

Tue  
1407

## The Finite Determinacy Theorem

We shall show that an <sup>isolated</sup> hyp. sing. is determined (up to right equiv.) by finitely many terms of its power series.

Def'n Let  $f \in \mathbb{C}\{x\}$ . Then, the  $k$ -jet of  $f$ ,  $\text{jet}(f, k) := f^{(k)} := f \bmod m^{k+1}$

Also  $J^{(k)} := \mathbb{C}\{x\}/m^{k+1}$ . Given  $f \in \mathbb{C}\{x\}$ ,  $f^{(k)} \in J^{(k)}$  is identified with the p.s. expansion up to order  $k$ .

Def'n  $f \in \mathbb{C}\{x\}$  is s.t.b. right  $k$ -determined, resp. contact  $k$ -determined, if  $\forall g \in \mathbb{C}\{x\}$  with  $f^{(k)} = g^{(k)}$ , we have  $f \overset{\sim}{\sim} g$ , resp.  $f \overset{\sim}{\sim} g$ .

Thm Let  $F \in \mathbb{C}\{x, t\} = \mathbb{C}\{x_1, \dots, x_n, t\}$ ,  $b, c \in \mathbb{Z}_{>0}$

(1) TFAE:

$$(a) \frac{\partial F}{\partial t} \in \langle x_1, \dots, x_n \rangle^b \left\langle \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n} \right\rangle + \langle x_1, \dots, x_n \rangle^c \cdot \langle F \rangle$$

(b)  $\exists \phi = (\phi_1, \dots, \phi_n) \in \mathbb{C}\{x, t\}^n$ ,  $u \in \mathbb{C}\{x, t\}$  st

(i)  $u(x, 0) = 1$

(ii)  $u(x, t) - 1 \in \langle x_1, \dots, x_n \rangle^c \cdot \mathbb{C}\{x, t\}$

(iii)  $\phi_i(x, 0) = x_i$ ,  $i = 1, \dots, n$

(iv)  $\phi_i(x, t) - x_i \in \langle x_1, \dots, x_n \rangle^b \mathbb{C}\{x, t\}$ ,  $i = 1, \dots, n$

(v)  $u(x, t) \cdot F(\phi(x, t), t) = F(x, 0)$

(2) The condition  $\frac{\partial F}{\partial t} \in \langle x_1, \dots, x_n \rangle^b \left\langle \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n} \right\rangle$  is equiv. to 1(b) with  $u = 1$ .

Remark:  $\phi_t(x) = \phi(x, t)$ ,  $u_t(x) = u(x, t)$

As  $\phi_0 = \text{id}$ , for  $|t| \ll 1$ ,  $\phi_t: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, \phi_t(0))$  is an isom, and similarly,  $u_t \in \mathbb{C}\{x\}^*$ ,  $|t| \ll 1$ .

(1)(b)  $\Rightarrow \phi_t: \mathcal{O}_{\mathbb{C}^n, 0} \xrightarrow{\sim} \mathcal{O}_{\mathbb{C}^n, \phi_t(0)}$ , mapping  $\langle F_0 \rangle$  to  $\langle F_t \rangle$ . Thus,  $(F_0^{-1}(0), 0) \cong (F_t^{-1}(0), \phi_t(0))$  being identity up to order  $b$ .

Pf (1)(a)  $\Rightarrow$  (b) Set  $\langle x \rangle = \langle x_1, \dots, x_n \rangle$

By (a),  $\exists Y_1, \dots, Y_n \in \langle x \rangle^b \cdot \mathbb{C}\{x, t\}$ ,  $Z \in \langle x \rangle^c \cdot \mathbb{C}\{x, t\}$  st.

$$\frac{\partial F}{\partial t} = \sum_{i=1}^n \frac{\partial F}{\partial x_i} Y_i - Z \cdot F$$

