Big Cohen-Macaulay algebras

Part 2: Absolute integral closure in prime characteristic

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2020-07-03

https://www.cmi.ac.in/~mkummini/notes/Rplus.pdf

These are expository lectures on the “big Cohen-Macaulay algebras” conjecture (Hochster) and its proof in the prime characteristic case.

Previous lecture: the conjecture and some applications.

This lecture: proof by Huneke and Lyubeznik in the prime characteristic case that the absolute integral closure is a big Cohen-Macaulay algebra.

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Let R be a domain. The *absolute integral closure* R^+ of R is the integral closure of R in an algebraic closure of its fraction field.

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Let R be domain of characteristic $p > 0$, that is a homomorphic image of a Gorenstein local ring. Then R^+ is a balanced (big) Cohen-Macaulay R-algebra.

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Theorem ([HL07, Corollary 2.3(a)])

Let R be as above. $\mathrm{H}^i_{\mathfrak{m}}(R^+) = 0$ *for every* $i < \dim R$.

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Theorem ([HL07, Corollary 2.3(a)])

Let R be as above. $\mathrm{H}^i_{\mathfrak{m}}(R^+) = 0$ *for every* $i < \dim R$.

We will sketch the proof of this implication now.

Local cohomology, 1

Let $I = (x_1, \ldots, x_d)$ be an R-ideal.

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Its right-derived functors $\mathrm{H}^i_I(-), i \in \mathbb{N}$ are called *local cohomology functors* (with support in I).

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Write $I_t = (x_1, \ldots, x_t)R$, $1 \leq t \leq d$.

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Hence, it suffices to show that every element of

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is a non-zero-divisor on $R^+/I_{j-1}R^+$.

Claim: $\mathfrak{m} \notin \operatorname{Ass}_R R^+/I_{j-1}R^+$ for each $2 \le j \le d$.

I.e., $\mathfrak{m}$ is not associated if we don't go modulo a full system of parameters.

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Let $\mathfrak{p} \in \operatorname{Ass}_R R^+/I_{j-1}R^+$.

Since $(R_\mathfrak{p})^+ = (R^+)_\mathfrak{p}$, it follows that $\mathfrak{p}R_\mathfrak{p} \in \operatorname{Ass}_{R_\mathfrak{p}}(R_\mathfrak{p})^+/I_{j-1}(R_\mathfrak{p})^+$.

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Apply the above claim to the local ring $(R_\mathfrak{p}, \mathfrak{p}R_\mathfrak{p})$ to see that $\mathfrak{p}$ is minimal over I_{j-1}.

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Apply the above claim to the local ring $(R_\mathfrak{p}, \mathfrak{p}R_\mathfrak{p})$ to see that $\mathfrak{p}$ is minimal over I_{j-1}.

I_{j-1} is a full system of parameters for $R_\mathfrak{p}$.

This shows that every element of

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is a non-zero-divisor on $R^+/(x_1, \ldots, x_{j-1})R^+$.

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is a non-zero-divisor on $R^+/(x_1, \dots, x_{j-1})R^+$.

In particular x_j is a non-zero-divisor on $R^+/(x_1, \dots, x_{j-1})R^+$.

To prove the claim that $\mathfrak{m} \notin \mathrm{Ass}_R R^+/I_{j-1}R^+$ for each $2 \leq j \leq d$,

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for each $i < d - t$. Apply with $i = 0$, $t = j - 1$. ■

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Therefore

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ETST each map in the directed system $\{\mathrm{H}^i_{\mathfrak{m}}(S)\}$ eventually is zero.

Main Theorem

Theorem ([HL07, Theorem 2.1])

Let $(R, \mathfrak{m})$ *be a* d*-dimensional local domain of characteristic* $p > 0$*, that is a homomorphic image of a Gorenstein local ring. Let* S *be a finite* R*-subalgebra of* R^+*. Let* $i < d$*. Then there exists a finite* S*-subalgebra* S' *of* R^+ *such that the map*

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is zero.

Consequently,

$$\mathrm{H}^i_{\mathfrak{m}}(R^+) = \lim_{\rightarrow} \mathrm{H}^i_{\mathfrak{m}}(S) = 0$$

for all $i < d$.

