COMPLEX ANALYSIS UG JAN-APR 2021. NOTES

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Contents

Outline

These are notes from an undergraduate course on complex analysis during Jan–Apr 2020 at CMI.

(1) Ahlfors, *Complex Analysis*.
(2) Conway, *Functions of one complex variable*.
(3) Kodaira, *Complex Analysis*.
(4) Lang, *Complex Analysis*.
(5) Rodríguez, Kra and Gilman, *Complex Analysis, in the spirit of Lipman Bers* (2nd ed.).

Lecture 1. Preliminaries

$\mathbb{C}$ as the ring $\mathbb{R}[X]/(X^2+1)$. Write $\imath$ for the image of X in $\mathbb{C}$.

Let $c \in \mathbb{C}$; then there exist unique $a, b \in \mathbb{R}$ such that $c = a + b\imath$. We call a the *real* part of c and b the *imaginary* part of c, and write $a = \Re(c)$ and $b = \Im(c)$. If $f : A \longrightarrow \mathbb{C}$ is a function (A being some set), then we write $\Re(f)$ and $\Im(f)$, respectively, for the functions $A \longrightarrow \mathbb{R}$, $a \mapsto \Re(f(a))$ and $a \mapsto \Im(f(a))$.

The function $|\cdot| : \mathbb{C} \longrightarrow \mathbb{R}_{\geq 0}$, $z \mapsto \sqrt{(\Re(z))^2 + (\Im(z))^2}$ is called the *modulus* or the *absolute value* function. This gives a metric on $\mathbb{C}$: take the distance between $c, c' \in \mathbb{C}$ to be $|c - c'|$. The function $\mathbb{C} \longrightarrow \mathbb{R}^2$, $c \mapsto (\Re(c), \Im(c))$ gives an isomorphism of real vector spaces and a homeomorphism[1] of metric spaces (with $\mathbb{R}^2$ given the usual metric). Therefore $\mathbb{C}$ is a complete metric space.

A subset $A \subseteq \mathbb{C}$ is *connected* if there are no open subsets U and V of $\mathbb{C}$ such that $A = (A \cap U) \cup (A \cap V)$ with $(A \cap U) \neq \varnothing \neq (A \cap V)$ and $(A \cap U \cap V) = \varnothing$.

Let $A \subseteq \mathbb{C}$, and $z_0, z_1 \in A$. A *path* in A from z_0 to z_1 is a continuous function $\gamma : [0,1] \longrightarrow A$ such that $\gamma(i) = z_i$, $i = 0, 1$. Say that A is *path-connected* if for every $z_0, z_1 \in A$, there is a path from z_0 to z_1.

1.1. **Proposition.** *An open subset of $\mathbb{C}$ is connected if and only if it is path-connected.*

Proof is left as an exercise.

1.2. **Definition.** By a *domain*, we mean a connected open subset of $\mathbb{C}$.

[1] Let X and Y be topological spaces, and $f : X \longrightarrow Y$ a function. We say that f is a *homeomorphism* if it is bijective and continuous, and its inverse function (which exists since f is bijective) is continuous.

When we talk of limits and convergence in $\mathbb{C}$, these are with respect to the metric topology. In particular, a sequence of complex numbers is convergent if and only if it is a Cauchy sequence. Consider a series $\sum_{i\in\mathbb{N}} a_i$ of complex numbers.[2] The sequence of *partial sums* for this series is the sequence $s_n = \sum_{i=0}^{n} a_i$, $n \in \mathbb{N}$. We say that the series *converges* if the sequence $s_0, s_1, s_2, \ldots$ converges. Now suppose that the series $\sum_{i\in\mathbb{N}} |a_i|$ of real numbers converges. (We say that $\sum_{i\in\mathbb{N}} a_i$ is *absolutely convergent* if this happens.) Let $\epsilon > 0$; then there exists N such that for every $n \geq m > N$, $\sum_{i=m}^{n} |a_i| < \epsilon$. Therefore $|s_n - s_m| < \epsilon$, i.e., the sequence (s_n) is Cauchy. Hence $\sum_{i\in\mathbb{N}} a_i$ is convergent. We have now shown that every absolutely convergent series is convergent.

1.3. **Notation.** Hereafter, when we write a complex number $c = a + b\imath$, it should be understood that $a = \Re(c)$ and $b = \Im(c)$. Similarly, when we write $f = u + v\imath$ for a $\mathbb{C}$-valued function f, $u = \Re(f)$ and $y = \Im(f)$. □

1.4. **Notation.** For $R \in \mathbb{R}_+ \cup \{+\infty\}$ and $c \in \mathbb{C}$, we denote by $B_{c,R}$ the open disc $\{z \in \mathbb{C} : |z-c| < R\}$ and by $\overline{B_{c,R}}$, its closure in $\mathbb{C}$.

Exercises.

1.1 Show that every connected open subset of $\mathbb{R}^n$ is path-connected. The "topologist's sine curve", i.e., the closure of

$$\left\{\left(x, \sin\frac{1}{x}\right) \mid x \in (0,1)\right\}$$

inside $\mathbb{R}^2$ is connected but not path-connected. (It is not open in $\mathbb{R}^2$).

1.2 Show that for every positive integer n, $\mathbb{R}^n$ with the usual metric is a complete metric space.

1.3 (Polar coordinates). For a nonzero $c \in \mathbb{C}$, there exist unique $r \in \mathbb{R}_+$ and non-unique $\theta \in \mathbb{R}$ so that $c = r(\cos\theta + \imath\sin\theta)$. (We still do not know what π is, or that $e^{\imath\theta} = (\cos\theta + \imath\sin\theta)$.) We refer to θ as *an argument* of c.

1.4 We think of z as the 'coordinate' for $\mathbb{C}$; This is related to the cartesian coordinates (x, y) of $\mathbb{R}^2$ by $x = \Re(z)$ and $y = \Im(z)$. We can also define another coordinate $\bar{z}$, with the property that $z = a + b\imath$ $(a, b \in \mathbb{R})$ is the same as the point given by $\bar{z} = a - b\imath$. Let n be a positive integer; express the equation $z^n = \bar{z}^n$ in polar coordinates and solve.

1.5 Prove the ratio test: Let $\sum_{i\in\mathbb{N}} a_i$ be a series of non-zero real numbers. If

$$L := \lim_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right|$$

exists, then the series converges if $L < 1$ and diverges if $L > 1$.

1.6 Prove the root test: Let $\sum_{i\in\mathbb{N}} a_i$ be a series of real numbers. Let

$$L := \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}.$$

Then the series converges if $L < 1$ and diverges if $L > 1$.

1.7 Consider the series $\sum a_n$ with

$$a_n = \begin{cases} 1, & n = 0, \\ \frac{a_{n-1}}{2}, & n \text{ odd}, \\ \frac{a_{n-1}}{8}, & n \geq 2 \text{ even}. \end{cases}$$

[2] By $\mathbb{N}$, we mean $\{0, 1, 2, \ldots\}$.

Show that the ratio test is inconclusive, while the root test concludes that the series converges.

Lecture 2. Differentiability

2.1. **Definition.** Let $c \in \mathbb{C}$ and f a (complex-valued) function defined in an open disc around c. Say that f is *(complex-)differentiable* at c if

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists. If this is the case, we call this limit the *derivative* of f at c, and denote it by $f'(c)$.

2.2. **Remark.** We will not explicitly say "complex-differentiable", hereafter, for $\mathbb{C}$-valued functions from subsets of $\mathbb{C}$. When we refer to such a function as being "differentiable", it should be understood as "complex-differentiable".

By a *constant function* we mean a function of the form $\mathbb{C} \longrightarrow \mathbb{C}, z \mapsto c$ for some $c \in \mathbb{C}$. It is immediate that constant functions are differentiable. The identity function on $\mathbb{C}$ (i.e., the map $z \mapsto z$) is differentiable. We could also consider the restrictions of these functions to some open $U \subseteq \mathbb{C}$. Before we construct more examples, we need some to see some rules of differentiation.

2.3. **Remark** (Rules of differentiation)**.** Let $c \in \mathbb{C}$, f and g functions defined on a neighbourhood [3] of c and differentiable at c, h a function defined on a neighbourhood of $f(c)$ and differentiable at $f(c)$, and $\alpha \in \mathbb{C}$. Then

(1) $(f + \alpha g)'(c) = f'(c) + \alpha g'(c)$.
(2) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.
(3) $(h \circ f)'(c) = h'(f(c))f'(c)$.
(4) $\left(\frac{1}{f}\right)'(c) = -\frac{f'(c)}{f(c)^2}$ if $f(c_1) \neq 0$ for every c_1 in a neighbourhood of c.

2.4. **Example.** We can now construct two more examples of differentiable functions. Let $p(X), q(X) \in \mathbb{C}[X]$ with $q(X) \neq 0$. The function

$$\mathbb{C} \longrightarrow \mathbb{C}, \; z \mapsto p(z)$$

(i.e., the polynomial p evaluated at z) is differentiable at all points in $\mathbb{C}$. Such functions are called *polynomial* functions. Let $U = \{z \in \mathbb{C} \mid q(z) \neq 0\}$. Since the set of zeros of $q(X)$ is finite, U is open. The function

$$U \longrightarrow \mathbb{C}, \; z \mapsto \frac{p(z)}{q(z)}$$

is differentiable at every point in U. These are called *rational* functions.

2.5. **Remark.** Let $c \in \mathbb{C}$ and f a (complex-valued) function defined in an open disc around c. If f is differentiable at c, then it is continuous at c. To see this, note that

$$f'(c) \cdot 0 = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \cdot \lim_{h \to 0} h = \lim_{h \to 0} f(c+h) - f(c).$$

[3] Let X be a topological space and $x \in X$. A *neighbourhood* of x in X is a subset V of X such that there exists an open subset U of X such that $x \in U \subseteq V$.

2.6. **Example.** Let f be a real-valued function defined in an open disc around $c \in \mathbb{C}$. Suppose that f is differentiable at c. Then, taking h to be real, we see that

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

is real. On the other hand, taking $h = \imath t$ to be purely imaginary, we get

$$f'(c) = \lim_{t \to 0} \frac{f(c+\imath t) - f(c)}{\imath t}$$

is purely imaginary. Hence $f'(c) = 0$. We will see this in a general context later. □

2.7. **Definition.** Let $U \subseteq \mathbb{C}$ be a domain, and $f : U \longrightarrow \mathbb{C}$. Say that f is *holomorphic* on U if it is (complex-)differentiable at every point in U. A function $f : \mathbb{C} \longrightarrow \mathbb{C}$ that is holomorphic on $\mathbb{C}$ is called *entire*.

2.8. **Theorem.** *Let U be a domain, $f : U \longrightarrow \mathbb{C}$ and $c = a + b\imath \in U$. Write f as $u(x, y) + v(x, y)\imath$. Then f is complex-differentiable at c, if and only if u and v are differentiable at (a, b) (as functions from $\mathbb{R}^2 \longrightarrow \mathbb{R}$) and their partial derivatives satisfy the* Cauchy-Riemann equations

$$u_x(a, b) = v_y(a, b) \text{ and } u_y(a, b) = -v_x(a, b). \tag{2.9}$$

Further, when this happens, $f'(c) = u_x(a, b) + \imath v_x(a, b) = v_y(a, b) - \imath u_y(a, b)$.

(Here $u_x(a, b)$ is the partial derivative $\frac{\partial u}{\partial x}(a, b)$, etc.)

Proof. Write $h = \Delta x + \imath \Delta y$ and $f(c + h) - f(c) = \Delta u + \imath \Delta v$. Assume that f is differentiable at c. Write $f'(c) = p + \imath q$; then

$$\Delta u + \imath \Delta v = (p + \imath q)(\Delta x + \imath \Delta y) + r(\Delta x + \imath \Delta y),$$

where $r(h)$ is a complex-valued function defined in a neighbourhood of $0 \in \mathbb{C}$, but possibly not at 0, such that $\lim_{h \to 0} \frac{r(h)}{h} = 0$. Write $r(z) = r_1(z) + \imath r_2(z)$. Thus

$$\begin{aligned} \Delta u &= p\Delta x - q\Delta y + r_1(h); \\ \Delta v &= q\Delta x + p\Delta y + r_2(h). \end{aligned}$$

Let $\epsilon > 0$. Then there exists $\delta > 0$ such that for each $h \in B_{0,\delta} \setminus \{0\}$, $|\frac{r(h)}{h}| < \epsilon$; since $r_1 = \Re(r)$ and $r_2 = \Im(r)$, $|\frac{r_1(h)}{h}| < \epsilon$ and $|\frac{r_2(h)}{h}| < \epsilon$. Therefore $\lim_{h \to 0} \frac{r_1(h)}{|h|} = 0$ and $\lim_{h \to 0} \frac{r_2(h)}{|h|} = 0$. Hence u and v are differentiable at (a, b) and (2.9) are satisfied.

Conversely, assume that u and v are differentiable at (a, b) and that (2.9) are satisfied. Write $p = u_x(a, b)$ and $q = v_x(a, b)$. Then

$$\begin{aligned} \Delta u &= p\Delta x - q\Delta y + r_1(h); \\ \Delta v &= q\Delta x + p\Delta y + r_2(h), \end{aligned}$$

where $\lim_{h \to 0} \frac{r_1(h)}{|h|} = 0$ and $\lim_{h \to 0} \frac{r_2(h)}{|h|} = 0$. Write $r(z) = r_1(z) + \imath r_2(z)$. Then $f(c + h) - f(c) = (p + \imath q)(\Delta x + \imath \Delta y) + r(z)$. Note that $\lim_{h \to 0} \frac{r(h)}{h} = 0$, by the triangle inequality. Hence $f'(c)$ exists and equals $p + \imath q$. □

Satisfying the Cauchy-Riemann equations alone is not a sufficient condition, in general, for f to be differentiable at a point; see the exercises.

2.10. **Remark.** Write $f_x = u_x + v_x\imath$ and $f_y = u_y + v_y\imath$ (wherever the partial derivatives on the right are defined). The Cauchy-Riemann equations can be rephrased in a more concise way, as $f_x = -\imath f_y$. Another description is given in the exercises.

Exercises.

2.1 Prove the rules of differentiation mentioned in class.

2.2 Let
$$f(z) = \begin{cases} z^5|z|^{-4}, & \text{if } z \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$
Write $z = x + y\imath$ and determine $\Re(f)$ and $\Im(f)$ as functions of the real variables x and y. Show that these satisfy the Cauchy-Riemann equations at $z = 0 \in \mathbb{C}$. Show that the limit
$$\lim_{h\to 0} \frac{f(h)}{h}$$
does not exist by considering first $h = r$ and then $h = (1+\imath)r$, with $r \in \mathbb{R}$. Hence f is not differentiable.

2.3 Define
$$f_z = \frac{1}{2}(f_x - \imath f_y), \text{ and}$$
$$f_{\bar{z}} = \frac{1}{2}(f_x + \imath f_y)$$
wherever the RHS is defined.
(1) Treating z and $\bar{z}$ as independent coordinates, show that this definition agrees with the formula one would get from applying the chain rule for the substitution $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2\imath}$.
(2) If f is differentiable at c, then $f'(c) = f_z(c)$; the Cauchy-Riemann equations simplify to give $f_{\bar{z}}(c) = 0$.

2.4 If $f = z^m\bar{z}^n$, with $m, n \geq 0$, then $f_z = mz^{m-1}\bar{z}^n$ and then $f_{\bar{z}} = nz^m\bar{z}^{n-1}$. Extend this to 'polynomials' in z and $\bar{z}$.

2.5 Show that the function
$$f(x + y\imath) = \begin{cases} \frac{xy^2(x+y\imath)}{x^2+y^4}, & \text{if}(x, y) \neq (0, 0) \\ 0, & \text{otherwise} \end{cases}$$
is not differentiable at 0.

2.6 Let $f(z)$ be a function defined in a neighbourhood of $c \in C$. Show that $f(z)$ is differentiable at c if and only if $\overline{f(\bar{z})}$ is differentiable at $\bar{c}$.

2.7 (Cauchy-Riemann equations in polar coordinates) Write $f = u + \imath v$, and express u and v as (real-valued) functions of r and θ. Since $x = r\cos\theta$ and $y = r\sin\theta$, we have $u_r = u_x\cos\theta + u_y\sin\theta$, $v_r = v_x\cos\theta + v_y\sin\theta$, $u_\theta = -u_x r\sin\theta + u_y r\cos\theta$, $v_\theta = -v_x r\sin\theta + v_y r\cos\theta$. Therefore the Cauchy-Riemann equations are
$$ru_r = v_\theta; rv_r = -u_\theta.$$

Lecture 3. Power series

3.1. **Definition.** A *(formal) power series* in the variable z is an expression of the form
$$\sum_{n\in\mathbb{N}} a_n z^n$$

where the a_n are complex numbers. A formal power series $\sum_{n\in\mathbb{N}} a_n z^n$ is said to *converge* (respectively, *diverge*) at $c \in \mathbb{C}$ if the series $\sum_{n\in\mathbb{N}} a_n c^n$ of complex numbers converges (respectively, diverges). For a $U \subseteq \mathbb{C}$, a power series is said to *converge* on U if it converges at c for every $c \in U$.

We will often drop the word 'formal' while talking about power series.

3.2. **Definition.** The *radius of convergence* of the series $\sum_{n\in\mathbb{N}} a_n z^n$ is

$$\left(\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}\right)^{-1}.$$

(Here, we mean that the radius of convergence is 0 (respectively $+\infty$) if the lim sup is $+\infty$ (respectively, 0).) A series is said to be *convergent* if its radius of convergence is positive.

3.3. **Theorem.** *Let R be the radius of convergence of the series $\sum_{n\in\mathbb{N}} a_n z^n$.*

(1) It converges absolutely in $B_{0,R}$; in particular, it converges in $B_{0,R}$.

(2) For every $0 \le \rho < R$, the sequence of functions

$$c \mapsto \sum_{n=0}^{m} a_n c^n, \ \ m \in \mathbb{N},$$

converges uniformly in $\overline{B_{0,\rho}}$.

(3) For every $c \in \mathbb{C} \smallsetminus \overline{B_{0,R}}$, the series is unbounded at c.

We will often abuse terminology and call a power series $\sum_{n\in\mathbb{N}} a_n z^n$ a function on $B_{0,R}$, by which we mean the function $c \mapsto \sum_{n\in\mathbb{N}} a_n c^n$ on $B_{0,R}$.

