Tue 12/07

The Finite Determinacy Theorem

We shall show that an isolated hyp. sing. is determined (up to right equiv.) by finitely many terms of its power series.

Def'n Let $f \in \mathbb{C}\{x\}$. Then, the k-jet of f, $\operatorname{jet}(f,k) := f^{(k)} := f \mod m^{k+1}$

Also $J^{(k)} := \mathbb{C}\{x\}/m^{k+1}$. Given $f \in \mathbb{C}\{x\}$, $f^{(k)} \in J^{(k)}$ is identified with the p.s. expansion up to order k.

Def'n $f \in \mathbb{C}\{x\}$ is s.t.b. right k-determined, resp. contact k-determined, if $\forall g \in \mathbb{C}\{x\}$ with $f^{(k)} = g^{(k)}$, we have $f \overset{r}{\sim} g$, resp. $f \overset{c}{\sim} g$.

Thm Let $F \in \mathbb{C}\{x,t\} = \mathbb{C}\{x_1, \dots, x_n, t\}$, $b, c \in \mathbb{Z}_{\geq 0}$

(1) TFAE:

(a) $\frac{\partial F}{\partial t} \in \langle x_1, \dots, x_n \rangle^b \langle \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n} \rangle + \langle x_1, \dots, x_n \rangle^c \cdot \langle F \rangle$

(b) $\exists\ \phi = (\phi_1, \dots, \phi_n) \in \mathbb{C}\{x,t\}^n$, $u \in \mathbb{C}\{x,t\}$ st

(i) $u(x,0) = 1$

(ii) $u(x,t) - 1 \in \langle x_1, \dots, x_n \rangle^c \cdot \mathbb{C}\{x,t\}$

(iii) $\phi_i(x,0) = x_i$, $i = 1, \dots, n$

(iv) $\phi_i(x,t) - x_i = \langle x_1, \dots, x_n \rangle^b\ \mathbb{C}\{x,t\}$, $i = 1, \dots, n$

(v) $u(x,t) \cdot F(\phi(x,t), t) = F(x,0)$

(2) The condition $\frac{\partial F}{\partial t} \in \langle x_1, \dots, x_n \rangle^b \langle \frac{\partial F_1}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n} \rangle$ is equiv. to 1(b) with $u = 1$.

<u>Rmk</u>: $\phi_t(x) = \phi(x,t)$, $u_t(x) = u(x,t)$

As $\phi_0 = id$, for $|t| \ll 1$, $\phi_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, \phi_t(0))$ is an isom, and similarly, $u_t \in \mathbb{C}\{x\}^*$, $|t| \ll 1$.

(1)(b) $\Rightarrow$ $\phi_t : \mathcal{O}_{\mathbb{C}^n,0} \xrightarrow{\sim} \mathcal{O}_{\mathbb{C}^n,\phi_t(0)}$, mapping $\langle F_0 \rangle$ to $\langle F_t \rangle$. Thus, $(F_0^{-1}(0), 0) \cong (F_t^{-1}(0), \phi_t(0))$ being identity up to order b.

<u>Pf</u> (1) (a) $\Rightarrow$ (b) Set $\langle x \rangle = \langle x_1, \dots, x_n \rangle$

By (a), $\exists\, Y_1, \dots, Y_n \in \langle x \rangle^b \cdot \mathbb{C}\{x,t\}$, $Z \in \langle x \rangle^c \cdot \mathbb{C}\{x,t\}$ st.

$$\frac{\partial F}{\partial t} = \cancel{x^\alpha g} - \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} Y_i - Z \cdot F$$

I. Set $Y = (Y_1, \dots, Y_n)$, and let $\phi = (\phi_1, \dots, \phi_n)$ be the unique soln ~~for~~ near t of the ODE

$$\frac{\partial \phi}{\partial t}(x,t) = Y(\phi(x,t), t), \quad \phi(x,0) = x.$$

