n})$ *with a local map* $R \to S$ *such that* S *is a balanced (big) Cohen-Macaulay* R*-algebra.*

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- In characteristic > 0, $R \to \widehat{R} \to \widehat{R}/\mathfrak{p} \to \left(\widehat{R}/\mathfrak{p}\right)^{+} =: S$, where $\mathfrak{p}$ is a prime ideal of $\widehat{R}$ of maximum dimension.

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- In characteristic 0, Artin approximation and reduction to characteristic > 0.

Vanishing of maps of Tor.

Theorem

Let $R \to S \to T$ be equi-characteristic noetherian rings, with R and T regular, R a domain, and S module-finite and torsion-free over R. Then for every R-module M and for every $i \geq 1$, the map

$$\mathrm{Tor}_i^R(M, S) \to \mathrm{Tor}_i^R(M, T)$$

is zero.

Proof:
We follow [Hun96, Chapter 9].

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May assume M a finitely generated R-module.

Localize R, S and M at the contraction of the maximal ideal of T to R, and complete.

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$\ker(S \to T)$ is a prime ideal of S, so it contains a minimal prime ideal $\mathfrak{p}$ of S. Note that $\mathfrak{p} \cap R = 0$ ($\because$ torsion-free).

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Given map factors as:

$$\begin{array}{ccc} \mathrm{Tor}_i^R(M, S) & \longrightarrow & \mathrm{Tor}_i^R(M, T) \\ & \searrow \quad \nearrow & \\ & \mathrm{Tor}_i^R(M, S/\mathfrak{p}) & \end{array}$$

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Replace S by $S/\mathfrak{p}$ and assume S complete local domain.

We have a commutative diagram

$$\begin{array}{ccccc} & & A & \longrightarrow & B \\ & & \uparrow & & \uparrow \\ R & \longrightarrow & S & \longrightarrow & T \end{array}$$

where A and B are balanced Cohen-Macaulay algebras for S and for T.

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Fact: Since R is regular, an R-algebra C is a balanced big Cohen-Macaulay R-algebra if and only if C is faithfully flat over R.

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Hence, A is faithfully flat over R, and B is faithfully flat over T.

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3. Let G be a finite group and V a finite-dimensional representation of G over a field $\Bbbk$ such that $|G|$ is invertible in $\Bbbk$. Let $S = \mathrm{Sym}\, V^*$ and $R = S^G$. Then $R \to S$ splits.

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Let $M = A/(x_1, \ldots, x_d)$.

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R is a free A-module, so it is a Cohen-Macaulay ring.

Direct summand conjecture

Conjecture

If $R \subseteq S$ is a module-finite extension of rings and R regular, then R is a direct summand of S as an R-module.

Vanishing of the maps of Tor implies the direct summand conjecture.

Tight closure

Definition

Let R be a domain of characteristic $p > 0$ and I an R-ideal. The *tight closure* of I is the set

$$I^* := \{z \in R \mid \exists c \in R \smallsetminus 0 \text{ such that for every } e \geq 0, cz^{p^e} \in I^{[p^e]}\}.$$

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If S is a module-finite extension of R, $IS \cap R \subseteq I^*$.

Question: Is $IR^+ \cap R = I^*$?

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Theorem ([Hoc94, Theorem 11.1])

Let $(R, \mathfrak{m})$ be a complete local domain of characteristic $p > 0$. Let I be an R-ideal. Let $x \in R$. Then $x \in I^*$ if and only if there exists a balanced Cohen-Macaulay R-algebra S such that $x \in IS$.

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Theorem ([Smi94, Theorem 5.1])

Let R be a locally excellent noetherian domain of characteristic $p > 0$. Let $x_1, \ldots, x_d$ be elements of R such that they form a part of a system of parameters in $R_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ containing $x_1, \ldots, x_d$. Write $I = (x_1, \ldots, x_d)$. Then $IR^+ \cap R = I^*$.

Mixed characteristic

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Thank you!