Proof of Theorem. (1): It suffices to prove the assertion about absolute convergence, i.e., that

$$\sum_{n\in\mathbb{N}} |a_n||z|^n$$

converges whenever $|z| < R$. Without loss of generality, the a_n are non-negative real numbers; we want to show that for $0 \le x < R$, $\sum_{n\in\mathbb{N}} a_n x^n$ converges. Let $x < y < R$. There exists $N \in \mathbb{N}$ such that $a_n^{\frac{1}{n}} < \frac{1}{y}$ for every $n \ge N$; hence $a_n x^n < (x/y)^n$ for every $n \ge N$. Hence $\sum_{n\in\mathbb{N}} a_n x^n$ converges.

(2): Let $\rho < \sigma < R$. Then, as earlier, $|a_n z^n| \le (\rho/\sigma)^n$ for all sufficiently large n. Write $s_m(z) = \sum_{n=0}^{m} a_n z^n$. Let $\epsilon > 0$. Then there exists N such that for every $m > k \ge N$, $|s_m(z) - s_k(z)| = |\sum_{n=k+1}^{m} a_n z^n| \le \sum_{n=k+1}^{m} |a_n z^n| \le \sum_{n=k+1}^{m} (\rho/\sigma)^n < \epsilon$, since the series $\sum_{n\in\mathbb{N}} (\rho/\sigma)^n$ converges. Note that by (1), $\sum_{n\in\mathbb{N}} a_n z^n$ converges in $B_{0,R}$ to give a function $f(z)$ on $B_{0,R}$. By taking $m \to \infty$ (keeping k fixed), we see that $|f(z) - s_k(z)| < \epsilon$, i.e., $\sum_{n\in\mathbb{N}} a_n z^n$ converges uniformly on $\overline{B_{0,\rho}}$.

(3): Let $|c| > y > R$. Then there are arbitrarily large n such that $a_n^{\frac{1}{n}} > \frac{1}{y}$. Hence

$$\lim_{n\longrightarrow\infty} |a_n c^n| \neq 0,$$

so the series does not converge. □

3.4. **Remark.** The radius of convergence of the complex power series $\sum_{n\in\mathbb{N}} a_n z^n$ and that of the real power series $\sum_{n\in\mathbb{N}} |a_n| x^n$ are the same. Hence the tests for determining the radius of convergence of real power series can be used to determine the radius of convergence of complex power series also.

3.5. **Example.** The radius of convergence of $\sum_{i=1}^{\infty} \frac{z^n}{n}$ is 1. The series does not coverge at $z = 1$, but converges at $z = -1$.

3.6. **Proposition.** *Let $\sum_{n\in\mathbb{N}} a_n z^n$ be a convergent power series with radius of convergence R. Then it is holomorphic on $B_{0,R}$, with derivative $\sum_{n\in\mathbb{N}} na_n z^{n-1}$. Further, the radius of convergence of the derivative is R.*

Proof. We will first prove that the radius of convergence of the series $\sum_{n\in\mathbb{N}} na_n z^{n-1}$ is R. Indeed,

$$\limsup_n (n|a_n|)^{\frac{1}{n}} = \lim_n n^{\frac{1}{n}} \limsup_n |a_n|^{\frac{1}{n}} = \limsup_n |a_n|^{\frac{1}{n}} = 1/R.$$

Write $f(z)$ and $f_1(z)$ respectively for the functions $\sum_{n\in\mathbb{N}} a_n z^n$ and $\sum_{n\in\mathbb{N}} na_n z^{n-1}$ on $B_{0,R}$. We want to show that $f'(c) = f_1(c)$ for every $c \in B_{0,R}$. Let $c \in B_{0,R}$. We will show that

$$\lim_{h\to 0} \left| \frac{f(c+h) - f(c)}{h} - f_1(c) \right| = 0.$$

Write $s_n(z) = \sum_{i=0}^n a_i z^i$. Then $s_n'(z) = \sum_{i=1}^n i a_i z^{i-1}$. Write $R_n(z) = f(z) - s_n(z)$ on $B_{0,R}$. Then for every sufficiently small r and every $z \in B_{c,r}$,

$$\frac{f(z) - f(c)}{z - c} - f_1(c) = \left(\frac{s_n(z) - s_n(c)}{z - c} - s_n'(c) \right) + \left(s_n'(c) - f_1(c) \right) + \left(\frac{R_n(z) - R_n(c)}{z - c} \right).$$

Choose r above such that $|c| + r < \rho < R$. Let $\epsilon > 0$.

Since

$$\frac{R_n(z) - R_n(c)}{z - c} = \frac{\sum_{i=n+1}^\infty a_i z^i - \sum_{i=n+1}^\infty a_i c^i}{z - c} = \frac{\sum_{i=n+1}^\infty a_i (z^i - c^i)}{z - c} = \sum_{i=n+1}^\infty a_i \sum_{j=0}^{i-1} z^j c^{i-1-j},$$

we see that

$$\left| \frac{R_n(z) - R_n(c)}{z - c} \right| \le \sum_{i=n+1}^\infty i|a_i|\rho^{i-1}.$$

We already observed that $\sum_{m=1}^\infty m a_m z^{m-1}$ converges in $B_{0,R}$. The same argument shows that there exists n_0 such that for each $n > n_0$,

$$\left| \frac{R_n(z) - R_n(c)}{z - c} \right| < \frac{\epsilon}{3}.$$

Similarly, there exists n_1 such that for each $n > n_1$,

$$\left| s_n'(c) - f_1(c) \right| < \frac{\epsilon}{3}.$$

Fix $n \ge \max\{n_0, n_1\}$. There exists $\delta > 0$ such that for all $z \in B_{c,\delta}$

$$\left| \frac{s_n(z) - s_n(c)}{z - c} - s_n'(c) \right| < \frac{\epsilon}{3}.$$

Hence for all $z \in B_{c,\delta}$

$$\left| \frac{f(z) - f(c)}{z - c} - f_1(c) \right| < \epsilon.$$

Therefore $f' = f_1$ on $B_{0,R}$. □

3.7. **Corollary.** *With notation as in the proposition, write $f(z)$ for the function $\sum_{n\in\mathbb{N}} a_n z^n$ on $B_{0,R}$. Then for every $k \ge 1$, the derivative $f^{(k)}(z)$ of $f(z)$ exists on $B_{0,R}$. Moreover, for every $k \in \mathbb{N}$, $k!a_k = f^{(k)}(0)$.*

Proof. Immediate from the proposition. □

3.8. **Proposition.** *Let $\sum_{n\in\mathbb{N}} a_n z^n$ be a convergent power series such that $a_m \neq 0$ for some m. Then there exists $R > 0$ such that for every $c \in B_{0,R}$ with $c \neq 0$, $\sum_{n\ge 1} a_n c^n \neq 0$.*

Proof. Let m be the smallest integer such that $a_m \neq 0$. Write the given series as $z^m \sum_{n\in\mathbb{N}} a_{n+m}z^n$. There exists $R > 0$ such that $\sum_{n\in\mathbb{N}} a_{n+m}c^n \neq 0$ for every $c \in B_{0,R}$, by continuity. Now note that for every $c \in B_{0,R}$, $c^m = 0$ only if $c = 0$. □

Exercises.

3.1 Read the statement of the Weierstrass M-test in Ahlfors, Chapter 2, Section 2.3 and understand its proof.

3.2 All the exercises in Ahlfors, Chapter 2, Section 2.4 ('Power series')

3.3 Show that the radius of convergence of $\sum_{i\in\mathbb{N}} a_i z^i$ is

$$\sup\{r \in \mathbb{R} \mid r \geq 0, \sum_{i\in\mathbb{N}} |a_i| r^i \text{ converges}\}.$$

3.4 Let $\sum_{i\in\mathbb{N}} a_i$ and $\sum_{i\in\mathbb{N}} b_i$ be convergent series of complex numbers, and $\alpha, \beta \in \mathbb{C}$. Show that the series $\sum_{i\in\mathbb{N}}(\alpha a_i + \beta b_i)$ is convergent and its value is $\alpha \sum_{i\in\mathbb{N}} a_i + \beta \sum_{i\in\mathbb{N}} b_i$.

3.5 Prove the properties of limits superior and inferior listed in Rodríguez, Kra and Gilman, Section 3.1.1.

3.6 Show that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ and that for every k, $\lim_{n\to\infty} \binom{n}{k}^{\frac{1}{n}} = 1$

3.7 The set $\mathbb{C}[[z]]$ of all formal power series $\sum_{n\in\mathbb{N}} a_n z^n$ form a commutative ring with

$$\text{addition} : \sum_{n\in\mathbb{N}} a_n z^n + \sum_{n\in\mathbb{N}} b_n z^n = \sum_{n\in\mathbb{N}} (a_n + b_n) z^n;$$

$$\text{multiplication} : \sum_{n\in\mathbb{N}} a_n z^n \cdot \sum_{n\in\mathbb{N}} b_n z^n = \sum_{n\in\mathbb{N}} \sum_{k=0}^{n} (a_k b_{n-k}) z^n.$$

It contains $\mathbb{C}$ as a subring identified with the 'constant' power series: $c \leftrightarrow c + 0z + 0z^2 + \cdots$. If $a_0 \neq 0$, then $\sum_{n\in\mathbb{N}} a_n z^n$ has an inverse in $\mathbb{C}[[z]]$.

3.8 The subset $\mathbb{C}\{z\}$ of $\mathbb{C}[[z]]$ consisting of all the convergent power series is a subring. If $\sum_{n\in\mathbb{N}} a_n z^n \in \mathbb{C}\{z\}$ and $a_0 \neq 0$, then its inverse in $\mathbb{C}[[z]]$ in fact belongs to $\mathbb{C}\{z\}$. (Hint: $\sum_{n\in\mathbb{N}} a_n z^n$ converges to something non-zero in a neighbourhood of 0.)

3.9 (Some ring-theoretic properties of $\mathbb{C}[[z]]$ and of $\mathbb{C}\{z\}$, not relevant for this course.) The map $\mathbb{C}[[z]] \longrightarrow \mathbb{C}$, $\sum_{n\in\mathbb{N}} a_n z^n \mapsto a_0$ is a surjective ring homomorphism; its kernel is generated by z; hence the ideal $\mathfrak{m}$ generated by z is a maximal ideal. Every element of $\mathbb{C}[[z]] \smallsetminus \mathfrak{m}$ is invertible in $\mathbb{C}[[z]]$, so $\mathfrak{m}$ is the unique maximal ideal of $\mathbb{C}[[z]]$. If I is a proper ideal of $\mathbb{C}[[z]]$, then $I = \mathfrak{m}^t$ (i.e., the ideal generated by z^t) for some $t \geq 1$. Similar statements for $\mathbb{C}\{z\}$ also.

Lecture 4. Analytic functions

4.1. **Definition.** Let U be a domain. We say that $f : U \longrightarrow \mathbb{C}$ is *(complex-)analytic* if for every $c \in U$, there exist $\delta > 0$ and a convergent power series $\sum_{i\in\mathbb{N}} a_i z^i$ such that $B_{c,\delta} \subseteq U$, $\sum_{i\in\mathbb{N}} a_i (z-c)^i$ converges on $B_{c,\delta}$ and $f(\zeta) = \sum_{i\in\mathbb{N}} a_i(\zeta - c)^i$ for every $\zeta \in B_{c,\delta}$.

4.2. **Remark.** The coefficients a_i in the expansion of f as a power series centred at $c \in U$ might depend on c. It might not be possible to choose a_i that will work at every $c \in U$. This is not surprising. We have seen that for any power series centred at $c \in C$, the set of points at which it converges contains an open disc $B_{c,R}$ and is contained inside the closed disc $\overline{B_{c,R}}$. However U might not be of this shape. We will see one such U (occurring in a natural way) when we discuss branches of the logarithm, later.

4.3. **Remark.** Let U be a domain and $f : U \longrightarrow \mathbb{C}$ an analytic function. Then for every $k \geq 1$, $f^{(k)}(z)$ is an analytic function on U. Moreover, for every $c \in U$, there exists a neighbourhood on which

$$f(z) = \sum_{n\in\mathbb{N}} f^{(n)}(c)(z-c)^n.$$

4.4. **Remark.** Every analytic function is holomorphic. After proving a version of the Cauchy integral formula for a disc (Theorem 14.1), we will show that every holomorphic function is analytic (Corollary 15.5). This is not the same situation for functions from $\mathbb{R}$ to $\mathbb{R}$. For every positive integer k, there exist $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that the kth order derivative $f^{(k)}$ exists, but is not continuous, so in particular $f^{(k+1)}$ does not exist. There are functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f^{(k)}$ exists for every positive integer k (such functions are called *smooth* functions) but f is smooth but not real-analytic, i.e., f does not have a power-series expansion on its domain.

4.5. **Proposition** (Lang, Chapter II, §4, Theorem 4.1)**.** *Let $\sum_{n\in\mathbb{N}} a_n z^n$ be a convergent power series with radius of convergence R. Then it is analytic on $B_{0,R}$.*

Proof. Write $f(z) = \sum_{n\in\mathbb{N}} a_n z^n$ on $B_{0,R}$. Let $c \in B_{0,R}$. We want to show that f can be represented by a convergent power series centred at c in a neighbourhood of c. To see this, choose $\epsilon > 0$ such that $B_{c,\epsilon} \subseteq B_{0,R}$. On $B_{c,\epsilon}$, we can write

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n(z-c+c)^n \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} a_n c^{n-k}(z-c)^k. \end{aligned}$$

Claim:

$$g(z) := \sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty}\binom{n}{k} a_n c^{n-k}\right)(z-c)^k.$$

converges and equals $f(z)$ in $B_{c,\epsilon}$.

To prove the claim, let $z \in B_{c,\epsilon}$. Note that $|c| + |z-c| < R$. Hence the series

$$\sum_n |a_n|(|c|+|z-c|)^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}|a_n||c|^{n-k}|z-c|^k.$$

converges. (Recall that inside the open disc of convergence, we have absolute convergence.) Hence we can change the order of summation:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}|a_n||c|^{n-k}|z-c|^k = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\binom{n}{k}|a_n||c|^{n-k}|z-c|^k.$$

Therefore $g(z)$ converges absolutely in $B_{c,\epsilon}$. The same argument also shows that $g(z) = f(z)$ on $B_{c,\epsilon}$. □

4.6. **Proposition.** *Let U be a domain and f an analytic function on U that is not identically zero. Then the zeros of f are isolated, i.e. for every $c \in U$ with $f(c) = 0$, there exists $\epsilon > 0$ such that $B_{c,\epsilon} \subseteq U$ and $f(\zeta) \neq 0$ for every $\zeta \in B_{c,\epsilon} \smallsetminus \{c\}$.*

Proof. Let $A = \{c \in U \mid f(c) \neq 0\}$. Since f is continuous and not identically zero, A is open and non-empty. We may assume that $A \neq U$. Write $\overline{A}$ for the closure of A in U. We want to show that the points in $U \smallsetminus A$ are isolated. We will show the following:

(1) For each $c \in \overline{A} \smallsetminus A$, there exists $\epsilon > 0$ such that $B_{c,\epsilon} \smallsetminus \{c\} \subseteq A$.
(2) $\overline{A} = U$.

Let $c \in \overline{A} \smallsetminus A$. Let $\sum_{n\in\mathbb{N}} a_n z^n$ be a convergent power series and $\epsilon > 0$ such that $f(z) = \sum_{n\in\mathbb{N}} a_n(z-c)^n$ on $B_{c,\epsilon} \subseteq U$. Since $B_{c,\epsilon} \cap A \neq \varnothing$, it follows that f is not identically zero on $B_{c,\epsilon}$. Therefore there exists m such that $a_m \neq 0$. By Proposition 3.8, we may assume that $f(\zeta) \neq 0$ for every $\zeta \in B_{c,\epsilon} \smallsetminus \{c\}$.

We now show that $\overline{A} = U$. By way of contradiction, assume that $\overline{A} \neq U$. We will show that $U \smallsetminus \overline{A}$ is closed. Let $c \in U$ be a limit point of $U \smallsetminus \overline{A}$. Now, if $c \in \overline{A} \smallsetminus A$, then by above, there exists $\epsilon > 0$ such that $B_{c,\epsilon} \smallsetminus \{c\} \subseteq A$. If $c \in A$, then there exists $\epsilon > 0$ such that $B_{c,\epsilon} \subseteq A$. In both cases, we cannot have a sequence in $U \smallsetminus \overline{A}$ converging to c. Hence $c \in U \smallsetminus \overline{A}$, so it is closed. This now leads to a contradiction, since U is connected and both $\overline{A}$ and $U \smallsetminus \overline{A}$ are non-empty and closed. Therefore $\overline{A} = U$. □

Exercises.

4.1 Let U be a domain, $c_0 \in \mathbb{C}$ and $\tau : U \longrightarrow \mathbb{C}$ be the map $c \mapsto c + c_0$. Then τ is continuous and injective; the inverse of τ on $\operatorname{Im}(\tau)$ (which exists since τ is injective) is continuous. $\operatorname{Im}(\tau)$ is a domain. If f is holomorphic (respectively, analytic) on U, then $f\tau^{-1}$ is holomorphic (respectively, analytic) on $\operatorname{Im}(\tau)$. (Using this, we can 'translate' many questions about the local behaviour of holomorphic or analytic functions at $c \in U$ to that of holomorphic or analytic functions at 0, in an appropriate neighbourhood of 0.)

4.2 Prove analogous statements when f is replaced by the composite $f \circ [\zeta \mapsto c\zeta]$ where c is a (fixed) non-zero complex number.

4.3 Let $f : U \longrightarrow \mathbb{C}$ be analytic on a domain U. Show that if $f^{(k)}(z) = 0$ for every $z \in U$, then f is given by a polynomial of degree at most k, hence, f can be extended to an entire function as follows:
   (1) There is a nonempty open subset of U on which f is given by a polynomial p of degree at most k.
   (2) $f - p$ is zero on a nonempty open subset of U, so it is zero on U.