We first show $\phi_i - x_i \in \langle x \rangle^b \cdot \mathbb{C}\{x,t\}$

Wma $b \geq 1$. Then, $Y(0,t) = 0$, for $|t| \ll 1$

Hence $\phi = 0$ is a soln of the ODE

$$\frac{\partial \phi}{\partial t}(0,t) = Y(\phi(0,t), t), \quad \phi(0,0) = 0$$

By uniqueness of the soln, $\phi(0,t) = 0$

$\Rightarrow$ $\phi_i(x,t) \in \langle x \rangle$. Since $Y_i \in \langle x \rangle^b$,

$$\frac{\partial \phi_i}{\partial t}(x,t) = Y_i(\phi(x,t), t) \in \langle x \rangle^b \cdot \mathbb{C}\{x,t\}$$

$\Rightarrow \quad \phi_i - x_i \in \langle x \rangle^b.$

II Observation : Set $\psi(x,t) = (\phi(x,t), t)$

$$(\text{v}) \iff \frac{\partial}{\partial t}\left[u \cdot (F \circ \psi)(x,t)\right] = 0$$

III Let u be the unique soln of the ODE

$$\frac{\partial u}{\partial t}(x,t) = u(x,t) \cdot (Z \circ \psi)(x,t), \quad u(x,0) = 1$$

Then, $Z \in \langle x \rangle^c \Rightarrow \frac{\partial u}{\partial t} \in \langle x \rangle^c \Rightarrow u - 1 \in \langle x \rangle^c$

$$\therefore \frac{\partial}{\partial t}\left\{u \cdot (F \circ \psi)\right\} = \frac{\partial u}{\partial t}(F \circ \psi) + u \cdot \frac{\partial (F \circ \psi)}{\partial t}$$

$$= u \cdot (Z \circ \psi)(F \circ \psi) + u \cdot \left(\sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \circ \psi \cdot \frac{\partial \phi_i}{\partial t} + \frac{\partial F}{\partial t} \circ \psi\right)$$

$$= u \cdot \left((Z \cdot F) \circ \psi + \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \circ \psi \circ \frac{\partial \phi_i}{\partial t_i} - \sum_{i=1}^{n} \left(\frac{\partial F}{\partial x_i} \cdot Y_i\right) \circ \psi - (Z \cdot F) \circ \psi\right)$$

$$= 0 \quad //$$

<u>(ii) $\Rightarrow$ (i)</u> Omitted.

Thm (Finite Determinacy Thm) Let $f \in \mathfrak{m} \subseteq \mathbb{C}\{x\}$

(1) f is right k-determined if

$$\mathfrak{m}^{k+1} \subseteq \mathfrak{m}^2 \cdot \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$$

(2) f is contact k-determined if

$$\mathfrak{m}^{k+1} \subseteq \mathfrak{m}^2 \cdot \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle + \mathfrak{m} \cdot \langle f \rangle$$

Pf Let $h \in \mathfrak{m}^{k+1}$, and consider

$$F(x,t) = f(x) + t \cdot h(x) \in \mathbb{C}\{x\}[t]$$

It suffices to show that $\forall\ t_0 \in \mathbb{C}$, the germ of ~~F~~ F in $\mathcal{O}_{\mathbb{C}^n \times \mathbb{C}, (0,t_0)}$ satisfies conds. 1(a) / (2) in the prev. Thm, since then $F_{t_0} \overset{c}{\sim} F_t$ / $F_{t_0} \overset{r}{\sim} F_t$ for $|t - t_0| \ll 1$
$\Rightarrow f = F_0 \sim F_1 = f + h$.

