Thu 14/07

# Classification of Simple Singularities

**Goal:** Simple sings.

$\Leftrightarrow$ $A_k$ : $x_1^{k+1} + x_2^2 + \dots + x_n^2$ , $k \geq 1$

$D_k$ : $x_1(x_2^2 + x_1^{k-2}) + x_3^2 + \dots + x_n^2$ , $k \geq 4$

$E_6$ : $x_1^3 + x_2^4 + x_3^2 + \dots + x_n^2$

$E_7$ : $x_1(x_1^2 + x_2^3) + x_3^2 + \dots + x_n^2$

$E_8$ : $x_1^3 + x_2^5 + x_3^2 + \dots + x_n^2$

## 1. Smooth germs

**Lemma** For $m \subseteq \mathbb{C}\{x\}$, TFAE

(i) $\mu(f) = 0$

(ii) $\tau(f) = 0$

(iii) $f$ is non-sing

(iv) $f \overset{r}{\sim} f^{(1)}$

(v) $f \overset{r}{\sim} x_1$

Pf : Ex.

## 2. Non-degenerate sings

Let $U \subseteq \mathbb{C}^n$ be open, $f : U \to \mathbb{C}$ holo. fcn.

Set $H(f) = \left( \dfrac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j=1}^{n} \in M_n(\mathbb{C}\{x\})$

the Hessian of $f$

Def'n A critical pt $p$ of $f$ is stb. non-degenerate if $\operatorname{rk} H(f)(p) = n$. The corank of $f$ at $p$ is the number $\operatorname{crk}(f,p) = n - \operatorname{rk} H(f)(p)$

Ex: Let $p$ be a crit pt. of $f$. If $\phi : (\mathbb{C}^n, p) \to (\mathbb{C}^n, p)$ is an isom. then

$$H(f\circ\phi)(p) = J(\phi)(p)^t \, H(f)(p) \cdot J(\phi)(p)$$

Thus, the notion of non-deg. crit pt. is indep. of the choice of coords.

Rmk: If $p$ is a reg. pt., $\operatorname{rk} H(f)(p)$ may depend on the choice of coords.

Thm (Morse Lemma) For $f \in \mathfrak{m}^2$, TFAE

(a) $\operatorname{crk}(f,0) = 0$

(b) $\mu(f) = 1$

(c) $\tau(f) = 1$

(d) $f \overset{r}{\sim} f^{(2)}$ and $f^{(2)}$ is non-deg

(e) $f \overset{r}{\sim} x_1^2 + \dots + x_n^2$ (f) $f \overset{c}{\sim} x_1^2 + \dots + x_n^2$

Pf Write $f(x) = \sum_{1 \le i,j \le n} h_{ij}(x) x_i x_j$, $h_{ij} \in \mathbb{C}\{x\}$,

with $(h_{ij}(0)) = \frac{1}{2} H(f)(0)$.

(a) $\Rightarrow$ (b). As $h_{ij}(0) = h_{ji}(0)$,

$$\frac{\partial f}{\partial x_\nu} = \sum_{i,j} \frac{\partial h_{ij}}{\partial x_\nu} x_i x_j + \sum_j h_{\nu j} x_j + \sum_i h_{i\nu} x_i \equiv 2\sum_{j=1}^n h_{\nu j}(0) \pmod{\mathfrak{m}^2}$$

As $H(f)(0)$ is nonsingular,

$$\left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle = \langle x_1, \dots, x_n \rangle \pmod{\mathfrak{m}^2}$$

NAK $\Rightarrow j(f) = m \Rightarrow \mu = 1$.

(b) $\Rightarrow$ (c) ~~Clear~~ Exercise

(c) $\Rightarrow$ (b) $\Rightarrow$ (d)

If $\tau = 1$, then $m = \langle f, j(f) \rangle$. But $f \in m^2$.

NAK $\Rightarrow m = j(f)$, so $\mu(f) = 1$.

$\Rightarrow f$ is right 2-det'd,

ie $f \overset{r}{\sim} f^{(2)}$

(d) $\Rightarrow$ (e) ~~Thus ds~~. As $H(f)(0)$ is symmetric, $\exists$ an orthogonal matrix $T$ st

$$T^t \tfrac{1}{2} H(f)(0) T = \mathbb{1}_n$$

Making the coord change $x \longmapsto T.x$ for $f = f^{(2)}$, we get

$$f(T.x) = x.T^t . \tfrac{1}{2} H(f)(0).T.x^t$$

$$= x_1^2 + \dots + x_n^2$$

(e) $\Rightarrow$ (f) Clear.

(f) $\Rightarrow \tau(f) = 1 \Rightarrow$ (c)

(e) $\Rightarrow$ (a) Clear.