4.4 Let $f : U \longrightarrow \mathbb{C}$ be analytic on a domain U, not identically zero. Let $A = \{c \in U \mid f^{(n)}(c) = 0 \text{ for every } n \in \mathbb{N}\}$. A is closed, since $\{c \in U \mid f^{(n)}(c) = 0\}$ is closed, for every $n \in \mathbb{N}$. A is open, since, for every $c \in A$, there is a neighbourhood in U on which f is identically zero, and, hence, this neighbourhood is a subset of A. Thus $A = \varnothing$. Now let $c \in U$ and $f(z) = \sum_{n\in\mathbb{N}} a_n(z-c)^n$ in a neighbourhood of c. Then there exists k such that $a_k \neq 0$. Thus there exists a neighbourhood V of c in U such that $f(z) \neq 0$ for every $z \in V \smallsetminus \{c\}$.

4.5 Consider the function $f(x) = e^{-x^{-2}}$ in a neighbourhood of 0 in $\mathbb{R}$. Show that $f^{(k)}$ exists in a neighbourhood of 0 and that $f^{(k)}(0) = 0$ for every $k \geq 0$. Hence f is not real-analytic in a neighbourhood of 0. This example was discovered by Cauchy and Hamilton.

4.6 (Not relevant for this course.) Let U be a domain and $\mathcal{A}(U)$ the set of analytic functions on U. It is a commutative ring with

$$\text{addition} : (f+g)(z) = f(z) + g(z);$$
$$\text{multiplication} : (fg)(z) = f(z)g(z).$$

It contains $\mathbb{C}$ as the subring of the constant functions on U. It is an integral domain. For $c \in U$, the set $\mathfrak{m}_c := \{f \in \mathcal{A}(U) \mid f(c) = 0\}$ is a maximal ideal of $\mathcal{A}(U)$. There is a ring homomorphism $\mathcal{A}(U) \longrightarrow \mathbb{C}\{z-c\}$ (the ring of convergent power series in the variable $z-c$) which factors through the localisation $\mathcal{A}(U)_{\mathfrak{m}_c}$.

Lecture 5. Exponential and logarithmic functions

By e^z or $\exp(z)$ we mean an analytic function $f(z)$ that $f'(z) = f(z)$ and $f(0) = 1$. Suppose that this has a solution. Then in a neighbourhood of 0, it can be written as a convergent power series $\sum_{n \in \mathbb{N}} a_n z^n$. Since $f'(z) = \sum_{n \in \mathbb{N}} n a_n z^{n-1}$ and $f(0) = a_0 = 1$, we conclude by induction that $a_n = \frac{1}{n!}$. Since

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 0$$

we see that the series $\sum_{n \in \mathbb{N}} \frac{z^n}{n!}$ converges everywhere on $\mathbb{C}$.

5.1. **Proposition.** *(1) $e^z e^{-z} = 1$ for every $z \in \mathbb{C}$.*
(2) In particular, $e^z \neq 0$ for every $z \in \mathbb{C}$.
(3) $e^{(z+c)} = e^z e^c$ for every $z, c \in \mathbb{C}$.

Proof. (1) NOTE: *A priori* e^{-z} is not $\frac{1}{e^z}$, but just the composite function $[z \mapsto \exp(z)] \circ [z \mapsto -z]$. Hence e^{-z} is analytic on $\mathbb{C}$,[4] and, hence, so is $e^z e^{-z}$. Its derivative is 0, so it is a constant function.[5] Now note that its value at 0 is 1.

(2) Follows immediately from (1).

(3) Fix c and consider

$$h(z) = \frac{e^{(z+c)}}{e^c}$$

as a function of z. It is analytic on $\mathbb{C}$ [6] and $h'(z) = h(z)$ and $h(0) = 1$. Hence $h(z) = e^z$. □

From the exponential function, we can define $\sin(z)$ and $\cos(z)$:

$$\cos z = \frac{e^{\imath z} + e^{-\imath z}}{2}$$
$$\sin z = \frac{e^{\imath z} - e^{-\imath z}}{2\imath}$$

5.2. **Proposition.** *(1) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$.*
(2) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$.
(3) $e^{\imath z} = \cos z + \imath \sin z$.
(4) $\cos^2 z + \sin^2 z = 1$.
(5) $\cos(-z) = \cos z$.
(6) $\sin(-z) = -\sin z$.
(7) $\cos'(z) = -\sin z$.
(8) $\sin'(z) = \cos z$.
(9) If x is real, then the new definitions of e^x, $\cos x$, $\sin x$ agree with the definitions in the case of real numbers.
(10) $e^{x+\imath y} = e^x(\cos y + \imath \sin y)$. In particular, $e^{\imath\pi} = 1$.

Proof of the above proposition is left as an exercise.

5.3. **Definition.** Let $z \in \mathbb{C} \setminus \{0\}$. By an *argument* $\arg z$ of z, we mean an real number θ such that $z = |z|e^{\imath\theta}$. Define the *principal argument* $\operatorname{Arg} z$ of z to be the argument in $(-\pi, \pi]$.

[4] Exercise 4.2
[5] Exercise 4.3
[6] Exercise 4.1

5.4. **Definition.** For a fixed choice of $\arg z$, we often write $\log z$ for $\log|z| + \imath \arg z$. Define

$$\operatorname{Log} z = \log|z| + \imath \operatorname{Arg} z$$

on $\mathbb{C} \smallsetminus (-\infty, 0]$.

5.5. **Remark.** Let $z \in \mathbb{C} \smallsetminus \{0\}$. If θ_1 and θ_2 are arguments of z, then $\theta_1 - \theta_2$ is a multiple of 2π. Note that $e^{\log z} = z$.

5.6. **Proposition.** *$\operatorname{Log} z$ is holomorphic on $\mathbb{C} \smallsetminus (-\infty, 0]$ with derivative $1/z$.*

Proof. On the given domain, the real and imaginary parts of $\operatorname{Log} z$, *viz.*, $\log|z|$ and $\operatorname{Arg} z$ are differentiable functions of the reals coordinates x and y. Hence it suffices to check that they satisfy the Cauchy-Riemann equations. For this, use the version in polar coordinates: $u = \log r$, $v = \theta$. Hence $ru_r = 1 = v_\theta$ and $rv_r = 0 = -u_\theta$.

Since $\operatorname{Log} z$ is holomorphic, we can use differentiate $e^{\operatorname{Log} z} = z$ to get $e^{\operatorname{Log} z}(\operatorname{Log}' z) = 1$, i.e., $(\operatorname{Log}' z) = 1/z$. □

Let U be a domain and f a continuous function on U. We say that f is a *branch of the logarithm on U* if $e^{f(z)} = z$ for every $z \in U$. A branch of the logarithm f on U is *principal* if $f(z) = \operatorname{Log}(z)$ for every $z \in U \cap \mathbb{C} \smallsetminus (-\infty, 0]$.

5.7. **Proposition.** *The power series*

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(z-1)^n}{n}$$

is the principal branch of the logarithm in $B_{1,1}$.

Proof. The given power series has radius of convergence 1, so it defines an analytic function $f(z)$ on $B_{1,1}$. Note that $f'(z) = \sum_{n\in\mathbb{N}} (-1)^n (z-1)^n = 1/z$. (Exercise: check last equality.) Let $g(z) = e^{f(z)}$. Then $g'(z) = e^{f(z)}/z$ and $g''(z) = 0$. Hence $g'(z) = \alpha$ a constant.[7] Since $g'(1) = 1$, it follows that $f(z)$ is a branch of the logarithm. Since $\operatorname{Log} 1 = f(1)$, it follows that $\operatorname{Log} z = f(z)$, because two branches differ by an integer multiple of $2\pi\imath$. □

Exercises.

5.1 Verify the properties of $\sin z$ and $\cos z$ listed in class.

5.2 Show that for $x > 0$, $x - \frac{x^3}{6} < \sin x < x$ and that $1 - \frac{x^2}{2} < \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}$. (Hint: Use the fact that $\sin x < 1$ and $\cos x < 1$ and integrate sin and cos alternately.)

5.3 Since $\cos 0 = 1$ and $\cos(\sqrt{3}) < 0$, there is a smallest real number θ such that $\cos\theta = 0$. Then $\sin\theta = \pm 1$. One can then define $\pi := 2\theta$.

5.4 Expand $\frac{1}{z}$ as a power series around $z = 1$. Find its radius of convergence.

5.5 Show that $\operatorname{Log}(z_1 z_2) = \operatorname{Log}(z_1) + \operatorname{Log}(z_2) + \delta$ for an appropriate δ.

5.6 Let U be a domain not containing 0 and f and g branches of the logarithm on U. Show that the function $h(z) := (f(z) - g(z))/(2\pi\imath)$ on U takes only integer values, by showing that $e^{2\pi\imath h(z)} = 1$. Hence there exists $n \in \mathbb{Z}$ such that $h(z) = n$ for every z. Hence $f(z) - g(z) = 2n\pi\imath$. Conversely, if $f(z) - g(z) = 2n\pi\imath$ for some n, and $f(z)$ is a branch of the logarithm if and only if $g(z)$ is.

[7] We cannot prove this with the results we proved so far. It is true that $g(z)$ is analytic, being the composite of the two analytic functions $f(z)$ and e^z; we can then use Exercise 4.3 from the last section. However the proof that the composite of two analytic functions is analytic is long, and we will not discuss it in class. Instead we use the fact (easily provable, using the chain rule) that the composite of two holomorphic functions is holomorphic and Proposition 7.5. The proof of Proposition 7.5 does not refer to anything in this section, so our argument is not circular.

5.7 Let U be a domain not containing 0 and f a branch of the logarithm on U. Show that f is holomorphic on U as follows. If $c \in U \setminus (\infty, 0]$, then there is a neighbourhood $B_{c,R}$ which does not intersect $(\infty, 0]$; on that neighbourhood, $f(z)$ differs from $\operatorname{Log}(z)$ by a holomorphic function, so $f(z)$ is holomorphic. If $c \in U \cap (\infty, 0]$, then 'rotate the domain on which Log is holomorphic' by an appropriate θ by using the function $\operatorname{Log}(e^{\imath\theta} z) - \imath\theta$. Conclude that $f'(z) = 1/z$.

Lecture 6. Path integrals, I

6.1. **Definition.** Let U be a domain. A *path* (also called an *arc*) in U is a continuous map $\gamma : [a, b] \longrightarrow U$. Let γ be a path. Say that γ is *closed* if $\gamma(b) = \gamma(a)$. By $-\gamma$, we mean the function $[a, b] \longrightarrow U, t \mapsto \gamma(a + b - t)$, and call it the *opposite path* of γ. Say that γ is *differentiable* if the functions $[a, b] \longrightarrow \mathbb{R}, t \mapsto \Re(\gamma(t))$ and $t \mapsto \Im(\gamma(t))$ are in $C^1([a, b])$. For a differentiable path γ, write $\gamma'(t)$ for $(\Re(\gamma(t)))' + \imath(\Im(\gamma(t)))'$. Say that γ is *piecewise differentiable* if there exists a partition $a = t_0 < t_1 < \cdots < t_n = b$ such that $\gamma|_{[t_i,t_{i+1}]}$ is in $C^1([t_i, t_{i+1}])$ for every $0 \leq i < n$; we also say that the partition $a = t_0 < t_1 < \cdots < t_n = b$ is *good* for γ to denote this fact.

6.2. **Proposition.** *Between any pair of points in a domain, there exists a piecewise-differentiable path connecting them.*

Proof. Let U be a domain and $c \in U$. We show that the set

$$A := \{\zeta \in U \mid \text{there exists a piecewise-differentiable path from } c \text{ to } \zeta\}$$

is both open and closed. Let $\zeta \in A$. Then there exists $R > 0$ such that $B_{\zeta,R} \subseteq U$. For every $\zeta' \in B_{\zeta,R}$, the radial straight line joining ζ and ζ' extends a piecewise-differentiable path from c to ζ; hence $B_{\zeta,R} \subseteq A$. Hence A is open. Now let $p \in \overline{A}$. Let $r > 0$. Let $\zeta \in B_{p,r} \cap A$. Then the radial straight line joining ζ and p extends a piecewise-differentiable path from c to ζ; hence $p \in A$, so A is closed.

Note that $c \in A$, so $A \neq \varnothing$. Now, since U is connected, we see that $A = U$. □

6.3. **Definition.** Let $a < b \in \mathbb{R}$ and $f : [a, b] \longrightarrow \mathbb{C}$ a continuous function. Define

$$\int_a^b f(t)\mathrm{d}t = \int_a^b \Re(f(t))\mathrm{d}t + \imath \int_a^b \Im(f(t))\mathrm{d}t.$$

6.4. **Lemma.** *Let $a = s_0 < s_1 < \cdots < s_m = b$ and $a = t_0 < t_1 < \cdots < t_n = b$ be good partitions for a piecewise-differentiable path $\gamma : [a, b] \longrightarrow U$. Let $u_0, \ldots, u_k$ be distinct elements of $\{s_0, \ldots, s_m, t_1, \ldots, t_{n-1}\}$ arranged in the ascending order. Then the partition $a = u_0 < \cdots < u_k = b$ is good for γ.*

Proof. We need to show that $\gamma|_{[u_i,u_{i+1}]}$ is in $C^1([u_i, u_{i+1}])$. Note that there exists j such that $[u_i, u_{i+1}] \subseteq [s_j, s_{j+1}]$ or $[u_i, u_{i+1}] \subseteq [t_j, t_{j+1}]$; this proves the assertion. □

6.5. **Definition.** Let U be a domain and $f : U \longrightarrow \mathbb{C}$. Let $\gamma : [a, b] \longrightarrow U$ be a piecewise differentiable path, with a good partition $a = t_0 < t_1 < \cdots < t_n = b$. Define

$$\int_\gamma f(z)\mathrm{d}z = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(\gamma(t))\gamma'(t)\mathrm{d}t.$$

This is independent of the choice of the good partition.

Exercises.

6.1 Show that the definition of $\int_\gamma f(z)\mathrm{d}z$ (Definition 6.5) does not depend on the choice of the partition.

6.2 Let $\gamma_1 : [a_1, b_1] \longrightarrow \mathbb{C}$ and $\gamma_2 : [a_2, b_2] \longrightarrow \mathbb{C}$ be paths in $\mathbb{C}$ such that $\gamma_1(b_1) = \gamma_2(a_2)$. Define a new path $\tilde{\gamma}_2 : [b_1, b_2 - a_2 + b_1] \longrightarrow \mathbb{C}$ by setting $\tilde{\gamma}_2(t) = \gamma_2(t + a_2 - b_1)$. Note that the images of γ_2 and $\tilde{\gamma}_2$ are the same; this is an example of reparametrization of a path, discussed in the next lecture. Define the *concatenation* of γ_1 and $\tilde{\gamma}_2$ to be the path $\gamma : [a_1, b_2 - a_2 + b_1] \longrightarrow \mathbb{C}$

$$t \mapsto \begin{cases} \gamma_1(t), & t \in [a_1, b_1], \\ \tilde{\gamma}_2(t), & t \in [b_1, b_2 - a_2 + b_1]. \end{cases}$$

Show that if γ_1 and γ_2 are piecewise-differentiable paths, then so is γ. This is used in the proof of Proposition 6.2 to a piecewise-differentiable path from c to ζ' (while showing that A is open) and from c to p (while showing that A is closed).

6.3 Let U be a domain. A *piecewise-linear* path in U is a continuous function $\gamma : [a, b] \longrightarrow U$ such that there exists a partition $0 = t_0 < t_1 < \ldots < t_n = b$ such that $\gamma|_{[t_i, t_{i+1}]} : t \mapsto \frac{(t-t_i)\gamma(t_{i+1}) + (t_{i+1}-t)\gamma(t_i)}{(t_{i+1}-t_i)}$. Show that for every pair points in U, there is a piecewise-linear path joining them.

Lecture 7. Path integrals, II

7.1. **Definition.** Let U be a domain and $\gamma : [a, b] \longrightarrow U$ a path. A *reparametrization* of γ is a path of the form $\gamma \circ \tau : [a', b'] \longrightarrow U$ where $\tau : [a', b'] \longrightarrow [a, b]$ is a continuous piecewise differentiable non-decreasing surjective function.

Note that $\mathrm{Im}(\gamma) = \mathrm{Im}(\gamma \circ \tau)$. The next example shows that this is not sufficient.

7.2. **Example.** Let $\gamma : [0, 1] \longrightarrow \mathbb{C}$, $t \mapsto e^{2\pi\iota t}$. Then $\gamma_1 : [0, 2] \longrightarrow \mathbb{C}$, $t \mapsto e^{\pi\iota t}$ is a reparametrization of γ. To see this, let $\tau_1 : [0, 2] \longrightarrow [0, 1]$ be the map $t \mapsto \frac{t}{2}$; then $\gamma_1 = \gamma \circ \tau_1$. On the other hand, $\gamma_2 : [0, 2] \longrightarrow \mathbb{C}$, $t \mapsto e^{2\pi\iota t}$, is not a reparametrization of γ. Intuitively, γ_2 involves going round the circle twice, while γ involves going round only once.