I. Set  $Y = (Y_1, \dots, Y_n)$ , and let  $\phi = (\phi_1, \dots, \phi_n)$  be the unique soln for near  $t=0$  of the ODE

$$\frac{\partial \phi}{\partial t}(x, t) = Y(\phi(x, t), t), \quad \phi(x, 0) = x. \quad \square$$

We first show  $\phi_i - x_i \in \langle x \rangle^b \cdot \mathbb{C}\{x, t\}$

Wma  $b \geq 1$ . Then,  $Y(0, t) = 0$ , for  $|t| \ll 1$

Hence  $\phi = 0$  is a soln of the ODE

$$\frac{\partial \phi}{\partial t}(0, t) = Y(\phi(0, t), t), \quad \phi(0, 0) = 0$$

By uniqueness of the soln,  $\phi(0, t) = 0$

$\Rightarrow \phi_i(x, t) \in \langle x \rangle$ . Since  $Y_i \in \langle x \rangle^b$ ,

$$\frac{\partial \phi_i}{\partial t}(x, t) = Y_i(\phi(x, t), t) \in \langle x \rangle^b \cdot \mathbb{C}\{x, t\}$$

$$\Rightarrow \phi_i - x_i \in \langle x \rangle^b.$$

II Observation: Set  $\psi(x,t) = (\phi(x,t), t)$

$$(v) \Leftrightarrow \frac{\partial}{\partial t} [u \cdot (F \circ \psi)(x,t)] = 0$$

III Let  $u$  be the unique soln of the ODE

$$\frac{\partial u(x,t)}{\partial t} = u(x,t) \cdot (Z \circ \psi)(x,t), \quad u(x,0) = 1$$

$$\text{Then, } Z \in \langle x \rangle^c \Rightarrow \frac{\partial u}{\partial t} \in \langle x \rangle^c \Rightarrow u^{-1} \in \langle x \rangle^c$$

$$\therefore \frac{\partial}{\partial t} (u \cdot (F \circ \psi)) = \frac{\partial u}{\partial t} (F \circ \psi) + u \cdot \frac{\partial (F \circ \psi)}{\partial t}$$

$$= u \cdot (Z \circ \psi)(F \circ \psi) + u \cdot \left( \sum_{i=1}^n \frac{\partial F \circ \psi}{\partial x_i} \cdot \frac{\partial \phi_i}{\partial t} + \frac{\partial F \circ \psi}{\partial t} \right)$$

$$= u \cdot \left( (Z \cdot F) \circ \psi + \sum_{i=1}^n \frac{\partial F \circ \psi}{\partial x_i} \circ \frac{\partial \phi_i}{\partial t} - \sum_{i=1}^n \left( \frac{\partial F}{\partial x_i} \cdot \gamma_i \right) \circ \psi \right)$$

$$= 0 \quad // \quad - (Z \cdot F) \circ \psi$$

(ii)  $\Rightarrow$  (i) Omitted.

Thm (Finite Determinacy Thm) Let  $f \in \mathfrak{m} \subseteq \mathbb{C}\{x\}$

(1)  $f$  is right  $k$ -determined if

$$\mathfrak{m}^{k+1} \subseteq \mathfrak{m}^2 \cdot \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$$

(2)  $f$  is contact  $k$ -determined if

$$\mathfrak{m}^{k+1} \subseteq \mathfrak{m}^2 \cdot \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle + \mathfrak{m} \cdot \langle f \rangle$$

Pf Let  $h \in \mathfrak{m}^{k+1}$ , and consider

$$F(x, t) = f(x) + t \cdot h(x) \in \mathbb{C}\{x\}[t]$$

It suffices to show that  $\forall t_0 \in \mathbb{C}$ , the germ of  $\# F$  in  $\mathbb{C}^n \times \mathbb{C}, (0, t_0)$  satisfies cond. 1(a) / (2) in the prev. Thm,

since then  $F_{t_0} \overset{\sim}{\sim} F_t / F_{t_0} \overset{\sim}{\sim} F_t$  for  $|t - t_0| \ll 1$   
 $\Rightarrow f = F_0 \sim F_1 = f + h$ .