For contact equiv, need to show

$$\frac{\partial F}{\partial t} = h \in \mathfrak{m}^2 \left\langle \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n} \right\rangle + \mathfrak{m}\langle F \rangle$$

As $h \in \mathfrak{m}^{k+1}$, we have $\mathfrak{m}^2 \frac{\partial h}{\partial x_i} + \mathfrak{m} \cdot h \subseteq \mathfrak{m}^{k+2}$

$$\therefore\ \mathfrak{m}^2 \left\langle \frac{\partial (f+th)}{\partial x_1}, \dots, \frac{\partial (f+th)}{\partial x_n} \right\rangle + \mathfrak{m} \cdot \langle f + th \rangle + \mathfrak{m}^{k+2}$$

$$= \mathfrak{m}^2 \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle + \mathfrak{m}\langle f \rangle + \mathfrak{m}^{k+2}$$

By hyp., $\mathfrak{m}^{k+1} \subseteq$ RHS. Rt. equiv is similar //

Cor If $f \in \mathbb{C}\{x\}$, $f(0) = 0$ has an isolated sing., then

(1) f is right $(\mu+1)$-det'd

(2) f is contact $(\tau+1)$-det'd.

Pf If $f \in m \setminus m^2$, then $\mu(f) = \tau(f) = 0$, and f is 1-det'd by the IFT. So let $f \in m^2$.

Then, $\dim m/\langle f, j(f)\rangle = \tau - 1$,

and

$$\frac{m}{\langle f, j(f)\rangle} \supseteq \frac{m^2 + \langle f, j(f)\rangle}{\langle f + j(f)\rangle} \supseteq \cdots$$

is a strictly decreasing seq.

$\Rightarrow m^\tau \subseteq \langle f, j(f)\rangle \Rightarrow m^{\tau+2} \subset m^2 j(f) + m\cdot\langle f\rangle$ //.

Thm (Mather–Yau) Let $f, g \in m \subseteq \mathbb{C}\{x\}$.

TFAE:

(a) $f \overset{c}{\sim} g$

(b) $\forall b \geq 0$, $\mathbb{C}\{x\}/\langle f, m^b j(b)\rangle \cong \mathbb{C}\{x\}/\langle g, m^b j(g)\rangle$ as $\mathbb{C}$-algs.

(c) $\exists b \geq 0$ st $\mathbb{C}\{x\}/\langle f, m^b j(b)\rangle \cong \mathbb{C}\{x\}/\langle g, m^b j(b)\rangle$ as $\mathbb{C}$-algs.

Thus, $f \overset{c}{\sim} g \iff T_f \cong T_g$, as $\mathbb{C}$-algs.

Pf (a) $\Rightarrow$ (b) Chain rule. - exercise

(b) $\Rightarrow$ (c) Clear

(c) $\Rightarrow$ (a) Let φ be an isom of the algebras in (c). Then, φ lifts to an isom $\tilde{\varphi}: \mathbb{C}\{x\} \to \mathbb{C}\{x\}$ s.t. $\tilde{\varphi}(\langle f, m^b j(f)\rangle) = \langle g, m^b j(b)\rangle$

As $\tilde{\varphi}(\langle f, m^b j(f)\rangle) = \langle \tilde{\varphi}(f), m^b j(\tilde{\varphi}(f))\rangle$,

Lemma $\langle f, m^b j(f)\rangle = \langle g, m^b j(g)\rangle$ (+)

Set $h = g - f$, and define

$$I_t = \left\langle f + th, m^b \left\langle \frac{\partial(f+th)}{\partial x_i} \right\rangle \right\rangle \subseteq \mathbb{C}\{x, t\}$$

By (+), $I_t \subseteq I_0 \cdot \mathbb{C}\{x,t\} = \langle f, m^b j(f)\rangle \cdot \mathbb{C}\{x,t\}$, and $I_0 = I_1$.