7.3. **Discussion** (invariance under reparametrization)**.** Let U be a domain and $\gamma : [a, b] \longrightarrow U$ a path and $\tau : [a', b'] \longrightarrow [a, b]$ a continuous piecewise differentiable non-decreasing surjective function. Write $\tilde{\gamma} = \gamma \circ \tau$. Let $a' = s_0 < s_1 < \cdots < s_m = b'$ be a good partition for τ and $a = t_0 < t_1 < \cdots < t_n = b$ be a good partition for γ. Let $u_0 < \ldots < u_k$ be the distinct elements of $\{s_0, \ldots, s_m\} \cup \{\tau^{-1}(t_0), \ldots, \tau^{-1}(t_n)\}$. Then $a' = u_0 < \cdots < u_k = b'$ is good for $\tilde{\gamma}$. Let $a = v_0 < \cdots < v_l = b$ be the distinct elements of $\{\tau(u_0), \ldots, \tau(u_k)\}$; this is good for γ. Thus,

$$\begin{aligned} \int_{\tilde{\gamma}} f(z)\mathrm{d}z &= \sum_{i=0}^{k-1} \int_{u_i}^{u_{i+1}} f(\tilde{\gamma}(s))\tilde{\gamma}'(s)\mathrm{d}s \\ &= \sum_{i=0}^{k-1} \int_{u_i}^{u_{i+1}} f(\gamma(\tau(s)))\gamma'(\tau(s))\tau'(s)\mathrm{d}s \\ &= \sum_{i=0}^{l-1} \int_{v_i}^{v_{i+1}} f(\gamma(t))\gamma'(t)\mathrm{d}t \\ &= \int_{\gamma} f(z)\mathrm{d}z. \end{aligned}$$

Question: where did we use the hypothesis that τ is a non-decreasing function? □

7.4. **Discussion** (integration along the opposite path). Let U be a domain and $\gamma : [a,b] \longrightarrow U$ a piecewise differentiable path, with a good partition $a = s_0 < \ldots < s_n = b$. For $0 \leq i \leq n$, write $t_i = (a+b) - s_{n-i}$. Then

$$-\gamma|_{[t_i,t_{i+1}]} = \gamma|_{[s_{n-i-1},s_{n-i}]} \circ (t \mapsto (a+b)-t)\,.$$

Therefore $a = t_0 < \ldots < t_n = b$ is a good partition for $-\gamma$.

$$\begin{aligned}\int_{-\gamma} f(z)\mathrm{d}z &= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f((-\gamma)(t))(-\gamma)'(t)\mathrm{d}t \\ &= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(\gamma(a+b-t))\,(\gamma'(a+b-t))\,(-1)\mathrm{d}t \\ &= \sum_{i=0}^{n-1} \int_{s_{n-i}}^{s_{n-i-1}} f(\gamma(s))\gamma'(s)\mathrm{d}s \\ &= -\int_{\gamma} f(z)\mathrm{d}z. \end{aligned}$$ □

7.5. **Proposition.** *Let U be a domain and f holomorphic on U. If f' is identically zero on U, then f is a constant function.*

Proof. Let $c_1, c_2 \in U$. We want to show that $f(c_1) = f(c_2)$. Let $\gamma : [a,b] \longrightarrow U$ a piecewise-differentiable path with $\gamma(a) = c_1$ and $\gamma(b) = c_2$. Let $a = t_0 < \ldots < t_n = b$ be a good partition for γ. It suffices to show that $f(\gamma(t_i)) = f(\gamma(t_{i+1}))$. Replacing a by t_i and b by t_{i+1}, we may assume that γ is differentiable on $[a,b]$.

The function

$$g : [a,b] \longrightarrow \mathbb{C}, t \mapsto f(\gamma(t))$$

is differentiable, with derivative $g'(t) = f'(\gamma(t))\gamma'(t) = 0$. Hence $f(c_2) = g(b) = g(a) = f(c_1)$. □

Exercises.

(1) Check that in Example 7.2, γ_2 is not a reparametrization of γ.
(2) Read Discussion 7.3 about reparametrization and understand where we used the hypothesis that τ is a non-decreasing function.

Lecture 8. Absolute value of a path integral

8.1. **Lemma.** *Let $f : [a,b] \longrightarrow \mathbb{C}$ be a continuous function and $c \in \mathbb{C}$. Then*

$$\int_a^b cf(t)\mathrm{d}t = c\int_a^b f(t)\mathrm{d}t.$$

Proof. Write $f(t) = u(t) + \imath v(t)$ and $c = \alpha + \imath\beta$. Then both sides of the asserted equality are equal to

$$\int_a^b (\alpha u(t) - \beta v(t))\,\mathrm{d}t + \imath \int_a^b (\alpha v(t) + \beta u(t))\,\mathrm{d}t.$$ □

8.2. **Corollary.** *Let $f : [a,b] \longrightarrow \mathbb{C}$ be a continuous function. Then*

$$\left|\int_a^b f(t)\mathrm{d}t\right| \leq \int_a^b |f(t)|\,\mathrm{d}t.$$

Proof. Without loss of generality, we may assume that $\int_a^b f(t)\mathrm{d}t \neq 0$. Let θ be an argument of $\int_a^b f(t)\mathrm{d}t$. Then

$$\begin{aligned}\left|\int_a^b f(t)\mathrm{d}t\right| &= \Re\left(e^{-\imath\theta}\int_a^b f(t)\mathrm{d}t\right)\\ &= \int_a^b \Re\left(e^{-\imath\theta}f(t)\right)\mathrm{d}t \qquad \text{(by Lemma 8.1)}\\ &\leq \int_a^b |f(t)|\,\mathrm{d}t.\end{aligned}$$

□

8.3. **Definition.** Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a piecewise-differentiable path and f a $\mathbb{C}$-valued function defined and continuous on $\mathrm{Im}(\gamma)$. The integral of f *with respect to arc length* denoted [8] by $\int_\gamma f|\mathrm{d}z|$ is

$$\int_a^b f(\gamma(t))|\gamma'(t)|\mathrm{d}t.$$

8.4. **Proposition.** *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a piecewise-differentiable path and f a $\mathbb{C}$-valued function defined and continuous on* $\mathrm{Im}(\gamma)$. *Then*

$$\int_{-\gamma} f|\mathrm{d}z| = \int_\gamma f|\mathrm{d}z|.$$

Proof. We repeat the argument from Discussion 7.4, with suitable changes.

$$\begin{aligned}\int_{-\gamma} f(z)|\mathrm{d}z| &= \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} f((-\gamma)(t))|(-\gamma)'(t)|\mathrm{d}t\\ &= \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} f(\gamma(a+b-t))|\gamma'(a+b-t)|\mathrm{d}t\\ &= \sum_{i=0}^{n-1} -\int_{s_{n-i}}^{s_{n-i-1}} f(\gamma(s))\gamma'(s)\mathrm{d}s\\ &= \int_\gamma f(z)|\mathrm{d}z|.\end{aligned}$$

□

8.5. **Proposition.** *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a piecewise-differentiable path and f a $\mathbb{C}$-valued function defined and continuous on* $\mathrm{Im}(\gamma)$. *Then*

$$\left|\int_\gamma f\mathrm{d}z\right| \leq \int_\gamma |f||\mathrm{d}z|.$$

Proof. Use Corollary 8.2 to see that

$$\left|\int_\gamma f\mathrm{d}z\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t\right| \leq \int_a^b |f(\gamma(t))||\gamma'(t)|\mathrm{d}t = \int_\gamma |f||\mathrm{d}z|.$$

□

8.6. **Definition.** Let γ be a piecewise-differentiable path. The *arc length* of γ is $\int_\gamma |\mathrm{d}z|$.

[8] Many textbooks, including Ahlfors, also use $\int_\gamma f\mathrm{d}s$ denote this, but we will avoid this usage, since sometimes we use s to denote a real or complex variable.

8.7. **Corollary.** *Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a piecewise-differentiable path and f a $\mathbb{C}$-valued function defined and continuous on $\operatorname{Im}(\gamma)$. Let $C \geq \max\{|f(z)| : z \in \operatorname{Im}(\gamma)\}$. Write L for the arc length of γ. Then*

$$\left|\int_\gamma f \mathrm{d}z\right| \leq CL$$

Proof. Observe that if g is a real-valued continuous function on $\operatorname{Im}(\gamma)$ taking non-negative real values, then $\int_\gamma g|\mathrm{d}z|$ is a non-negative real number. Now apply this observation with $g = C - |f|$ to see that

$$\left|\int_\gamma f \mathrm{d}z\right| \leq \int_\gamma |f||\mathrm{d}z| \leq CL. \qquad \square$$

Exercises.

(1) Show that the arc length of a piecewise-linear path is the sum of the lengths of the line segments in it. (See Exercise 6.3 in Lecture 6.)

(2) Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a piecewise-differentiable path. Then the arc length of γ is the supremum of the set

$$\left\{\sum_{i=0}^{n-1} |\gamma(t_{i+1}) - \gamma(t_i)| : n \geq 1, a = t_0 < t_1 < \ldots < t_n = b\right\}.$$

(3) Let γ be a piecewise-differentiable path in $\mathbb{C}$ and $\tilde{\gamma}$ a reparametrization. Show that the arc lengths of γ and $\tilde{\gamma}$ equal each other.

Lecture 9. Primitives

9.1. **Definition.** Let U be a domain and $f : U \longrightarrow \mathbb{C}$. A *primitive* F of f on U is a (holomorphic) function $F : U \longrightarrow \mathbb{C}$ such that $F' = f$ on U.

Note that f need not have a primitive on U; see Proposition 9.4.

9.2. **Proposition.** *If F_1 and F_2 are primitives of a function f on a domain U, then $F_1 - F_2$ is a constant function.*

Proof. Note that $(F_1 - F_2)' = F_1' - F_2' = 0$; now apply Proposition 7.5 to $F_1 - F_2$. $\square$

9.3. **Example.** Let $m \in \mathbb{Z}$ and $f(z) = z^m$ (wherever it can be defined). If $m \geq 0$, then $z^{m+1}/(m+1)$ is a primitive of z^m on $\mathbb{C}$. If $m < -1$, then $z^{m+1}/(m+1)$ is a primitive of z^m on $\mathbb{C} \smallsetminus \{0\}$. Now suppose $m = -1$. If there is a branch of the logarithm on U, then it is a primitive of $f(z)$. (Branches of the logarithm are holomorphic, with derivative $\frac{1}{z}$; see exercise in Section 5.) Hence $\frac{1}{z}$ has a primitive on $\mathbb{C} \smallsetminus (-\infty, 0]$, while, using the next proposition, one of the exercises will show that it does not have a primitive on $\mathbb{C} \smallsetminus \{0\}$. $\square$

9.4. **Proposition.** *Let U be a domain and $f : U \longrightarrow \mathbb{C}$ be a continuous function. Then the following are equivalent:*

(1) f has a primitive on U.

(2) There exists a function $F : U \longrightarrow \mathbb{C}$ such that for every piecewise-differentiable path $\gamma : [a, b] \longrightarrow U$, $\int_\gamma f(z)\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$.

(3) For every piecewise-differentiable closed path γ in U, $\int_\gamma f(z)\mathrm{d}z = 0$.

Proof. (1) $\implies$ (2): Let F be a primitive of f on U. Then $\int_\gamma f(z)\mathrm{d}z = \int_\gamma F'(z)\mathrm{d}z$; we want to show that its value is $F(\gamma(b)) - F(\gamma(a))$. Let $a = t_0 < t_1 < \cdots < t_n = b$ be a good partition for γ. It suffices to show that for every i

$$\int_{t_i}^{t_{i+1}} F'(\gamma(t))\gamma'(t)\mathrm{d}t = F(\gamma(t_{i+1})) - F(\gamma(t_i)).$$

Without loss of generality, we may assume that γ is a differentiable path. Write $G = F \circ \gamma$. Then $G'(t) = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t)$ is a continuous function, so we see that

$$\int_a^b G'(t)\mathrm{d}t = G(b) - G(a)$$

by evaluating its real and imaginary parts (which are continuous, and, hence the fundamental theorem of calculus applies).

(2) $\implies$ (1): We prove that F is a primitive of f on U. Let $c \in U$. Let $\epsilon > 0$. We want to show that there exists $\delta > 0$ such that for all $h \in \mathbb{C}$ with $|h| < \delta$,

$$\left|\frac{F(c+h) - F(c)}{h} - f(c)\right| < \epsilon \tag{9.5}$$

First, choose δ such that $B_{c,\delta} \subseteq U$. Then, for every h with $c + h \in B_{c,\delta}$, we can evaluate $F(c + h) - F(c)$ as $\int_\tau f(z)\mathrm{d}z$, where τ is the function

$$[0,1] \longrightarrow \mathbb{C}, \ t \mapsto t(c+h) + (1-t)c.$$

(That is, we are going from c to $c + h$ along the line segment joining c to $c + h$ at a constant speed.) Write $f(z) = f(c) + \phi(z)$ on $B_{c,\delta}$. Using one of the exercises (or equation (3) of Ahlfors, Chapter 4, Section 1.1 ('Line integrals')) we see that

$$\begin{aligned}\left|\int_\tau \phi(z)\mathrm{d}z\right| &= \left|\int_0^1 \phi(\tau(t))\tau'(t)\mathrm{d}t\right| \\ &= |h|\int_0^1 |\phi(t(c+h) + (1-t)c)|\mathrm{d}t\end{aligned}$$

Since f is continuous, we may assume, possibly replacing δ by a smaller real number, that $|\phi(z)| < \epsilon$ for every $z \in B_{c,\delta}$. Now

$$\left|\frac{F(c+h) - F(c)}{h} - f(c)\right| = \left|\frac{\int_\tau f(z)\mathrm{d}z}{h} - f(c)\right| = \left|\frac{f(c)\cdot h + \int_\tau \phi(z)\mathrm{d}z}{h} - f(c)\right| < \epsilon,$$

thus proving (9.5).

★ mk: [Rewriting the above argument using integration w.r.t. arc length:] Since f is continuous, we may assume, possibly replacing δ by a smaller real number, that $|\phi(z)| < \epsilon$ for every $z \in \overline{B_{c,\delta}}$. By Corollary 8.7

$$\left|\int_\tau \phi(z)\mathrm{d}z\right| < \epsilon|h|.$$

Now

$$\left|\frac{F(c+h) - F(c)}{h} - f(c)\right| = \left|\frac{\int_\tau f(z)\mathrm{d}z}{h} - f(c)\right| = \left|\frac{f(c)\cdot h + \int_\tau \phi(z)\mathrm{d}z}{h} - f(c)\right| < \epsilon,$$

thus proving (9.5).

(2) $\iff$ (3): Exercise. □

Exercises.

(1) Let $f : [a, b] \longrightarrow \mathbb{C}$ be a function. Show that

$$\left|\int_a^b f \mathrm{d}t\right| \leq \int_a^b |f| \mathrm{d}t$$

(This is proved in equation (3) of Ahlfors, Chapter 4, Section 1.1 ('Line integrals').)

(2) Prove the assertion (2) $\iff$ (3) in Proposition 9.4.

(3) Let r be a positive real number. Let $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$, $t \mapsto re^{\imath t}$. Show that $\int_\gamma (1/z)\mathrm{d}z = 2\pi$. On the other hand, if γ is a piecewise-differentiable closed path that avoids some ray in $\mathbb{C}$ (i.e., $\{re^{\imath\alpha} \mid r \in \mathbb{R}, r \geq 0\}$ for some fixed α) then $\int_\gamma (1/z)\mathrm{d}z = 0$.

Lecture 10. Cauchy integral theorem, I

10.1. **Theorem** (Cauchy integral theorem for a rectangle). *(Ahlfors, Chapter 4, Section 1.4, Theorem 2, p. 109) Let U be a domain and f a holomorphic function on U. Let $R \subseteq U$ be a rectangle. Then*

$$\int_{\partial R} f(z)\mathrm{d}z = 0.$$

Note that ∂R is the union of four line segments, parallel to the real and imaginary axes. It is thought of as a closed curve in U, starting from one corner, and going once along the line segments.

Proof. Proof given in Ahlfors (due to Goursat). □

The following lemma should help clarify the estimation of $|\eta(R_n)|$ in equation (16) and the following paragraph on page. 111 of Ahlfors' book. In the proof of (9.5), we estimated

$$\left|\int_\tau \phi(z)\mathrm{d}z\right|$$

where τ is a linear path, i.e., a line segment parametrized by a linear function. We want to do a similar for ∂R_n, which is a piecewise-linear path.

10.2. **Lemma.** *Let U be a domain, $g : U \longrightarrow \mathbb{C}$ a continuous function and $\gamma : [a, b] \longrightarrow U$ a piecewise-linear path, i.e., a continuous function such that there exists a partition $a = t_0 < t_1 < \ldots < t_n = b$ such that $\gamma|_{[t_i,t_{i+1}]} : t \mapsto \frac{(t-t_i)\gamma(t_{i+1})+(t_{i+1}-t)\gamma(t_i)}{(t_{i+1}-t_i)}$. Then*

$$\left|\int_\gamma g(z)\mathrm{d}z\right| \leq \sum_{i=0}^{n-1} \frac{|\gamma(t_{i+1}) - \gamma(t_i)|}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} |g(\gamma(t))|\mathrm{d}t.$$

In particular, if $C \geq |g(z)|$ for every $z \in \operatorname{Im}(\gamma)$, then

$$\left|\int_\gamma g(z)\mathrm{d}z\right| \leq C \sum_{i=0}^{n-1} |\gamma(t_{i+1}) - \gamma(t_i)|.$$

Proof. Since

$$\int_\gamma g(z)\mathrm{d}z = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} g(\gamma(t))\gamma'(t)\mathrm{d}t.$$

it follows that

$$\left|\int_\gamma g(z)\mathrm{d}z\right| \leq \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |g(\gamma(t))||\gamma'(t)|\mathrm{d}t.$$

Now note that

$$\gamma'(t) = \frac{\gamma(t_{i+1}) - \gamma(t)}{t_{i+1} - t_i}$$

on $[t_i, t_{i+1}]$, proving the first assertion. The second assertion follows immediately from the first. □

10.3. **Corollary.** *With notation as in Ahlfors' book,* $|\eta(R_n)| \leq \epsilon L_n d_n$.

Proof. Note that

$$\int_{\partial R_n} [f(z) - f(z^*) - (z - z^*)f'(z^*)]\,\mathrm{d}z = \int_{\partial R_n} f(z)\mathrm{d}z - f(z^*)\int_{\partial R_n} \mathrm{d}z - f'(z^*)\int_{\partial R_n} (z - z^*)\mathrm{d}z.$$
$$= \eta(R_n)$$

since 1 and $(z - z^*)$ have primitives on $\mathbb{C}$. Hence we want to estimate

$$|\eta(R_n)| = \left| \int_{\partial R_n} [f(z) - f(z^*) - (z - z^*)f'(z^*)]\,\mathrm{d}z \right|.$$

Note that n is large enough so that

$$|f(z) - f(z^*) - (z - z^*)f'(z^*)| < \epsilon|z - z^*| < \epsilon d_n.$$

for all $z \in \partial R_n$. Now apply Lemma 10.2. ★ mk: [Or, directly] ∂R_n is a piecewise-linear path, and its arc length is the length L_n of the perimeter of R_n (Exercise 1 of Lecture 8). Now apply Corollary 8.7. □

Exercises.

(1) Show that in the proof of the theorem (with notation as in Ahlfors' book),

$$\left| \bigcap_n R_n \right| = 1.$$

Lecture 11. Cauchy integral theorem, II

11.1. **Theorem** (Cauchy integral theorem for a disc)**.** *(Ahlfors, Chapter 4, Section 1.5, Theorem 4, p.113) Let U be an open disc, f a holomorphic function on U. Then f has a primitive on U. In particular,*

$$\int_\gamma f(z)\mathrm{d}z = 0,$$

for every piecewise-differentiable closed path γ in U.