For contact equiv, need to show

$$\frac{\partial F}{\partial t} = h \in \mathfrak{m}^2 \cdot \left\langle \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n} \right\rangle + \mathfrak{m} \langle F \rangle$$

As  $h \in \mathfrak{m}^{k+1}$ , we have  $\mathfrak{m}^2 \cdot \frac{\partial h}{\partial x_i} + \mathfrak{m} \cdot h \subseteq \mathfrak{m}^{k+2}$

$$\therefore \mathfrak{m}^2 \cdot \left\langle \frac{\partial (f+th)}{\partial x_1}, \dots, \frac{\partial (f+th)}{\partial x_n} \right\rangle + \mathfrak{m} \cdot \langle f+th \rangle + \mathfrak{m}^{k+2}$$

$$= \mathfrak{m}^2 \cdot \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle + \mathfrak{m} \langle f \rangle + \mathfrak{m}^{k+2}$$

By hyp.,  $\mathfrak{m}^{k+1} \subseteq \text{RHS}$ . Rt. equiv is similar //

Cor If  $f \in \mathbb{C}\{x\}$ ,  $f(0) = 0$  has an isolated sing., then

(1)  $f$  is right  $(\mu+1)$ -det'd

(2)  $f$  is contact  $(\tau+1)$ -det'd.

Pf If  $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ , then  $\mu(f) = \tau(f) = 0$ , and  $f$  is 1-det'd by the IFT. So let  $f \in \mathfrak{m}^2$

Then,  $\dim \mathfrak{m} / \langle f, j(f) \rangle = \tau - 1$ ,

and  $\frac{\mathfrak{m}}{\langle f, j(f) \rangle} \supseteq \frac{\mathfrak{m}^2 + \langle f, j(f) \rangle}{\langle f + j(f) \rangle} \supseteq \dots$

is a strictly decreasing seq.

$\Rightarrow \mathfrak{m}^\tau \subseteq \langle f, j(f) \rangle \Rightarrow \mathfrak{m}^{\tau+2} \subset \mathfrak{m}^2 \cdot j(f) + \mathfrak{m} \cdot \langle f \rangle //$

Thm (Mather - Yan) Let  $f, g \in \mathfrak{m} \subseteq \mathbb{C}\{x\}$ .

TFAE:

(a)  $f \sim^c g$

(b)  $\forall b \geq 0$ ,  $\mathbb{C}\{x\} / \langle f, \mathfrak{m}^b j(f) \rangle \cong \mathbb{C}\{x\} / \langle g, \mathfrak{m}^b j(g) \rangle$   
as  $\mathbb{C}$ -algs.

(c)  $\exists b \geq 0$  st  $\mathbb{C}\{x\} / \langle f, \mathfrak{m}^b j(f) \rangle \cong \mathbb{C}\{x\} / \langle g, \mathfrak{m}^b j(g) \rangle$   
as  $\mathbb{C}$ -algs.

Thus,  $f \sim^c g \Leftrightarrow T_f \cong T_g$ , as  $\mathbb{C}$ -algs.

Pf (a)  $\Rightarrow$  (b) Chain rule. - exercise

(b)  $\Rightarrow$  (c) Clear

(c)  $\Rightarrow$  (a) Let  $\varphi$  be an isom of the algebras  
in (c). Then,  $\varphi$  lifts to an isom  $\tilde{\varphi}: \mathbb{C}\{x\} \rightarrow \mathbb{C}\{x,t\}$   
s.t.  $\tilde{\varphi}(\langle f, m^b_j(f) \rangle) = \langle g, m^b_j(g) \rangle$

$$\text{As } \tilde{\varphi}(\langle f, m^b_j(f) \rangle) = \langle \tilde{\varphi}(f), m^b_j(\tilde{\varphi}(f)) \rangle,$$

$$\text{wma } \langle f, m^b_j(f) \rangle = \langle g, m^b_j(g) \rangle \quad (*)$$

Set  $h = g - f$ , and define

$$I_t = \left\langle f + th, m^b \left\langle \frac{\partial (f+th)}{\partial x_i} \right\rangle \right\rangle \subseteq \mathbb{C}\{x, t\}$$

By (\*),  $I_t \subseteq I_0 \cdot \mathbb{C}\{x, t\} = \langle f, m^b_j(f) \rangle \cdot \mathbb{C}\{x, t\}$ ,

and  $I_0 = I_1$ .