3. Corank 1 Singularities

Thm (Splitting Lemma) If $f \in m^2 \subseteq \mathbb{C}\{x\}$ has $\mathrm{rk}\, H(f)(0) = k$, then

$$f \overset{r}{\sim} x_1^2 + \dots + x_k^2 + g(x_{k+1}, \dots, x_n)$$

with $g \in m^3$, residual part. It is det'd up to rt. equiv

Allows us to reduce us to germs in $m^3$.

Pf: As $H(f)(0)$ has rank $k$,

$f^{(2)} \overset{r}{\sim} x_1^2 + \dots + x_k^2$ by a linear change of coords, as above.

So, wma

$$f(x) = x_1^2 + \dots + x_k^2 + f_3(x_{k+1}, \dots, x_n) + \sum_{i=1}^{k} x_i \cdot g_i(x_1, \dots, x_n),$$

$g_i \in m^2$, $f_3 \in m^3$.

Make the coord. change $x_i \mapsto x_i - \frac{g_i}{2}$, $i = 1, \dots, k$ and $x_i \mapsto x_i$, $i > k$ :

Exercise

$$\begin{cases} f(x) = x_1^2 + \dots + x_k^2 + f_3(x_{k+1}, \dots, x_n) \\ \qquad + f_4(x_{k+1}, \dots, x_n) + \sum_{i=1}^{k} x_i \cdot h_i(x) \\ \text{with } h_i \in m^3, f_4 \in m^4. \end{cases}$$

Continue with $h_i$ instead of $g_i$. The last term can thus be made of arb. high order, hence 0 in the limit.

If $f$ has isolated sing, use Finite Determinacy Thm. In general, need to verify convergence

Exercise: Uniqueness. Use ~~use~~ Mather-Yau for $M_f$.

**Thm** Let $f \in m^2 \subseteq \mathbb{C}\{x\}$, $k \geq 1$. TFAE:

(a) $\operatorname{crk}(f) \leq 1$, $\mu(f) = k$

(b) $f \overset{r}{\sim} x_1^{k+1} + x_2^2 + \ldots + x_n^2$, ie $f$ is of type $A_k$

(c) $f \overset{c}{\sim} x_1^{k+1} + x_2^2 + \ldots + x_n^2$

Further, $\operatorname{crk}(f) = 0$ iff $f$ is $A_1$

$\operatorname{crk}(f) = 1$ iff $f$ is $A_k$, $k \geq 2$.

Pf: (b) $\Rightarrow$ (c) $\Rightarrow$ (a) Clear

We prove (a) $\Rightarrow$ (b)

By Splitting Lemma, woma

$$f = g(x_1) + x_2^2 + \ldots + x_n^2$$
$$= u \cdot x_1^{k+1} + x_2^2 + \ldots + x_n^2, \quad u \in \mathbb{C}\{x_1\}^*$$

Then, $x_1' = \sqrt[k+1]{u} \cdot x_1$, $x_i' = x_i$, $i \geq 2$, transforms $f$ to $A_k$

**Cor** $A_k$-sings are right simple: there is a nbhd of $f$ in $m^2$ which only meets orbits of sings. of type $A_l$, $l \leq k$.

Pf As ~~crk~~ $\operatorname{crk}(f)$ is u.s.c. on $m^2$, a nbhd of $A_k$ contains only ~~$A_l$~~- sings. of $\operatorname{crk} \leq 1$. As $\mu$ is u.s.c. too, these are singularities with $\mu = l \leq k = \mu(f)$

## 4. Corank 2 Singularities

**Prop** Let $f \in m^3$. Then, $\exists$ linear aut $\varphi \in \mathcal{E}$ of $\mathbb{C}\{x,y\}$

cf. $f^{(3)}$ has one of the following forms:

(1) $xy(x+y)$

(2) $x^2y$

(3) $x^3$

Pf. Ex.

Exercise: Tangent cone of $f$.

If $f \overset{c}{\sim} g$, $u\varphi(f) = g$, then $\operatorname{ord}(f) = \operatorname{ord}(g)$ and $u(0)\, \varphi^{(1)}(f_d) = g_d$

$\Rightarrow f_d \overset{c}{\sim} g_d \Rightarrow f_d \overset{r}{\sim} g_d$ so lie in some $GL(n, \mathbb{C})$ orbit of $\mathfrak{m}^d/\mathfrak{m}^{d+1}$