Proof. The second assertion follows from the first and Proposition 9.4; therefore we will prove the first. Without loss of generality, we may assume that U is centred at 0 (Exercise). Define $F : U \longrightarrow \mathbb{C}$ by $\zeta \mapsto \int_\sigma f(z)\mathrm{d}z$, where σ is the piecewise-differentiable path from 0 to ζ that goes from 0 to $(\Re(\zeta), 0)$ (the line segment parallel to the real axis) and from there to ζ (the line segment parallel to the imaginary axis).

We will show that F is holomorphic on U with $F' = f$. Let $c \in U$. Let $\epsilon > 0$. We want to show that there exists $\delta > 0$ such that for all $h \in \mathbb{C}$ with $|h| < \delta$,

$$\left| \frac{F(c+h) - F(c)}{h} - f(c) \right| < \epsilon. \tag{11.2}$$

There exists $\delta > 0$ such that $B_{c,\delta} \subseteq U$, that $|f(z) - f(c)| < \frac{\epsilon}{2}$ for every $z \in B_{c,\delta}$, since f is continuous. Let $h \in B_{0,\delta}$. Let σ (respectively, σ_1) be the piecewise-differentiable path from 0 to

c (respectively, $c+h$) that goes from 0 to $(\Re(c), 0)$ (respectively, to $(\Re(c+h), 0)$) and from there to c (respectively $c+h$). Let τ be the piecewise-differentiable path from c to $(\Re(c+h), \Im(c))$ and from there to $c+h$. Applying Theorem 10.1 to the rectangle with vertices $(\Re(c), 0)$, $(\Re(c+h), 0)$, $(\Re(c+h), \Im(c))$ and c we see that

$$F(c+h) = \int_{\sigma_1} f(z)\mathrm{d}z = \int_{\sigma} f(z)\mathrm{d}z + \int_{\tau} f(z)\mathrm{d}z$$
$$= F(c) + \int_{\tau} f(z)\mathrm{d}z.$$

Write $\phi(z) = f(z) - f(c)$ on $B_{c,\delta}$. Now,

$$\int_{\tau} f(z)\mathrm{d}z = \int_{\tau} f(c)\mathrm{d}z + \int_{\tau} \phi(z)\mathrm{d}z$$
$$= f(c)[(c+h) - c] + \int_{\tau} \phi(z)\mathrm{d}z$$
$$= hf(c) + \int_{\tau} \phi(z)\mathrm{d}z.$$

(We have used the fact constant functions have primitives on $\mathbb{C}$.) Hence we can rewrite (11.2) as

$$\left| \frac{\int_{\tau} \phi(z)\mathrm{d}z}{h} \right| < \epsilon. \tag{11.3}$$

Let τ_1 (respectively τ_2) be the piecewise-differentiable path from c to $(\Re(c+h), \Im(c))$ (respectively, from $(\Re(c+h), \Im(c))$ to $c+h$). Then τ as the concatenation of τ_1 and τ_2. Therefore the arc length of τ is at most $|\Re(h)| + |\Im(h)| < 2|h|$. Now apply Corollary 8.7 after noting that $|\phi(z)| < \epsilon/2$ on $\mathrm{Im}(\gamma)$ to obtain (11.3). ☐

Exercises.

(1) Show that in the proof of Theorem 11.1, we can assume that the centre of U is 0 as follows: Let c be the centre of U. Let $\tau : U \longrightarrow \mathbb{C}$ be the function $z \mapsto z - c$. Let $U_1 = \mathrm{Im}(\tau)$. Then τ maps U homeomorphically to U_1. Let $f_1 = f \circ \tau^{-1}$ and $\gamma_1 = \tau \circ \gamma$. Then $\int_{\gamma} f(z)\mathrm{d}z = \int_{\gamma_1} f_1(z)\mathrm{d}z$.

(2) Depending on the generality of Green's theorem that you are familiar with, one can establish a version of Cauchy integral theorem, as follows. Let γ be a Jordan curve in $\mathbb{C}$ (i.e., a closed piece-wise differentiable path that is injective, except at the end-points). Let U be a domain that contains γ and the open subset of $\mathbb{C}$ bounded by γ. Let f be a holomorphic function on U. Write $z = x + \imath y$, $f = u(x,y) + \imath v(x,y)$. We showed that if f is holomorphic, then u and v are differentiable on U.
 (a) $f\mathrm{d}z = u\mathrm{d}x - v\mathrm{d}y + \imath(v\mathrm{d}x + u\mathrm{d}y)$.
 (b) Suppose that f' is continuous. Then $\int_{\gamma} f\mathrm{d}z = 0$.

(3) Let U be a domain containing $\overline{B_{0,1}}$ and $\gamma : [0,1] \longrightarrow \mathbb{C}$, $t \mapsto e^{2\pi\imath t}$. Compute

$$\int_{\gamma} \frac{1}{z - \frac{1}{2}}\mathrm{d}z$$

as follows. (Note that neither is the integrand holomorphic on U nor is γ centred at $\frac{1}{2}$, so earlier arguments do not apply immediately.)

(a) Let $0 < r \ll 1$ and $\sigma : [0,1] \longrightarrow \mathbb{C}, t \mapsto \frac{1}{2} + re^{2\pi\iota t}$. Compute

$$\int_\sigma \frac{1}{z-\frac{1}{2}} \mathrm{d}z$$

(b) For $0 < \epsilon \ll 1$, define the following four points and paths in U: $p \in \operatorname{Im}\gamma$ with $\Re(p) > 0$ and $\Im(p) = \epsilon$; $q \in \operatorname{Im}\gamma$ with $\Re(q) > 0$ and $\Im(q) = -\epsilon$; $a \in \operatorname{Im}\sigma$ with $\Re(p) > \frac{1}{2}$ and $\Im(p) = \epsilon$; $b \in \operatorname{Im}\sigma$ with $\Re(q) > \frac{1}{2}$ and $\Im(q) = -\epsilon$; γ_1 from p to q counter-clockwise, following the same path as γ; γ_1 from a to b counter-clockwise, following the same path as σ; τ_1 from a to p, parallel to the real axis; τ_2 from b to q, parallel to the real axis. Let Γ be the closed piecewise differentiable path at p obtained by concatenating $\gamma_1, -\tau_2, -\sigma_1$ and τ_1. Show that

$$\int_\Gamma \frac{1}{z-\frac{1}{2}} \mathrm{d}z = 0.$$

(Hint: $\Gamma \subseteq U \smallsetminus \{\frac{1}{2} + r \mid r \in \mathbb{R}, r \geq 0\}$, on which $\frac{1}{z-\frac{1}{2}}$ has a primitive.)

(c) Show that

$$\lim_{\epsilon \longrightarrow 0} \int_{\gamma_1} \frac{1}{z-\frac{1}{2}} \mathrm{d}z = \int_\gamma \frac{1}{z-\frac{1}{2}} \mathrm{d}z.$$

$$\lim_{\epsilon \longrightarrow 0} \int_{\sigma_1} \frac{1}{z-\frac{1}{2}} \mathrm{d}z = \int_\sigma \frac{1}{z-\frac{1}{2}} \mathrm{d}z.$$

$$\lim_{\epsilon \longrightarrow 0} \int_{\tau_1} \frac{1}{z-\frac{1}{2}} \mathrm{d}z = \lim_{\epsilon \longrightarrow 0} \int_{\tau_2} \frac{1}{z-\frac{1}{2}} \mathrm{d}z$$

(d) Conclude that

$$\int_\gamma \frac{1}{z-\frac{1}{2}} \mathrm{d}z = \int_\sigma \frac{1}{z-\frac{1}{2}} \mathrm{d}z.$$

(e) Generalize the result, after replacing $\frac{1}{2}$ by an arbitrary $c \in B_{0,1}$.

Lecture 12. Cauchy integral theorem, III

General background: We need to show that if $g(z)$ is holomorphic on $B_{c,R}$ ($R > 0$) and γ is the closed path $[0,1] \longrightarrow \mathbb{C}, t \mapsto c + re^{2\pi\iota t}$ (with $0 < r < R$) then

$$\frac{1}{2\pi\iota} \int_\gamma \frac{g(z)}{z-\zeta} \mathrm{d}z = g(\zeta).$$

for every $\zeta \in B_{c,r}$. One way to evaluate the integral (a la Lang or Rodríguez-Kra-Gilman) is to observe that γ can be 'continuously deformed' to a closed path $\gamma_1 : [0,1] \longrightarrow \mathbb{C}, t \mapsto \zeta + \rho e^{2\pi\iota t}$ with a small ρ (so that γ_1 is inside $B_{c,r}$), and therefore we may try to evaluate the integral on γ_1.

Another option (a la Ahlfors) is to look at the function

$$f(z) = \frac{g(z) - g(\zeta)}{z - \zeta}$$

and observe that it is holomorphic on $B_{c,R}$ except at ζ, where it has the property that $\lim_{z\to\zeta}(z-\zeta)f(z) = 0$. We now strengthen CIT 11.1 to include such functions with this property. In fact, we will see later that we can extend f to a holomorphic function which is defined also at ζ.

12.1. **Theorem.** *Ahlfors, p. 113, Theorem 5 (the version with 'mild singularities'.) Let U be an open disc, U' an open subset of U obtained by omitting finitely many points of U, f a holomorphic function on U' and γ a closed path in U'. Assume that* $\lim_{z\to\zeta}(z-\zeta)f(z) = 0$ *for every* $\zeta \in U \smallsetminus U'$. *Then*

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

Proof. Without loss of generality, U is centred at 0. Define $F : U' \longrightarrow \mathbb{C}, \zeta \mapsto \int_\sigma f(z)\mathrm{d}z$, where σ be an rectilinear path in U' (a path consisting of finitely many segments, parallel to the real and the imaginary axes) from 0 to ζ. (This path needs to avoid the points in $U \smallsetminus U'$.) We need to show that

(1) the value of $F(\zeta)$ does not depend on the choice of σ, for every $\zeta \in U'$
(2) F is holomorphic with $F' = f$.

To prove the first assertion, we will prove an analogous version of Theorem 10.1, in which some points in the interior of the rectangle are omitted; see Theorem 12.2 below. The holomorphicity of F can be proved exactly as in the proof of Theorem 11.1, since for every $\zeta \in U'$, there exists $\delta > 0$ such that $B_{\zeta,\delta} \subset U'$. □

12.2. **Theorem** (Cauchy integral theorem for a rectangle, with 'mild singularities')**.** *(Ahlfors, Chapter 4, Section 1.4, Theorem 3, p. 111) Let U be a domain, U' an open subset of U obtained by omitting finitely many points of U. Let f a holomorphic function on U' such that* $\lim_{z\to\zeta}(z-\zeta)f(z) = 0$ *for every* $\zeta \in U \smallsetminus U'$. *Let $R \subseteq U$ be a rectangle, such that $\partial R \subseteq U'$. Then*

$$\int_{\partial R} f(z)\mathrm{d}z = 0.$$

Proof. Proof given in Ahlfors, p. 112. After subdividing R, we may assume that R contains exactly one element of $U \smallsetminus U'$; call this element ζ. Let $R_0 \subseteq R$ be a square of size $2a$ (with sides parallel to the axes) with centre ζ. Then

$$\int_{\partial R} f(z)\mathrm{d}z = \int_{\partial R_0} f(z)\mathrm{d}z.$$

Let $\epsilon > 0$. We may choose a such that

$$|(z-\zeta)f(z)| < \epsilon$$

for every $z \in R_0$. Therefore for each $z \in \partial R_0$, $|z-\zeta| > a$ and so

$$|f(z)| < \frac{\epsilon}{a}.$$

The length of the perimeter of R_0 is $8a$. Hence by Corollary 8.7

$$\left|\int_{\partial R_0} f(z)\mathrm{d}z\right| < \frac{\epsilon}{a} \cdot 8a = 8\epsilon.$$

Therefore

$$\int_{\partial R} f(z)\mathrm{d}z = 0.$$ □

12.3. **Remark.** We will later see that in this situation, $\lim_{z\to\zeta} f(z)$ exists for every $\zeta \in U'$. Hence we can extend the function to a holomorphic function on U, by setting $f(\zeta) = \lim_{z\to\zeta} f(z)$ for every $\zeta \in U'$. The proof of this result will require some knowledge about the local behaviour of holomorphic functions, for which we need to know this result. Otherwise, the argument would be circular.

Lecture 13. Index of a point

The following proposition generalizes Exercise 3 of Lecture 11.

13.1. **Proposition.** *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ a closed piecewise differentiable path. Let $\zeta \in \mathbb{C} \smallsetminus \operatorname{Im}\gamma$. Then there exists $n(\zeta,\gamma) \in \mathbb{Z}$ such that*

$$\int_\gamma \frac{\mathrm{d}z}{z-\zeta} = n(\zeta,\gamma)\cdot 2\pi\imath.$$

Proof. For $s \in [a,b]$, write

$$h(s) = \int_a^s \frac{\gamma'(t)\mathrm{d}t}{\gamma(t)-\zeta}.$$

This is a continuous function on $[a,b]$. Since $\gamma'(t)$ is continuous except on a finite subset of $[a,b]$,

$$h'(s) = \frac{\gamma'(s)}{\gamma(s)-\zeta}$$

on the complement of that finite set. Therefore

$$h_1(t) := \frac{\gamma(t)-\zeta}{e^{h(t)}}$$

is differentiable except on a finite subset of $[a,b]$. Note that

$$h_1'(t) := \frac{\gamma'(t)}{e^{h(t)}} - \frac{(\gamma(t)-\zeta)h'(t)}{e^{h(t)}} = 0.$$

Since $h_1(s)$ is continuous, it is constant, so

$$e^{h(t)} = \frac{\gamma(t)-\zeta}{\gamma(a)-\zeta}$$

for every $t \in [a,b]$. Since $\gamma(a) = \gamma(b)$, we conclude that $e^{h(a)} = e^{h(b)} = 1$. Therefore there exists $n(\zeta,\gamma) \in \mathbb{Z}$ such that

$$\int_\gamma \frac{\mathrm{d}z}{z-\zeta} = h(b) = n(\zeta,\gamma)\cdot 2\pi\imath. \qquad \square$$

13.2. **Lemma.** *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a closed piecewise differentiable path. Then the function $\mathbb{C} \smallsetminus \operatorname{Im}\gamma \longrightarrow \mathbb{Z}$, $\zeta \mapsto n(\zeta,\gamma)$ is locally constant.*

Proof. Let $\zeta \in \mathbb{C} \smallsetminus \operatorname{Im}\gamma$. We want to show that there exists $\delta > 0$ such that for every $\zeta' \in B_{\zeta,\delta}$,

$$\int_\gamma \frac{\mathrm{d}z}{z-\zeta} = \int_\gamma \frac{\mathrm{d}z}{z-\zeta'}. \tag{13.3}$$

Let $\delta > 0$ be such that $B_{\zeta,\delta} \cap \operatorname{Im}\gamma = \varnothing$. Let $\zeta' \in B_{\zeta,\delta}$. Let $f(z) : \mathbb{C} \smallsetminus \{\zeta'\} \longrightarrow \mathbb{C}$ be the function

$$f(z) = \frac{z-\zeta}{z-\zeta'}.$$

Let L be the line segment joining ζ and ζ' and $U = \mathbb{C} \smallsetminus L$. Then $f(U) \cap (-\infty, 0] = \varnothing$, i.e., for every $z \in U$, $\Im(f(z)) \neq 0$ or $\Re(f(z)) > 0$ (Exercise). Hence we can define

$$g : \mathbb{C} \smallsetminus L \longrightarrow \mathbb{C}, z \mapsto \operatorname{Log}(f(z)).$$

Note that g is holomorphic on U and that

$$g'(z) = \frac{f'(z)}{f(z)} = \frac{1}{z-\zeta} - \frac{1}{z-\zeta'}.$$

Since U is a domain and γ is a closed path in U, it follows that

$$\int_\gamma \left[\frac{1}{z-\zeta} - \frac{1}{z-\zeta'}\right] \mathrm{d}z = 0,$$

establishing (13.3). □

The following corollary recovers Exercise 3 of Lecture 11.

13.4. **Corollary.** *Let $c \in \mathbb{C}$, $r > 0$ and $\gamma : [0,1] \longrightarrow \mathbb{C}$ the path $t \mapsto c + re^{2\pi\imath t}$. Then for every $\zeta \in B_{c,r}$, $n(\zeta, \gamma) = 1$.*

Proof. Note that $n(c, \gamma) = 1$. Let $U = \{\zeta \in B_{c,r} \mid n(\zeta, \gamma) = 1\}$. By the lemma, U and $B_{c,r} \setminus U$ are open. Since $B_{c,r}$ is connected and U non-empty, $B_{c,r} = U$. □

Exercises.

(1) Show that $f(U) \cap (-\infty, 0] = \varnothing$ in the proof of Lemma 13.2.

(2) Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a piecewise-differentiable path. Show that $n(\zeta, \gamma) = 0$ for all $\zeta \in \mathbb{C}$ with $|\zeta| \gg 0$.

(3) Ahlfors, Chapter 4, Section 2.1 ('Index of a point ...'), Exercise 3 (p. 118). (proof of the Jordan curve theorem).