Suppose  $f, g$  are holomorphic in  $V \subseteq \mathbb{C}^n$ , and  
Consider the coherent  $\mathcal{O}_{V \times \mathbb{C}}$ -module

$$\mathcal{F} := \langle I_0 \cdot \mathbb{C}\{x, t\} \rangle /$$

$$\mathcal{F} := \left\langle f, m^b \left\langle \frac{\partial f}{\partial x_i} \right\rangle \right\rangle / \left\langle f + th, m^b \left\langle \frac{\partial (f+th)}{\partial x_i} \right\rangle \right\rangle$$

Then,  $\text{supp } \mathcal{F} \subseteq V \times \mathbb{C}$  is closed and analytic.

$$\Rightarrow \text{supp } \mathcal{F} \cap (\{0\} \times \mathbb{C}) = \{t \in \mathbb{C} \mid \mathcal{F}_{(0,t)} \neq 0\}$$

$$= \{t \in \mathbb{C} \mid I_0 \neq I_t\}$$

is a discrete set in  $\mathbb{C} = \{0\} \times \mathbb{C}$ .

$\Rightarrow U = \{t \mid I_0 = I_t\}$  is open, connected;  $0, 1 \in U$ .

$$\therefore \frac{\partial (f+th)}{\partial t} = h \in I_0 = I_t = \langle f+th, m^b \cdot j(f+th) \rangle$$

$$\Rightarrow f+th \sim f+t'h \text{ for } t, t' \in U, |t-t'| \ll 1. \quad \forall t \in U.$$

$$\Rightarrow f+th \sim f \quad \forall t \in U. \Rightarrow f \sim g.$$

Cor Let  $f, g \in m \subseteq \mathbb{C}\{x\}$ , with  $f$  defining an isolated sing.

$$(1) \langle g, j(g) \rangle \subseteq \langle f, j(f) \rangle \Rightarrow f+tg \sim f \text{ for almost all } t \in \mathbb{C}$$

$$(2) \langle g, j(g) \rangle \subseteq m \langle f, j(f) \rangle \Rightarrow f+tg \sim f \quad \forall t \in \mathbb{C}$$

Pf By assumption,  $\exists$  matrix  $A(x) = (a_{ij})$  st.

$$\left\langle f+tg, \frac{\partial (f+tg)}{\partial x_1}, \dots, \frac{\partial (f+tg)}{\partial x_n} \right\rangle = \left( f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right) (I + tA(x))$$

Case 1  $\det(1+tA(0)) = 0$  for at most  $(n+1)$  values of  $t$ .  $\Rightarrow T_f = T_{f+tg}$  if  $\det \neq 0$ .

$$\Rightarrow f+tg \sim f \text{ if } \det \neq 0$$

Case 2  $\det(1+tA(0)) \neq 0$  as  $-a_{ij} \in m$ .

Remark  
Example/exercise:

$$F_t(x, y) = x^4 + y^5 + tx^2y^3$$

Then,  $F_t \sim F_1$  for only finitely many  $t$ .

But, if  $t \neq 0$ , have isoms

$$\varphi_t: \mathbb{C}\{x, y\} \rightarrow \mathbb{C}\{x, y\}, \quad x \mapsto x/t^{5/2}, \quad y \mapsto y/t^2$$

$$\text{st } \varphi_t(j(F_t)) = j(F_1). \Rightarrow M_{F_t} \cong M_{F_1} \quad \forall t \neq 0.$$