Ex: Let $f = \alpha_d x^d + \alpha_{d-1} x^{d-1} + \dots + \alpha \in A[x]$

If $\beta := \alpha_{d-1}/d\alpha_d \in A$, subs. $x - \beta$ for $x$ yields a poly:

$$\alpha_d x^d + \beta_{d-2} x^{d-2} + \dots + \beta_0 \in A[x]$$

Tschirnhaus transformation

Thm Let $f \in \mathfrak{m}^3 \subseteq \mathbb{C}\{x,y\}$, $k \geq 4$. TFAE

(a) $f^{(3)}$ factors into at least 2 different factors, and $\mu(f) = k$

(b) $f \overset{r}{\sim} x(y^2 + x^{k-2})$, ie $f$ is of type $D_k$

(c) $f \overset{c}{\sim} x(y^2 + x^{k-2})$

Further, $f^{(3)}$ factors into 3 different factors iff $f$ is $D_4$

P6 (b) $\Leftrightarrow$ (c). OK

(b) $\Rightarrow$ (a) Clear

We shall prove (a) $\Rightarrow$ (b)

Assume $f^{(3)}$ factors into 3 distinct factors.
Then, by the Prop'n, $f^{(3)} \overset{r}{\sim} xy(x+y) = g$

Claim: $\mathfrak{m}^4 \subseteq \mathfrak{m}^2 J(g)$. $\}$ ~~Thus~~

Thus $g$ is r-3 determined. $\Rightarrow f \overset{r}{\sim} g$

If $f^{(3)}$ factors into exactly 2 different factors, then, wma $f^{(3)} = x^2y$. Observe $f \neq f^{(3)}$, for o/w $\mu(f) = \infty$.

Let $m = \operatorname{ord}(f - f^{(3)})$. Consider

$$f^{(m)} = x^2y + \alpha y^m + \beta x y^{m-1} + x^2 h(x,y), \quad (\dagger)$$

$\alpha, \beta \in \mathbb{C}$, $h \in \mathfrak{m}^{m-2}$. Apply the Tschirnhaus transf's

$x = x - \beta/2 \cdot y^{m-2}$, $\quad y = y - h(x,y)$, get

$$f^{(m)}(x,y) = x^2y + \alpha y^m \quad (\dagger\dagger)$$

Case I If $\alpha = 0$, consider $f^{(m+1)}$ which has the form $(\dagger)$, so it can be transformed to $(\dagger\dagger)$. ~~If still $\alpha = 0$~~, continue until $\alpha \neq 0$. Indeed, this must eventually happen as

$\alpha = 0$

$$\Rightarrow \mu(f) \geq \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/(J(f) + \mathfrak{m}^{m-1})$$
$$= \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/(J(f^{(m)}) + \mathfrak{m}^{m-1}) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{\langle x^2, xy, y^{m-1}\rangle}$$

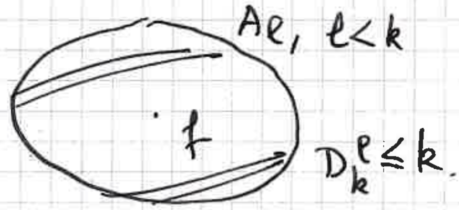
Case II If $\alpha \neq 0$, $y \rightsquigarrow \frac{y}{\alpha^{1/m}}$, $x \rightsquigarrow \alpha^{2/m} x$, get

$$f^{(m)}(x,y) = x^2 y + y^m, \quad m\text{-det'd}$$

$$\Rightarrow f \overset{r}{\sim} y(x^2 + y^{m-1}), \quad D_{m+1}\text{-sing.}$$

Cor $D_k$-sings are right.

Pf: Exercise.




Thm Let $f \in \mathfrak{m}^3 \subset \mathbb{C}\{x,y\}$. TFAE

(a) $f^{(3)}$ has a unique linear factor of mult 3, and $\mu(f) \leq 8$

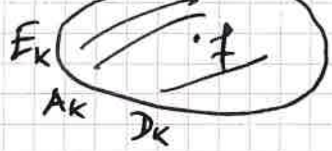
(b) $f^{(3)} \overset{r}{\sim} x^3$ and if $f^{(3)} = x^3$ then $f \notin \langle x, y^2 \rangle^3$

(c) $f \overset{r}{\sim} g$ with $g \in \{x^3 + y^3, x^3 + xy^3, x^3 + y^5\}$, ie $g$ is $E_6, E_7$, or $E_8$

(d) $f \overset{c}{\sim} g$ ——— " ——— " ———

Further, $\mu(E_k) = k$, $k = 6, 7, 8$.

Cor $E_6, E_7, E_8$ are right simple.




$k \leq 8$.

Thus, not ADE

$\Leftrightarrow$ (1) $\operatorname{crk}(f) \geq 3$, or

(2) $\operatorname{crk}(f) = 2$, $f \overset{r}{\sim} g(x_1, x_2) + x_3^2 + \dots + x_n^2$, w/

(i) $g \in \mathfrak{m}^4$ or