(4) Here is another proof of Lemma 13.2. Let ζ, ϵ be such that $B_{\zeta,\epsilon} \subseteq \mathbb{C} \setminus \operatorname{Im}(\gamma)$. Let $0 < \delta \ll \epsilon$ and $\zeta' \in B_{\zeta,\delta}$. For every $z \in \operatorname{Im}(\gamma)$,

$$\left|\frac{1}{z-\zeta} - \frac{1}{z-\zeta'}\right| = \left|\frac{\zeta - \zeta'}{(z-\zeta)(z-\zeta')}\right| < \frac{\delta}{\epsilon(\epsilon - \delta)}.$$

Let L be the arc length of γ. Then

$$\left|\int_\gamma \left[\frac{1}{z-\zeta} - \frac{1}{z-\zeta'}\right] \mathrm{d}z\right| < \frac{\delta L}{\epsilon(\epsilon-\delta)} < 2\pi.$$

Lecture 14. Cauchy integral formula

14.1. **Theorem** (Cauchy integral formula for a circular path). *(Ahlfors, Chapter 4, (22), p.119, for circles.) Let U be a domain, $c \in U$, $r > 0$ such that $\overline{B_{c,r}} \subseteq U$. Let $\zeta_1, \dots, \zeta_m \in U$ and $U' = U \setminus \{\zeta_1, \dots z_m\}$. Let f be a holomorphic function on U' such that $\lim_{z \to \zeta_i}(z - \zeta_i)f(z) = 0$ for every $1 \leq i \leq m$. Let γ be the circular path on the boundary of $B_{c,r}$. Then for all $\zeta \in B_{c,r} \cap U'$,*

$$f(\zeta) = \frac{1}{2\pi\imath}\int_\gamma \frac{f(z)\mathrm{d}z}{z-\zeta}.$$

Proof. Let

$$g(z) = \frac{f(z) - f(\zeta)}{z - \zeta}.$$

Then g is holomorphic on $U \setminus \{\zeta, \zeta_1, \dots, \zeta_m\}$, and $\lim_{z\to a}(z-a)g(z) = 0$ for every $a \in \{\zeta, \zeta_1, \dots, \zeta_m\}$. Therefore

$$\int_\gamma \frac{f(z)\mathrm{d}z}{z-\zeta} = \int_\gamma \frac{f(\zeta)\mathrm{d}z}{z-\zeta}.$$

Now apply Corollary 13.4. □

14.2. **Lemma.** *Let γ be a piecewise-differentiable closed path in $\mathbb{C}$. Let $g : \operatorname{Im}(\gamma) \longrightarrow \mathbb{C}$ be a continuous function. For positive integers n, define $F_n : \mathbb{C} \setminus \operatorname{Im}(\gamma) \longrightarrow \mathbb{C}$ by*

$$z \mapsto \int_\gamma \frac{g(\zeta)}{(\zeta - z)^n} \mathrm{d}\zeta.$$

Then for each $n \geq 1$, F_n is holomorphic on $\mathbb{C} \setminus \operatorname{Im}(\gamma)$ with $F_n' = nF_{n+1}$.

Proof. We will prove that F_1 is holomorphic with $F_1' = F_2$. For the rest, read the proof of Ahlfors, Chapter 4, Section 2.3 ('Higher derivatives'), Lemma 3.

Let z_0, ϵ be such that $B_{z_0,\epsilon} \subseteq \mathbb{C} \setminus \operatorname{Im}(\gamma)$. Let $0 < \delta \ll \epsilon$ and $z \in B_{z_0,\delta}$.

Step 1: $\lim_{z \to z_0} F_1(z) = F_1(z_0)$. Proof: Note that

$$F_1(z) - F_1(z_0) = (z - z_0) \int_\gamma \frac{g(\zeta)}{(\zeta - z)(\zeta - z_0)} \mathrm{d}\zeta. \tag{14.3}$$

For every $\zeta \in \operatorname{Im}(\gamma)$, $|(\zeta - z_0)| > \epsilon$ and $|(\zeta - z)| > \epsilon - \delta$. By Proposition 8.5

$$|F_1(z) - F_1(z_0)| < \frac{\delta}{\epsilon(\epsilon - \delta)} \int_\gamma |g(\zeta)||\mathrm{d}\zeta|.$$

Therefore $\lim_{z \to z_0} |F_1(z) - F_1(z_0)| = 0$.

Step 2: $F_1'(z_0) = F_2(z_0)$. Proof: Consider the function

$$G(z) = \int_\gamma \frac{g(\zeta)}{(\zeta - z)(\zeta - z_0)} \mathrm{d}\zeta$$

on $\mathbb{C} \setminus \operatorname{Im}(\gamma)$. Applying the previous step with $g(\zeta)/(\zeta - z_0)$ replacing $g(z)$, we see that $\lim_{z \to z_0} G(z) = G(z_0) = F_2(z_0)$. Now by (14.3)

$$\lim_{z \to z_0} \frac{F_1(z) - F_1(z_0)}{z - z_0} = \lim_{z \to z_0} G(z) = F_2(z_0).$$

□

14.4. **Corollary.** *With notation as in Theorem 14.1,*

$$f^{(n)}(\zeta) = \frac{n!}{2\pi\imath} \int_\gamma \frac{f(z)\mathrm{d}z}{(z - \zeta)^{n+1}}.$$

In particular, f is infinitely complex-differentiable on U'.

Proof. Write

$$F_{n+1} = \int_\gamma \frac{f(z)\mathrm{d}z}{(z - \zeta)^{n+1}}.$$

By Theorem 14.1, $F_1 = f$. By Lemma 14.2, $F_{n+1} = \frac{1}{n!} f^{(n)}$. □

Exercises.

(1) Complete the proof of Lemma 14.2. (Ahlfors, Chapter 4, Section 2.3 ('Higher derivatives'), Lemma 3 (p. 121))

(2) Let U be a domain and γ a piecewise-differentiable path in U. If f_n is a sequence of continuous functions on U converging uniformly to f, then

$$\lim_n \int_\gamma f_n(z)\mathrm{d}z = \int_\gamma f(z)\mathrm{d}z.$$

If $\sum_n f_n$ converges uniformly to f, then

$$\sum_n \int_\gamma f_n(z)\mathrm{d}z = \int_\gamma f(z)\mathrm{d}z.$$

(3) Let $r \in \mathbb{R}$ and $f_r : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $(x, y) \mapsto (x^2 + y^2 - 1)r$. Show that for each r, f_r is a real-analytic function. $f_r(p)$ does not depend on r if $p \in \partial B_{0,1}$, but depends on r if $p \in B_{0,1}$. This is in contrast with the behaviour of holomorphic functions (on a domain containing $\overline{B_{0,1}}$).

Lecture 15. Holomorphic functions are analytic.

In this lecture, we will prove that holomorphic functions are analytic.

15.1. **Lemma.** *Let U be a domain, $c \in U$ and f holomorphic on $U' := U \setminus \{c\}$. Then the following are equivalent:*

(1) $\lim_{z\to c} f(z)$ *exists (in $\mathbb{C}$).*
(2) $\lim_{z\to c}(z - c)f(z) = 0$.
(3) *there exists a holomorphic function $\tilde{f}$ on U such that $\tilde{f}|_{U'} = f$;*

Morever, in this situation, $\tilde{f}$ is uniquely determined by f.

Proof. (1) $\implies$ (2): $\lim_{z\to c}(z - c)f(z) = \lim_{z\to c}(z - c) \lim_{z\to c} f(z) = 0$.

(2) $\implies$ (3): Let $r > 0$ be such that $\overline{B_{c,r}} \subseteq U$. Let $\gamma : [0, 1] \longrightarrow U$ be the path $t \mapsto c + e^{2\pi\imath t}$. Define $\tilde{f} : U \longrightarrow \mathbb{C}$ by

$$\tilde{f}(\zeta) = \begin{cases} f(\zeta), & \text{if } \zeta \in U', \\ \int_\gamma \frac{f(z)\mathrm{d}z}{z-c}, & \text{if } \zeta = c. \end{cases}$$

We need to show that $\tilde{f}$ is holomorphic on U; for which it suffices to check that it is differentiable at c. We may therefore restrict our attention to $B_{c,r}$. Using Cauchy integral formula (Theorem 14.1) for $U' \cap B_{c,r}$, we can rewrite $\tilde{f}$ on $B_{c,r}$ as

$$\tilde{f}(\zeta) = \int_\gamma \frac{f(z)\mathrm{d}z}{z - \zeta}$$

Now apply Lemma 14.2, $\tilde{f}$ is holomorphic on $B_{c,r}$.

(3) $\implies$ (1): $\lim_{z\to c} = \tilde{f}(c)$.

Proving uniqueness is left as an exercise. ☐

15.2. **Definition.** Let U be a domain, $c \in U$ and f holomorphic on $U' := U \setminus \{c\}$. We say that c is a *removable singularity* of f if the equivalent conditions of the previous lemma are satisfied.

15.3. **Theorem.** *Let U be a domain, $c \in U$ and f a holomorphic on U. Let $n \in \mathbb{N}$. Then there exists a holomorphic function $f_n(z)$ on U such that*

$$f(\zeta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(\zeta - c)^k + (\zeta - c)^n f_n(\zeta)$$

on U. Let γ be the circular path around the boundary of $B_{c,R}$ where $R > 0$ is such that $\overline{B_{c,R}} \subseteq U$. Then

$$f_n(\zeta) = \frac{1}{2\pi\imath} \int_\gamma \frac{f(z)\mathrm{d}z}{(z - c)^n(z - \zeta)}$$

on $B_{c,R}$.

Proof. (Ahlfors, Chapter 4, Section 3.1, pp. 124ff.) The function $\frac{f(z)-f(c)}{z-c}$ (on $U \setminus \{c\}$) has a removable singularity at $z = c$, so there exists a holomorphic function $f_1(z)$ on U such that $f_1(z) = \frac{f(z)-f(c)}{z-c}$ on $U \setminus \{c\}$. Repeating this argument for f_1, and by induction, we see that for each positive integer k, there exists a holomorphic function f_{k+1} on U such that $f_{k+1}(z) = \frac{f_k(z)-f_k(c)}{z-c}$ on $U \setminus \{c\}$. Putting this together, we get the following:

$$\begin{aligned} f(z) &= f(c) + (z-c)f_1(z) \\ &= f(c) + (z-c)f_1(c) + (z-c)^2 f_2(z) \\ &= f(c) + (z-c)f_1(c) + (z-c)^2 f_2(c) + \cdots + (z-c)^{n-1} f_{n-1}(c) + (z-c)^n f_n(z) \end{aligned}$$

Note that $f^{(k)}(z)$ is the k-th order derivative of $(z-c)^k f_k(z)$, so $f^{(k)}(c) = k! f_k(c)$. This proves the first assertion.

Now note that on $B_{c,R}$

$$\begin{aligned} f_n(\zeta) &= \frac{1}{2\pi\imath}\int_\gamma \frac{f_n(z)}{z-\zeta}\mathrm{d}z \\ &= \frac{1}{2\pi\imath}\int_\gamma \frac{f(z)}{(z-c)^n(z-\zeta)}\mathrm{d}z - \sum_{k=0}^{n-1}\frac{f^{(k)}(c)}{2\pi\imath k!}\int_\gamma \frac{1}{(z-c)^{n-k}(z-\zeta)}\mathrm{d}z. \end{aligned} \tag{15.4}$$

Let $\zeta_1, \zeta_2 \in B_{c,R}$ and

$$G_m(\zeta_1,\zeta_2) := \int_\gamma \frac{1}{(z-\zeta_1)^m(z-\zeta_2)}\mathrm{d}z.$$

as a function of ζ_1, with ζ_2 fixed. We first show that $G_1(\zeta_1,\zeta_2) = 0$. First assume that $\zeta_1 \neq \zeta_2$. Then

$$\int_\gamma \frac{1}{(z-\zeta_1)(z-\zeta_2)}\mathrm{d}z = \frac{1}{\zeta_1-\zeta_2}(n(\zeta_1,\gamma) - n(\zeta_2,\gamma)) = 0.$$

Now assume that $\zeta_1 = \zeta_2$. Since $\frac{1}{(z-\zeta_1)^2}$ has a primitive on $\mathbb{C} \setminus \{0\}$, we see that

$$\int_\gamma \frac{1}{(z-\zeta_1)^2}\mathrm{d}z = 0.$$

By Lemma 14.2 applied to the function $\frac{1}{(z-\zeta_2)}$, we see that G_m for $m \geq 2$ are successive derivatives of G_1 (thought of as a function of ζ_1), so $G_m = 0$ for each $m \geq 1$. Therefore in (15.4), we obtain

$$\int_\gamma \frac{1}{(z-c)^{n-k}(z-\zeta)}\mathrm{d}z = 0$$

for each $k = 0, \ldots, n-1$, thus completing the proof of the theorem. □

15.5. **Corollary.** *Let U be a domain and f holomorphic on U. Let c, R be such that $\overline{B_{c,R}} \subseteq U$. Then*

$$f(\zeta) = \sum_{k\in\mathbb{N}} \frac{f^{(k)}(c)}{k!}(\zeta-c)^k$$

on $B_{c,R}$. In particular, every holomorphic function on U is analytic on U.

Proof. Let γ be the circular path on the boundary of $B_{c,R}$. To prove the assertion, it suffices to show that for every $\epsilon > 0$, there exists N such that for every $n > N$,

$$\left|(\zeta - c)^n \frac{1}{2\pi\imath} \int_\gamma \frac{f(z)\mathrm{d}z}{(z-c)^n(z-\zeta)}\right| < \epsilon.$$

Let $M = \sup\{f(z) \mid z \in \mathrm{Im}(\gamma)\}$. Then

$$\left|(\zeta - c)^n \frac{1}{2\pi\imath} \int_\gamma \leq \frac{f(z)\mathrm{d}z}{(z-c)^n(z-\zeta)}\right| \leq \frac{M|\zeta - c|^n}{R^{n-1}(R - |\zeta - c|)}.$$

The assertion now follows, since $|\zeta - c| < R$. □

Exercises.

(1) Prove the uniqueness of $\tilde{f}$ in Lemma 15.1.
(2) Show that kth order derivative of $(z-c)^k g(z)$ at $z = c$ is $k!g(c)$.
(3) With fixed $\zeta_2 \in B_{c,R}$, use an appropriate result to conclude that $G_1(\zeta_1, \zeta_2)$ is a holomorphic function of $\zeta_1 \in B_{c,R}$, the proof of Theorem 15.3. Then show that $G_1(\zeta_2, \zeta_0) = 0$ by taking a limit.
(4) Let c be a removable singularity of f (which is defined on $U \smallsetminus \{c\}$ for some open neighbourhood U of c). Show that there exists $m \in \mathbb{N}$ and a holomorphic function f_1 on U such that $f(z) = (z-c)^m f_1(z)$ such that $f_1(c) \neq 0$.
(5) Let U be a domain and f a holomorphic function on U. Then the zeros of f are isolated. Show that f has only finitely many zeroes in any compact subset of U. If $c \in U$ is a zero, then there exists a unique positive integer m such that $f(z) = (z-c)^m f_1(z)$ on U, where f_1 is holomorphic on U and $f_1(c) \neq 0$. It is called the *order* (or *multiplicity*) of the zero at c.

Lecture 16. Morera's theorem, Liouville's theorem

16.1. **Corollary** (Morera's theorem). *Let U be a domain and $f : U \longrightarrow \mathbb{C}$ a continuous function. If $\int_\gamma f(z)\mathrm{d}z = 0$ for every closed piecewise-differentiable path γ in U, then f is analytic.*

Proof. By Corollary 15.5, it suffices to show that f is holomorphic. By Proposition 9.4, f has a primitive F on U. Since F is holomorphic, it is infinitely complex-differentiable by Corollary 14.4; in particular, $f = F'$ is holomorphic. □

16.2. **Proposition** (Liouville's theorem). *Every bounded entire function is constant.*

Proof. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a bounded holomorphic function. Let $M \in \mathbb{R}$ be such that $|f(z)| < M$ for every $z \in \mathbb{C}$. Let $\zeta \in \mathbb{C}$ and $r > 0$. By Corollary 14.4,

$$f'(\zeta) = \frac{1}{2\pi\imath} \int_{\partial B_{\zeta,r}} \frac{f(z)}{(z-\zeta)^2}\mathrm{d}z$$

Hence $|f'(\zeta)| \leq Mr^{-1}$, so $f' = 0$ on $\mathbb{C}$. Now apply 7.5. □

16.3. **Theorem** (Fundamental theorem of algebra). *$\mathbb{C}$ is an algebraically closed field.*

We need to show that complex polynomials of positive degree have a zero. First we prove a lemma about such polynomials.

16.4. **Lemma.** *Let $f \in \mathbb{C}[X]$ be a polynomial of positive degree. Then for every positive real number M, there exists $R > 0$ such that $|f(z)| > M$ for every z with $|z| > R$.*

Proof. Write $f(X) = \sum_{i=0}^{d} a_i X^i$, with $d > 0$ and $a_d \neq 0$. For every $z \in \mathbb{C}$, $|f(z)| \geq |a_d||z|^d - \sum_{i=0}^{d-1} |a_i||z|^i$. (Use $a^d z^d = f(z) - \sum_{i=0}^{d-1} a_i z^i$.) The assertion now follows from Exercise 1 below. □

Proof of Theorem 16.3. Let $f \in \mathbb{C}[X]$ be a polynomial of positive degree. We want to show that there exists $c \in \mathbb{C}$ such that $f(c) = 0$. By way of contradiction, assume that this is false. Hence $g(z) = \frac{1}{f(z)}$ is an entire function. We now claim that g is bounded. Assume the claim. Then g and, therefore, f are constant functions by Proposition 16.2, contradicting the hypothesis that f has positive degree.

To prove the claim, assume, on the contrary, that for each positive integer n, there exists $c_n \in \mathbb{C}$ such that $|g(c_n)| > n$. Then $|f(c_n)| < \frac{1}{c_n}$. If the sequence c_n is bounded (i.e., contained in a compact subset of $\mathbb{C}$), then it would have a convergent subsequence c_{n_m}, $m \geq 1$. (We implicitly assume that the function $m \mapsto n_m$ is an increasing function.) Then $f(\lim_m c_{n_m}) = \lim_m f(c_{n_m}) = 0$, a contradiction. Hence for every positive real number R, there exists n such that $|c_n| > R$. Now use Lemma 16.4 to obtain a contradiction. □

Exercises.

(1) Let $\sum_{i=0}^{d} b_i X^i \in \mathbb{R}[X]$ with $b_d > 0$. Then for every positive real number M, there exists $R > 0$ such that $g(x) > M$ for every $x \in \mathbb{R}$ with $x > R$.
(2) Show that Lemma 16.4 does not hold for entire functions in general, by looking at the exponential function.

Lecture 17. Isolated singularities

Let U be a domain, $c \in U$ and f a holomorphic function on $U \smallsetminus \{c\}$. We say that c is an *isolated singularity* of f. Recall that an isolated singularity c is said to be a removable singularity if $\lim\limits_{z \to c} (z - c) f(z) = 0$ (Definition 15.2).

(We do not rule out the situation that f is defined or is differentiable at c.)