Suppose f, g are holomorphic in $V \subseteq \mathbb{C}^n$, and Consider the coherent $\mathcal{O}_{V\times\mathbb{C}}$ - module

~~$\mathcal{F} := \mathcal{O}\ I_0 \cdot \mathbb{C}\{x,t\}$~~

$$\mathcal{F} := \left\langle f, m^b \cdot \left\langle \frac{\partial f}{\partial x_i} \right\rangle \right\rangle \Big/ \left\langle f + th, m^b \cdot \left\langle \frac{\partial(f+th)}{\partial x_i} \right\rangle \right\rangle$$

Then, $\operatorname{supp} \mathcal{F} \subseteq V \times \mathbb{C}$ is closed and analytic.

$$\Rightarrow \operatorname{supp}\mathcal{F} \cap (\{0\}\times\mathbb{C}) = \{t \in \mathbb{C} \mid \mathcal{F}_{(0,t)} \neq 0\}$$
$$= \{t \in \mathbb{C} \mid I_0 \neq I_t\}$$

is a discrete set in $\mathbb{C} = \{0\} \times \mathbb{C}$.

$\Rightarrow U = \{t \mid I_0 = I_t\}$ is open, connected; $0, 1 \in U$.

$$\therefore \frac{\partial(f+th)}{\partial t} = h \in I_0 = I_t = \langle f+th, m^b \cdot j(f+th)\rangle \quad \forall t \in U.$$

$\Rightarrow f+th \overset{c}{\sim} f+t'h$ for $t, t' \in U$, $|t-t'| \ll 1$.

$\Rightarrow f+th \overset{c}{\sim} f \;\forall t \in U \implies f \overset{c}{\sim} g$.

<u>Cor</u> Let $f, g \in m \subseteq \mathbb{C}\{x\}$, with f defining an isolated sing.

(1) $\langle g, j(g)\rangle \subseteq \langle f, j(f)\rangle \Rightarrow f+tg \overset{c}{\sim} f$ for almost all $t \in \mathbb{C}$

(2) $\langle g, j(g)\rangle \subseteq m\langle f, j(f)\rangle \Rightarrow f+tg \overset{c}{\sim} f \;\forall t \in \mathbb{C}$

<u>Pf</u> By assumption, $\exists$ matrix $A(x) = (a_{ij})$ st.

$$\left\langle f+tg, \frac{\partial(f+tg)}{\partial x_1}, \dots, \frac{\partial(f+tg)}{\partial x_n}\right\rangle = \left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)(I + tA(x))$$

<u>Case 1</u> $\det(1+tA(0)) = 0$ for at most $(n+1)$ values of t. $\Rightarrow T_f = T_{f+tg}$ if $\det \neq 0$.

$\Rightarrow f+tg \overset{c}{\sim} f$ if $\det \neq 0$

<u>Case 2</u> $\det(1+tA(0)) \neq 0$ as $a_{ij} \in m$.

<u>~~Example~~ Remark</u>/exercise: $F_t(x,y) = x^4 + y^5 + tx^2y^3$

Then, $F_t \overset{r}{\sim} F_1$ for only finitely many t.

But, if $t \neq 0$, have isoms

$\varphi_t : \mathbb{C}\{x,y\} \to \mathbb{C}\{x,y\}$, $x \mapsto x/t^{5/2}$, $y \mapsto y/t^2$

st $\varphi_t(j(F_t)) = j(F_1)$. $\Rightarrow M_{F_t} \cong M_{F_1} \;\forall t \neq 0$.

(2). Equip $M_{f,b} = \mathbb{C}\{x\}/_{m^b j(f)}$ with a $\mathbb{C}\{t\}$-alg str. via $t \cdot g = f \cdot g \mod (m^b j(f))$

Then, the following holds ~~true~~ :

<u>Thm</u> Let $f, g \in m \subseteq \mathbb{C}\{x\}$ be hyp. sings. TFAE:

(1) $f \overset{r}{\sim} g$

(2) $\forall\, b \geq 0$, $M_{f,b} \cong M_{g,b}$ as $\mathbb{C}\{t\}$-algs.

(3) For some $b \geq 0$, ——— " ———

(3) <u>Thm</u> Let $f, g \in m \subseteq \mathbb{C}\{x\}$ have isolated sings. Then, if f is quasihomogeneous, $\forall\, g \in m$, $f \overset{r}{\sim} g \iff M_f \cong M_g$ as $\mathbb{C}$-algs.