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Extended Čech (or *stable Koszul*) complex

$$\check{C}^\bullet(x_1, \ldots, x_d): \qquad 0 \to R \to \bigoplus_{1 \le i \le d} R_{x_i} \to \bigoplus_{1 \le i < j \le d} R_{x_i x_j} \to \cdots \to R_{x_1 x_2 \cdots x_d} \to 0$$

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Fact: For all R-modules M,

$$\mathrm{H}^i_I(M) = \mathrm{H}^i(\check{C}^\bullet(x_1, \ldots, x_d) \otimes_R M),$$

if $\sqrt{I} = \sqrt{(x_1, \ldots, x_d)}$.

The Frobenius map $r \mapsto r^p$ commutes with localization: for any multiplicatively closed set $U \subseteq R$,

$$\begin{array}{ccc} R & \longrightarrow & U^{-1}R \\ {\scriptstyle F}\downarrow & & {\scriptstyle F}\downarrow \\ R & \longrightarrow & U^{-1}R \end{array}$$

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Hence it induces a map of complexes $F : \check{C}^{\bullet}(x_1, \ldots, x_d) \to \check{C}^{\bullet}(x_1, \ldots, x_d)$ and on $F : H^i_I(R) \to H^i_I(R)$.

$\alpha \in \mathrm{H}^i_I(R)$ is represented by a cycle

$$\left(\frac{a}{b}\right) \in \bigoplus_{1 \le j_1 \le \cdots \le j_i \le n} R_{x_{j_1} \cdots x_{j_i}} = \check{\mathrm{C}}^i$$

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Then $\alpha^p := F(\alpha)$ is the element of $\mathrm{H}^i_I(R)$ represented by the cycle

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Write α^{p^e} for $F^e(\alpha)$, eth iterate of F.

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Recall our assertion: $\exists\, S'$ such that the map $\mathrm{H}^i_{\mathfrak{m}}(S) \to \mathrm{H}^i_{\mathfrak{m}}(S')$ is zero.

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First find a finite S-subalgebra $\tilde{S}$ of $R^+ = S^+$ such that

$$\mathrm{Im}(\mathrm{H}^i_{\mathfrak{m}}(S) \to \mathrm{H}^i_{\mathfrak{m}}(\tilde{S}))$$

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Hence

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is stable under Frobenius.

Since

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is finitely generated module, one proves that there exists a finite $\tilde{S}$-subalgebra S' of R^+ such that the composite map

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One proves this for each generator of

$$\mathrm{Im}(\mathrm{H}^i_{\mathfrak{m}}(S) \to \mathrm{H}^i_{\mathfrak{m}}(\tilde{S}))$$

and takes the compositum.

Step 2 of the proof

Lemma ('Equational lemma')

Let R be a noetherian domain of characteristic $p > 0$. Let I be an R-ideal and $\alpha \in \mathrm{H}^i_I(R)$ be an element such that $\{\alpha^{p^e} \mid e \geq 0\}$ belong to a finitely generated submodule of $\mathrm{H}^i_I(R)$. Then there exists a finite R-subalgebra R' of R^+ such that α goes to zero under the map

$$\mathrm{H}^i_I(R) \to \mathrm{H}^i_I(R').$$

Since $\sum_{i=0}^{t} R\alpha^{p^i}$, $t \geq 0$ form an ascending chain inside a finitely generated R-module, there exists s such that

$$\alpha^{p^s} = \sum_{i=1}^{s-1} r_i \alpha^{p^{s-i}}, \ r_i \in R, \forall i.$$

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$d^{i-1} : \check{C}^{i-1} \to \check{C}^i$

We show that $\beta = g(\beta') \in \check{C}^{i-1}(R'')$ for finite extension R''.