17.1. **Definition.** An isolated singularity c is said to be a *pole* of f if it is not a removable singularity of f and it is a removable singularity of $1/f$.

17.2. **Example.** z^m with $m < 0$ has a pole at 0.

17.3. **Remark.** With notation as above, let U be a neighbourhood of c such that f is defined and holomorphic on $U \smallsetminus \{c\}$. Since the zeros of a holomorphic function are isolated (use Corollary 15.5 and Proposition 4.6), we may assume, by replacing U by a smaller neighbourhood if necessary, that $f(\zeta) \neq 0$ for each $\zeta \in U$, $\zeta \neq c$. Hence we can talk of $\frac{1}{f}$ in $U \smallsetminus \{c\}$ and consider whether c is a removable singularity or not.

17.4. **Proposition.** *Let c be a pole of f. Then*

(1) $\lim\limits_{z \to c} \frac{1}{f(z)} = 0$.
(2) There exists a positive integer N and a neighbourhood V of c in U such that

$$f(z) = \sum_{k=-N}^{\infty} a_k (z - c)^k$$

on $V \smallsetminus \{c\}$.
(3) For every positive real number M, there exists $\delta > 0$ such that $|f(z)| > M$ for every $\zeta \in B_{c,\delta}$.

Proof. (1): By Lemma 15.1, there exists a neighbourhood V of c and a holomorphic function g on V such that $g(z) = \frac{1}{f(z)}$ on $V \smallsetminus \{c\}$. If $g(c) \neq 0$, then $\lim_{z\to c} f(z) = \frac{1}{g(c)}$, which would imply that f has a removable singularity at c. Hence $\lim_{z\to c} g(z) = g(c) = 0$.

(2): Since c is a removable singularity of $\frac{1}{f}$, we can write $\frac{1}{f(z)} = (z-c)^N f_1(z)$ for some $N \in \mathbb{N}$ and a holomorphic function $f_1(z)$ in a neighbourhood of c with $f_1(c) \neq 0$. (See Exercise 4 in Lecture 15.) Since $\lim_{z\to c} \frac{1}{f(z)} = 0$, $N > 0$. Note that $\frac{1}{f_1(z)}$ is holomorphic in a neighbourhood of c, so it admits a convergent power series expansion around c.

(3): Exercise. □

The next two propositions characterise zeros and poles of holomorphic functions by looking at the limit of $|z-c|^n|f(z)|$ for various n. Their proofs are left as exercises.

17.5. **Proposition.** *Let U be a domain, $c \in U$ and f a non-zero holomorphic function on $U \smallsetminus \{c\}$. Then the following are equivalent:*

(1) *f can be extended to a holomorphic function $\tilde{f}$ on U with $\tilde{f}(c) = 0$;*
(2) *$\lim_{z\to c} f(z) = 0$;*
(3) *there exist $m, n \in \mathbb{Z}$ with $m < 0$ and $n < 0$ such that $\lim_{z\to c} |z-c|^m|f(z)| = 0$ and $\lim_{z\to c} |z-c|^n|f(z)| = \infty$*
(4) *there exists $N \in \mathbb{Z}$ with $N < 0$ such that $\lim_{z\to c} |z-c|^m|f(z)| = 0$ for every $m > N$ and $\lim_{z\to c} |z-c|^n|f(z)| = \infty$ for every $n < N$.*

17.6. **Proposition.** *Let U be a domain, $c \in U$ and f a holomorphic function on $U \smallsetminus \{c\}$. Then the following are equivalent:*

(1) *c is pole of f;*
(2) *there exist $m, n \in \mathbb{N}$ such that $\lim_{z\to c} |z-c|^m|f(z)| = 0$ and $\lim_{z\to c} |z-c|^n|f(z)| = \infty$*
(3) *there exists $N \in \mathbb{Z}$ with $N > 0$ such that $\lim_{z\to c} |z-c|^m|f(z)| = 0$ for every $m > N$ and $\lim_{z\to c} |z-c|^n|f(z)| = \infty$ for every $n < N$.*

17.7. **Definition.** Let U be a domain, $c \in U$ and f a holomorphic function on $U \smallsetminus \{c\}$. Say that c is an *essential singularity* of f if it is not a removable singularity or a pole of f.

17.8. **Proposition.** *c is an essential singularity of f if and only if $\lim_{z\to c} |z-c|^n|f(z)|$ does not exist for any $n \in \mathbb{Z}$.*

Proof. 'If': by definition. 'Only if': By way of contradiction, assume that $\lim_{z\to c} |z-c|^N|f(z)|$ exists. Then c is a removable singularity of $(z-c)^N f(z)$. If $N \le 0$, then c is a removable singularity of f. If $N > 0$, then c is a pole of f. □

17.9. **Proposition.** *Let U be a domain, $c \in U$ and f a holomorphic function on $U \smallsetminus \{c\}$. Suppose that c is an essential singularity of f. Then for every $A \in \mathbb{C}$, every $\epsilon > 0$ and every $\delta > 0$, there exists $\zeta \in B_{c,\epsilon}$ such that $|f(\zeta) - A| < \delta$.*

Proof. By way of contradiction, let $A \in \mathbb{C}$, $\epsilon > 0$, $\delta > 0$ be such that for every $\zeta \in B_{c,\epsilon} \smallsetminus \{\zeta\}$, $|f(\zeta) - A| \ge \delta$. Then for every $n < 0$, $\lim_{z\to c} |z-c|^n|f(z) - A| = \infty$, so c is not an essential singularity of $f(z) - A$. Then there exists $m > 0$ such that $\lim_{z\to c} |z-c|^m|f(z) - A| = 0$. Note that

$$\lim_{z\to c} |z-c|^m|f(z)| \le \lim_{z\to c} |z-c|^m|f(z) - A| + \lim_{z\to c} |z-c|^m|A| = 0$$

so c is not an essential singularity of f. □

17.10. **Definition.** Let U be a domain. By a *meromorphic function* on U, we mean a a holomorphic function $f : U' \longrightarrow \mathbb{C}$ where $U' \subseteq U$, and points in $U \smallsetminus U'$ are isolated in U and are poles of f.

17.11. **Example.** If f is a holomorphic function on a domain U then its zeros are isolated, by Corollary 15.5 and Proposition 4.6; hence $\frac{1}{f}$ is meromorphic on U. E.g., $\frac{1}{z}$ is a meromorphic function on $\mathbb{C}$. Every rational function is meromorphic on every domain in $\mathbb{C}$.

Exercises.

(1) Let c be a pole of f. For every positive real number M, there exists $\delta > 0$ such that $|f(z)| > M$ for every $\zeta \in B_{c,\delta}$.

(2) Prove Proposition 17.5.

(3) Prove Proposition 17.6.

(4) Let U be a domain, $c \in U$ and f holomorphic on $U \smallsetminus \{c\}$. Suppose that c is a pole of f. Write

$$f(z) = \sum_{k=-N}^{\infty} a_k (z-c)^k$$

in a punctured neighbourhood $V \smallsetminus \{c\}$ of c, with $N > 0$ and $a_{-N} \neq 0$. Let $r > 0$ be such that $\overline{B_{c,r}} \subseteq V$. Let $\gamma : [0,1] \longrightarrow V$ be the path $t \mapsto c + re^{2\pi\imath t}$. Then

$$\int_\gamma f \mathrm{d}z = 2\pi\imath a_{-1}.$$

We say that a_{-1} is the *residue* of f at c, and denote it by $\operatorname{Res}_f(c)$.

(5) Let U be a disc, f a meromorphic function on U and γ a piecewise-differentiable closed path in U. Let $\{\zeta_j\}$ be the poles of f. Assume that γ does not pass through ζ_j for any j. Show that $n(\zeta_j, \gamma) = 0$ except for finitely many j and that

$$\frac{1}{2\pi\imath}\int_\gamma f \mathrm{d}z = \sum_j n(\zeta_j,\gamma)\operatorname{Res}_f(\zeta_j).$$

(6) Let U be a domain and $\mathcal{M}(U)$ be the set of meromorphic functions on U. For $f, g \in \mathcal{M}(U)$ and $c \in U$, let

$$(f+g)(c) = \lim_{z\longrightarrow c}(f(z)+g(z))$$
$$(fg)(c) = \lim_{z\longrightarrow c}(f(z)g(z))$$

Show that c is a pole of $f+g$ if and only if c is a pole of f or of g. State and prove a similar characterization of poles of fg.

(7) (Not relevant for this course.) Let U be a domain. Then $\mathcal{M}(U)$ is the field of fractions of $\mathcal{A}(U)$.

(8) Let N be an integer and suppose that c is a removable singularity of $(z-c)^N f(z)$. If $N \leq 0$, then c is a removable singularity of $f(z)$. If $N > 0$, then c is a pole of $f(z)$ of order $\leq N$.

(9) $e^{\frac{1}{z}}$ has an essential singularity at 0.

(10) Let $f(z), g(z)$ be a holomorphic functions defined in a neighbourhood of $c \in \mathbb{C}$. Suppose that $f(z)$ has an essential singularity at c. Show that $f(z)g(z)$ has an essential singularity at c.

Lecture 18. Local mapping

18.1. **Example.** Consider the holomorphic function $f(z) = z^m$, $m > 0$, on $\mathbb{C}$. It has a zero at $z = 0$, of order m. Note that for every $b \in \mathbb{C}$, there exist m solutions (counted with multiplicity) for the equation $f(z) = b$. □

We now show that every holomorphic function exhibits the same behaviour locally. Here is the result:

18.2. **Proposition.** *Let U be a domain and f a non-constant holomorphic function on U. Let $\zeta \in U$. Write $a = f(\zeta)$. Let m be the order of the zero of $f(z) - a$ at $z = \zeta$. Then for every $0 < \epsilon \ll 1$, there exists $\delta > 0$ such that for every $b \in B_{a,\delta}$, there exists m solutions in $B_{\zeta,\epsilon}$ to the equation $f(z) = b$.*

As an immediate corollary, we get the following:

18.3. **Corollary.** *Let U be a domain and f a non-constant holomorphic function on U. Then $f(U)$ is an open subset of $\mathbb{C}$. In other words, every non-constant holomorphic function is an open map.*

Proof. With notation as in Proposition 18.2, $B_{f(\zeta),\delta} \subseteq f(B_{\zeta,\epsilon})$ for every $\zeta \in U$ and every $0 < \epsilon \ll 1$. Since the open discs $B_{\zeta,\epsilon}$ form a basis for the topology of U, f is open. □

Before proving Proposition 18.2, we need to develop a method to count zeros. In the above example with $f = z^m$, we note that if γ is a closed piecewise-differentiable curve in $\mathbb{C}$ not passing through 0, then

$$\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z = m\int_\gamma \frac{\mathrm{d}z}{z} = 2\pi\imath \cdot n(0,\gamma)\cdot m.$$

In particular, if γ is a circular path such that 0 is in the bounded region, then $m = \frac{1}{2\pi\imath}\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z$.

18.4. **Proposition.** *Let U be a disc and f holomorphic on U. Let $\{\zeta_j\}$ be the distinct zeros of f; denote the order of ζ_j by m_j. Let γ be a closed piecewise differentiable path in U, not passing through any of the ζ_j. Then*

(1) $n(\zeta_j,\gamma) = 0$ for all but finitely many j.

(2)
$$\frac{1}{2\pi\imath}\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z = \sum_j n(\zeta_j,\gamma)\cdot m_j.$$

Proof. We will use Exercise 5 from Lecture 15 in this proof. (1): Since $\operatorname{Im}\gamma$ is compact, there exists an open subset V of U such that its closure $\overline{V}$ in $\mathbb{C}$ contains $\operatorname{Im}\gamma$ and is a subset of U. Note that f has only finitely many zeros in $\overline{V}$. For any $\zeta \in \mathbb{C} \smallsetminus V$, $n(\zeta,\gamma) = 0$, since the function $\frac{1}{z-\zeta}$ is holomorphic on V.

(2): By (1), we may assume that the (distinct) zeros of f are $\zeta_1,\dots,\zeta_r$, with orders $m_1,\dots m_r$. Write $f(z) = \prod_{j=1}^r (z-\zeta_j)^{m_j} g(z)$ where g is holomorphic on an open set V containing $\operatorname{Im}\gamma$ and does not have any zeros. Then

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^r \frac{m_j}{z-\zeta_j} + \frac{g'(z)}{g(z)}.$$

The assertion now follows from noting that $\frac{g'(z)}{g(z)}$ is holomorphic on V. □

Proof of Proposition 18.2. Let $0 < \epsilon \ll 1$. Then $B_{\zeta,\epsilon} \subseteq U$. Let $\gamma : [0,1] \longrightarrow U, t \mapsto \zeta + \epsilon e^{2\pi\imath t}$. Let $\Gamma = f \circ \gamma$. Moreover, since $\epsilon \ll 1$, we may assume that ζ is the only solution to $f(z) = a$ in $\overline{B_{\zeta,\epsilon}}$; in particular, $a \notin \operatorname{Im}\Gamma$. Let $\delta > 0$ be such that $B_{a,\delta} \cap \operatorname{Im}\Gamma = \varnothing$.

Let $b \in B_{a,\delta}$. Let $\{\zeta_j\}$ be the distinct zeros of $f(z) - b$ in $B_{\zeta,\epsilon}$, with m_j the order of ζ_j. Then

$$\sum_j n(\zeta_j, \gamma) \cdot m_j = \frac{1}{2\pi\imath} \int_\gamma \frac{f'(z)}{f(z) - b} dz = \frac{1}{2\pi\imath} \int_\Gamma \frac{dw}{w - b} = n(b, \Gamma);$$

$$m = \frac{1}{2\pi\imath} \int_\gamma \frac{f'(z)}{f(z) - a} dz = \frac{1}{2\pi\imath} \int_\Gamma \frac{dw}{w - a} = n(a, \Gamma).$$

In both rows, the first equality is by Proposition 18.4 and the second by the substitution $w = f(z)$. Observe that $n(a, \Gamma) = n(b, \Gamma)$ since a and b belong to the same connected component of $\mathbb{C} \setminus \operatorname{Im}(\Gamma)$. Now note that $n(\zeta_j, \gamma) = 1$ for each ζ_j. Hence $\sum_j m_j = m$. □

Exercises.

(1)

Lecture 19. Maximum principle, definite integrals etc.

In this short lecture, we tie various loose ends.

19.1. **Proposition.** *Let U be a domain and f a non-constant holomorphic function on U.*

(1) *There does not exist $\zeta \in U$ such that $|f(\zeta)| = \sup\{|f(z)| : z \in U\}$*
(2) *Assume that U is bounded and that f can be extended to a continuous function $\tilde{f}$ on $\overline{U}$. There exists $\zeta \in \partial U$ such that $|\tilde{f}(\zeta)| = \sup\{|f(z)| : z \in U\}$.*

Proof. (1): Let $\zeta \in U$. For every $0 < \epsilon \ll 1$, there exists δ such that $B_{f(\zeta),\delta} \subseteq f(B_{\zeta,\epsilon})$, by Proposition 18.2. Now note that there exists $b \in B_{f(\zeta),\delta}$ such that $|b| > |f(\zeta)|$.

(2): Since $\overline{U}$ is compact, there exists $\zeta \in \overline{U}$ such that $|\tilde{f}(\zeta)| = \sup\{|\tilde{f}(z)| : z \in \overline{U}\} = \sup\{|f(z)| : z \in U\}$. □

Here is a generalization of Proposition 18.4 to meromorphic functions.

19.2. **Proposition.** *Let U be a disc and f meromorphic on U. Let a_i be the distinct zeros of f, with orders l_i, respectively; let b_j be the distinct poles of f, with orders m_j, respectively. Let γ be a closed piecewise differentiable path in U, not passing through any of the a_i and any of the b_j. Then*

$$\frac{1}{2\pi\imath} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_i l_i \cdot n(a_i, \gamma) - \sum_j m_j \cdot n(b_j, \gamma)$$

Proof. As in the proof of Proposition 18.4, we may assume that the number of zeroes and the number of poles are finite. Hence we may write

$$f(z) = \prod_{\substack{i \\ \text{finite}}} (z - a_i)^{l_i} \prod_{\substack{j \\ \text{finite}}} (z - b_b)^{-m_j} g(z)$$

in some open subset V containing $\operatorname{Im} \gamma$, where $g(z)$ is holomorphic on V and does not have any zeros. Hence

$$\frac{f'(z)}{f(z)} = \sum_{\substack{i \\ \text{finite}}} \frac{l_i}{z - a_i} - \sum_{\substack{j \\ \text{finite}}} \frac{m_i}{z - b_j} + \frac{g'(z)}{g(z)},$$

from which the assertion follows. □

We now look at an example of evaluating definite real integrals using complex integration.

19.3. **Example.** Integrate

$$\int_0^{\pi} \frac{\mathrm{d}\theta}{a+\cos\theta}$$

where $a > 1$ is a real number. Write b for its value. Note that

$$2b = \int_0^{2\pi} \frac{\mathrm{d}\theta}{a+\cos\theta}$$

Write $z = e^{\imath\theta}$. Then $\mathrm{d}\theta = -\imath\frac{\mathrm{d}z}{z}$ and $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$. Let $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}, \theta \mapsto e^{2\pi\imath\theta}$. Then

$$2b = -\imath\int_\gamma \frac{\mathrm{d}z}{z^2+2az+1}$$

The meromorphic function $\frac{1}{z^2+2az+1}$ has two poles $\alpha = -a+\sqrt{a^2-1}$ and $\beta = -a-\sqrt{a^2-1}$. Since $|\alpha| < 1$, $n(\alpha,\gamma) = 1$; Since $|\beta| > 1$, $n(\alpha,\gamma) = 0$; Note that

$$\frac{1}{z^2+2az+1} = \frac{1}{\alpha-\beta}\left(\frac{1}{z-\alpha} - \frac{1}{z-\beta}\right).$$

Hence

$$2b = \frac{(-\imath)(2\pi\imath)}{\alpha-\beta}, \text{ i.e., } b = \frac{\pi}{\sqrt{a^2-1}}.$$

□

Exercises.

(1)

Lecture 20. Conformality

20.1. **Definition.** Let $n \geq 2$ be an integer, $U \subseteq \mathbb{R}^n$ an open subset and $p \in U$. A function $f : U \longrightarrow \mathbb{R}^n$ is said to be *conformal* at p if it is differentiable at p and it preserves angles and orientation at p. We say that f is conformal on U if it is conformal at p for every $p \in U$.