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$$z_{j_1,\ldots,j_{i-1}} \in R^+$$

such that

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$$\beta = \left(\frac{r_{j_1,\ldots,j_{i-1}}}{(x_{j_1} \cdots x_{j_{i-1}})^e} \right) \in \check{C}^{i-1}(R)$$

For each $(i-1)$-tuple $1 \leq j_1 < \cdots < j_{i-1} \leq d$ there exists

$$z_{j_1,\ldots,j_{i-1}} \in R^+$$

such that

$$g\left(\frac{z_{j_1,\ldots,j_{i-1}}}{(x_{j_1} \cdots x_{j_{i-1}})^e} \right) = \frac{r_{j_1,\ldots,j_{i-1}}}{(x_{j_1} \cdots x_{j_{i-1}})^e}$$

If we expand this out, and clear denominators by multiplying by $(x_{j_1} \cdots x_{j_{i-1}})^{ep^s}$, we get a monic polynomial expression of $z_{j_1,\ldots,j_{i-1}}$ over R.

Adjoining these finitely many $z_{j_1,\ldots,j_{i-1}}$, we get a finite R-subalgebra R'' of R^+ and

$$\beta' \in \check{C}^{i-1}(R'')$$

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$$g(\beta') = \beta.$$

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$$g(\bar{\alpha}) = g(\tilde{\alpha}) - g(d^{i-1}(\beta')) = d^{i-1}(\beta) - d^{i-1}(g(\beta')) = 0$$

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Hence α goes to zero in $\mathrm{H}^i_I(R')$.

This outlines Step 2 of the proof. ∎

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$\mathcal{D}$ is an exact functor, and takes finite-length A-modules to finite-length A-modules.

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Equivalently,

$$\operatorname{Im}(\operatorname{Ext}^{n-i}_A(\tilde{S}, A) \to \operatorname{Ext}^{n-i}_A(S, A))$$

is a finite-length R- (or A-) module.

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Hence there exists a finite $S_{\mathfrak{p}}$-algebra $S'^{\mathfrak{p}}$ such that the map

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Matlis duality for $A_{\mathfrak{p}}$ gives that the map

$$\operatorname{Ext}^{\operatorname{ht}\mathfrak{p}-i}_{A_{\mathfrak{p}}}(S'^{\mathfrak{p}}, A_{\mathfrak{p}}) \to \operatorname{Ext}^{\operatorname{ht}\mathfrak{p}-i}_{A_{\mathfrak{p}}}(S_{\mathfrak{p}}, A_{\mathfrak{p}})$$

is zero.

Clear denominators to get a finite S-subalgebra $\tilde{S}^{\mathfrak{p}}$ of R^+ such that

$$\mathrm{Im}(\mathrm{Ext}_A^{n-i}(\tilde{S}^{\mathfrak{p}}, A) \to \mathrm{Ext}_A^{n-i}(S, A))$$

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Do this for each $\mathfrak{p}$ in the finite set

$$\operatorname{Ass}_A \operatorname{Ext}_A^{n-i}(S, A) \smallsetminus \{\mathfrak{n}\}$$

to get a finite S-subalgebra $\tilde{S}$ of R^+ such that

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In other words,

$$\operatorname{Im}(\operatorname{Ext}_A^{n-i}(\tilde{S}, A) \to \operatorname{Ext}_A^{n-i}(S, A))$$

has finite length. ∎

Separability

Theorem (Sannai-Singh [SS12])

Let $(R, \mathfrak{m})$ be a d-dimensional local domain of characteristic $p > 0$, that is a homomorphic image of a Gorenstein local ring. Let $i < d$.

1. *[SS12, Theorem 1.3(2)] Let S be a finite R-subalgebra of R^+. Then there exists a finite S-subalgebra S' of R^+ such that the map*

$$\mathrm{H}^i_{\mathfrak{m}}(S) \to \mathrm{H}^i_{\mathfrak{m}}(S')$$

   is zero and the field extension $[\mathrm{Frac}(S') : \mathrm{Frac}(S)]$ is Galois.

2. *[SS12, Corollary 3.3] Write $R^{+\mathrm{sep}}$ for the elements of R^+ separable over $\mathrm{Frac}(R)$. Then $\mathrm{H}^i_{\mathfrak{m}}(R^{+\mathrm{sep}}) = 0$. Consequently, $R^{+\mathrm{sep}}$ is a balanced big Cohen-Macaulay algebra.*

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Thank you!