What does this mean? Let $e_1, \ldots, e_n$ denote the standard basis for $\mathbb{R}^n$, and let $x_1, \ldots, x_n$ be the coordinates of $\mathbb{R}^n$ with respect to this basis. Write $f = (f_1, \ldots, f_n)$, with respect to this basis. Let $1 \leq i \leq n$. Consider the curve $\gamma : (-\epsilon, \epsilon) \longrightarrow U, t \mapsto p + te_i$. Since f is differentiable at p, the composite curve $f\gamma$ is differentiable at 0, with derivative

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_i}(p) \\ \frac{\partial f_2}{\partial x_i}(p) \\ \vdots \\ \frac{\partial f_n}{\partial x_i}(p) \end{bmatrix}$$

This is the ith column of the jacobian matrix J of f at p.

The jacobian of f at p gives the map $\mathrm{d}f : \Omega(U)_p \longrightarrow \Omega(\mathbb{R}^n)_{f(p)}$, when these are identified with $\mathbb{R}^n$. Here, $\Omega(-)$ is the cotangent bundle, and $\Omega(-)_q$ the cotangent space at q.

Saying that f preserves angles at p is same as saying that J preserves angles, i.e.,

$$\frac{v \cdot w}{|v||w|} = \frac{Jv \cdot Jw}{|Jv||Jw|}$$

for every non-zero $v, w \in \mathbb{R}^n$. We now have the following:

20.2. **Proposition.** *Let J be an $n \times n$ real matrix. Then J preserves angles if and only if there exist $\lambda > 0$ and an orthogonal matrix A such that $J = \lambda A$.*

Proof. 'Only if': Note that Je_i and Je_j are orthogonal to each other if $i \neq j$, so $Je_i \cdot Je_j = 0$ for $i \neq j$. Write $\lambda_i = |Je_i|, 1 \leq i \leq n$. Note that $\lambda_i > 0$ for each i. Let A be the matrix whose ith column is $\frac{Je_i}{\lambda_i}$. Since the columns of J are orthogonal to each other, it follows that A is an orthogonal matrix. Therefore $|Av| = |A^t v| = |v|$ for every $v \in \mathbb{R}^n$.

Now consider the linear transformation $A^t J$. For every $v, w \in \mathbb{R}^n$,

$$\frac{A^t Jv \cdot A^t Jw}{|A^t Jv||A^t Jw|} = \frac{Jv \cdot AA^t Jw}{|Jv||Jw|} = \frac{Jv \cdot Jw}{|Jv||Jw|} = \frac{v \cdot w}{|v||w|},$$

i.e., $A^t J$ preserves angles. Note that $A^t Je_i = \lambda_i A^t Ae_i = \lambda_i e_i$ for each i, i.e., $A^t J$ is a diagonal matrix (with respect to the basis e_i). Hence it must be a multiple of I_n (Exercise), i.e, $\lambda_i = \lambda_j$ for all i, j. Set $\lambda = \lambda_i$.

'If':

$$\frac{Jv \cdot Jw}{|Jv||Jw|} = \frac{\lambda Av \cdot \lambda Aw}{|\lambda Av||\lambda Aw|} = \frac{Av \cdot Aw}{|Av||Aw|} = \frac{v \cdot A^t Aw}{|v||w|} = \frac{v \cdot w}{|v||w|}.$$ □

An *orientation* on U is a choice of a basis (i.e. a non-zero vector) in $\wedge^n \Omega(U)$. Since we have already looked at the jacobian matrix with respect to $x_1, \ldots, x_n$, let us take $dx_1 \wedge \cdots \wedge dx_n$. Then the induced map $\wedge^n \Omega(U)_p \longrightarrow \wedge^n \Omega(\mathbb{R}^n)_{f(p)}$ is multiplication by $\det J$. To preserve orientation is to say that $\det J > 0$.

We summarise this discussion as follows.

20.3. **Proposition.** *Let $n \geq 2$ be an integer, $U \subseteq \mathbb{R}^n$ an open subset and $f : U \longrightarrow \mathbb{R}^n$ a differentiable function. Then f is conformal on U if and only if the jacobian matrix of f at p is a multiple of an orthogonal matrix and its determinant is positive, for every $p \in U$.*

We now restrict out attention to dimension 2. Let x, y be coordinates of $\mathbb{R}^2$. Write $f = (u, v)$. Then

$$J = \begin{bmatrix} u_x(p) & u_y(p) \\ v_x(p) & v_y(p) \end{bmatrix}$$

Since the columns are orthogonal to each other, there exists $\lambda \neq 0$ such that $u_y(p) = -\lambda v_x(p)$ and $v_y(p) = \lambda u_x(p)$. Then $\det J = \lambda(u_x(p)^2 + v_x(p)^2)$, so $\lambda > 0$. Thus

$$J = \begin{bmatrix} u_x(p) & -\lambda v_x(p) \\ v_x(p) & \lambda u_x(p) \end{bmatrix}$$

For the rows of J to be orthogonal, $\lambda^2 = 1$, so $\lambda = 1$. Hence

$$J = \begin{bmatrix} u_x(p) & -v_x(p) \\ v_x(p) & u_x(p) \end{bmatrix}$$

Summarising this, we get the following relation between conformality and holomorphicity.

20.4. **Proposition.** *Let $U \subseteq \mathbb{C}$ be open and $f : U \longrightarrow \mathbb{C}$. Then the following are equivalent:*

(1) *f is holomorphic and $f'(z)$ has no zeroes on U;*
(2) *f is conformal on U.*

20.5. **Remark.** Some books might require conformal maps to be injective, by definition.

20.6. **Remark.** A conformal map preserves angles and orientation only, but not length. To see this, let $U \subseteq \mathbb{C}$ be a domain and f holomorphic on U. Suppose that $f'(p) \neq 0$. Let $\gamma : (-\epsilon, \epsilon) :$ U be a C^1-path with $\gamma(0) = p$. Write $\Gamma = f\gamma$. Then $|\Gamma'(0)| = |f'(p)||\gamma'(0)|$. Hence the length of an infinitesimal arc through p gets multiplied by $|f'(p)|$.

Exercises.

(1) Show that the map $z \mapsto \bar{z}$ preserves angles, but not orientation.
(2) Orientation in the case of $\mathbb{R}^2$. Let v_1, v_2 be a basis of $\mathbb{R}^2$. Plot them as vectors based at 0. We can think of orientation as the direction (clockwise, or counter-clockwise) in which we have to go from v_1 to v_2 traversing the smaller of the angles between them. (One of these angles must be in $(0, \pi)$; this is the smaller angle.) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear transformation. Show that f preserves orientation if and only if the direction in which one has to traverse smaller angle from $f(v_1)$ to $f(v_2)$ is the same as the direction in which one has to traverse the smaller angle from v_1 to v_2.
(3) Show that if $f : U \longrightarrow \mathbb{C}$ is conformal, then for every $p \in U$, it maps a neighbourhood of p homeomorphically onto its image.
(4) Show that if $U \subseteq \mathbb{C}$ is a domain and f is an injective holomorphic function on U, then f is conformal.

Lecture 21. Riemann sphere

We want to discuss Moebius transformations next.

Consider the unit sphere S^2 in $\mathbb{R}^3$, with the map

$$\sigma : S^2 \smallsetminus \{(0,0,1)\} \longrightarrow \mathbb{C}, (x_1, x_2, x_3) \mapsto \frac{x_1 + \imath x_2}{1 - x_3}.$$

What is this map? Identify the hyperplane $x_3 = 0$ with $\mathbb{C}$, with x_1 as the real part and x_2 the imaginary part. Then $\sigma((x_1, x_2, x_3))$ is the point where the line through $(0,0,1)$ and (x_1, x_2, x_3) meets $\mathbb{C}$. I.e., we need to solve for λ in

$$(1-\lambda)(0,0,1) + \lambda(x_1, x_2, x_3) = (y_1, y_2, 0).$$

Hence $\lambda = \frac{1}{1-x_3}$. This gives the above description of σ. σ is a homeomorphism, with S^2 given the subspace topology of $\mathbb{R}^3$. Points of $S^2 \smallsetminus (0,0,1)$ with $x_3 > 0$ are mapped to $\mathbb{C} \smallsetminus \overline{B_{0,1}}$, the points with $x_3 < 0$ are mapped to $B_{0,1}$. With this, S^2 is a one-point compactification of $\mathbb{R}^2 = \mathbb{C}$ (Exercise). The map σ is called *stereographic projection*.

By the *Riemann sphere*, we mean S^2, with a *complex manifold* structure given on it. Cover S^2 with two open subsets, $U := S^2 \smallsetminus \{(0,0,1)\}$ and $V := S^2 \smallsetminus \{(0,0,-1)\}$. We identify U with $\mathbb{C}$, using σ. Note that $\sigma((0,0,-1)) = 0$. We can now define $\tau : V \longrightarrow \mathbb{C}$ by

$$\tau(p) = \begin{cases} 0, & \text{if } p = (0,0,1) \\ \frac{1}{\sigma(p)}, & \text{otherwise.} \end{cases}$$

This identifies V with $\mathbb{C}$, and on $\mathbb{C} \smallsetminus \{0\}$, the map $\tau\sigma^{-1}$ is $z \mapsto \frac{1}{z}$, which is a *biholomorphic* map, i.e., a bijective holomorphic map whose inverse is holomorphic. We will write $\widehat{\mathbb{C}}$ for the Riemann sphere.

Using the Riemann sphere, we can reinterpret the notion of poles. Identify $\mathbb{C}$ with U using σ, and write ∞ for the point $(0,0,1)$. This is sometimes called the *point at infinity* (w.r.t this identification). Let $p \in \mathbb{C}$ and f a holomorphic function defined in a neighbourhood W of p, with a pole at p. The function $f : W \smallsetminus \{p\} \longrightarrow \mathbb{C} = U$ extends to a function $\tilde{f} : W \longrightarrow S^2$, with $\tilde{f}(p) = \infty$. Shrink W so that f does not have a zero in W. Hence $\operatorname{Im} \tilde{f} \subseteq V$. The composite $\tau\tilde{f}$ is the holomorphic map $z \mapsto \frac{1}{f(z)}$ on W.

Exercises.

(1) Show that the map σ in the definition of the Riemann sphere is a homeomorphism and that the Riemann sphere is a one-point compactification of $\mathbb{R}^2 = \mathbb{C}$.
(2) Let $c \in (-1, 1)$. Show that σ maps $S^2 \cap \{x_3 = c\}$ to a circle in $\mathbb{C}$ and $S^2 \cap \{x_3 < c\}$ to the open set bounded by the circle.

Lecture 22. Moebius transformations

References for this lecture are Ahlfors Chapter 2, Section 1.4, and Chapter 3, Section 3. See also Rodríguez, Kra and Gilman, Chapter 8, especially the early parts.

NOTE: We identify $\mathbb{C}$ with $\widehat{\mathbb{C}} \smallsetminus \{\infty\}$ through the stereographic projection σ. When you read this lecture and the next, you should keep this in mind. Sometimes, we will switch between $\mathbb{C}$ and its image under σ without explicitly mentioning it.

By a *Moebius transformation*, we mean a meromorphic function on $\mathbb{C}$ given by a rational function of the form

$$f(z) = \frac{az+b}{cz+d}$$

where a, b, c, d are complex numbers with $ad \neq bc$.

22.1. **Remark.** We make the following observations (notation as above):

(1) $f(z)$ is holomorphic on $\mathbb{C}$ if and only if $c = 0$; otherwise f has exactly one pole, at $-\frac{d}{c}$.
(2) $f(z)$ is injective on the complement of the pole. (Exercise)
(3) We can think of f as being given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix}$$

We can extend f to bijective function from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$, still denoted by f, by setting

$$\begin{cases} f(\infty) = \infty, & \text{if } c = 0; \\ f(-\frac{d}{c}) = \infty \text{ and } f(\infty) = \frac{a}{c}, & \text{otherwise.} \end{cases}$$

22.2. **Proposition.** *The extended function $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ is a homeomorphism.*

Proof. The extended function is bijective (check). I will show that it is continuous and open, treating the two cases $c = 0$ and $c \neq 0$ separately for your convenience.

$c = 0$: Then $d \neq 0$. Replacing a by $\frac{a}{d}$ and b by $\frac{b}{d}$, we may assume that f is given by the polynomial $az + b$. The inverse function is $z \mapsto \frac{z-b}{a}$. It is continuous and open. Hence f is a homeomorphism of $\mathbb{C}$ to itself. Now consider the extension of f to $\widehat{\mathbb{C}}$. Let U be an open subset of $\widehat{\mathbb{C}}$. We want to show that $f(U)$ and $f^{-1}(U)$ are open. If $U \subset \mathbb{C}$, then $f(U)$ and $f^{-1}(U)$ are open. Otherwise, i.e. if $\infty \in U$, then $\widehat{\mathbb{C}} \smallsetminus U$ is compact and, hence, closed, so

$$f(U) = \widehat{\mathbb{C}} \smallsetminus f\left(\widehat{\mathbb{C}} \smallsetminus U\right)$$
$$f^{-1}(U) = \widehat{\mathbb{C}} \smallsetminus f^{-1}\left(\widehat{\mathbb{C}} \smallsetminus U\right)$$

are open. (Note: f is bijective.)

$c \neq 0$: Let us compute f^{-1}: Write $w = \frac{az+b}{cz+d}$. Rewrite as $(cz + d)w = (az + b)$, so we get $(cw - a)z = -dw + b$. Define a meromorphic function $g : \mathbb{C} \longrightarrow \mathbb{C}$ by

$$g(z) = \frac{-dz + b}{cz - a}.$$

Note that
$$\begin{bmatrix} -d & b \\ c & -a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = -(ad-bc)I_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -d & b \\ c & -a \end{bmatrix}$$
Hence on $\mathbb{C} \smallsetminus \{-\frac{d}{c}, \frac{a}{c}\}$, $(fg)(z) = (gf)(z)$, i.e, g is the inverse of f. It is easy to check that its extension to $\widehat{\mathbb{C}}$ is the inverse of f (on $\widehat{\mathbb{C}}$). Since $\widehat{\mathbb{C}}$ is a metric space, we can check that f is continuous by taking limits. Let $\zeta \in \widehat{\mathbb{C}}$. Let ζ_i be a sequence converging to ζ. If $\zeta = \infty$, then $|\sigma(\zeta_i)| \to \infty$ (see the description of σ in the previous lecture). Hence $f(\zeta_i) \to \frac{a}{c}$. Similarly, if $\zeta = -\frac{d}{c}$, then $|f(\zeta_i)| \to \infty$. Hence f is continuous. This applies to every Moebius transformation, including to $g = f^{-1}$. Hence f is a homeomorphism. □

We will come back to Moebius transformations in the next lecture. For now, we look at rational functions, in general.

Let $p, q \in \mathbb{C}[z]$ be relatively prime non-zero polynomials. Let $m = \deg p$ and $n = \deg q$. Then we get a meromorphic function
$$f(z) = \frac{p(z)}{q(z)}$$
on $\mathbb{C}$. We can extend f to $\widehat{\mathbb{C}}$ as follows. Let $\zeta \in \widehat{\mathbb{C}}$. If $\zeta \in \mathbb{C}$ and $q(\zeta) \neq 0$, $f(\zeta) = \frac{p(\zeta)}{q(\zeta)}$ (nothing new here). For $\zeta \in \mathbb{C}$ is a zero of q, then for every sequence $\zeta_i \longrightarrow \zeta$, $|\frac{p(\zeta_i)}{q(\zeta_i)}| \longrightarrow \infty$, so we can define $f(\zeta) = \infty$.

Now assume $\zeta = \infty$. Write $p(z) = a_m z^m + \cdots + a_0$ and $q(z) = b_n z^n + \cdots + b_0$. If $m > n$, then $|\frac{p(\zeta_i)}{q(\zeta_i)}| \longrightarrow \infty$, so we can define $f(\infty) = \infty$. If $m = n$, then we can define $f(\infty) = \frac{a_m}{b_m}$, since $\frac{p(\zeta_i)}{q(\zeta_i)} \longrightarrow \frac{a_m}{b_m}$. If $m < n$, then $\frac{p(\zeta_i)}{q(\zeta_i)} \longrightarrow 0$, so define $f(\infty) = 0$. Consider the neighbourhood $V \subseteq \widehat{\mathbb{C}}$ of ∞, from the last lecture. We identify V with $\mathbb{C}$ using τ. Since $\zeta_i \to \infty$, $\tau(\zeta_i) \longrightarrow 0$. On a neighbourhood $W \subseteq V$ of ∞, f is holomorphic, and has the form
$$\frac{a_m(\frac{1}{z})^m + a_{m-1}(\frac{1}{z})^{m-1} + \cdots + a_0}{b_n(\frac{1}{z})^n + b_{n-1}(\frac{1}{z})^{n-1} + \cdots + b_0} = z^{n-m}\frac{a_m + a_{m-1}z^1 + \cdots + a_0 z^m}{b_n + b_{n-1}z^1 + \cdots + b_0 z^n}.$$
Hence the order of the zero of f at ∞ is $n - m$.

Exercises.

(1) Let a, b, c, d be complex numbers such that $ad - bc \neq 0$. Let $U = \mathbb{C}$ if $c = 0$; let $U = \mathbb{C} \smallsetminus \{-\frac{d}{c}\}$, otherwise. Show that the function
$$f : U \longrightarrow \mathbb{C},\ z \mapsto \frac{az+b}{cz+d}$$
is injective. Hint: Look at the linear map $\mathbb{C}^2 \longrightarrow \mathbb{C}^2$ given by the matrix
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$
(2) Complete the proof (by filling in the missing steps) of Proposition 22.2.

(3) Read Ahlfors, Chapter 2, Section 1.4 about partial fraction expansions. Do Exercise 1 of that section.

(4) Let $p, q \in \mathbb{C}[z]$ be relatively prime non-zero polynomials. Assume without loss of generality that $\deg q > 0$. Let f be the meromorphic function
$$\frac{p(z)}{q(z)}$$

on $\mathbb{C}$. Show that f' is meromorphic and the poles of f' are exactly the poles of f. If ζ is a pole of f of order m, then the order of the pole of f' at ζ is $m+1$.

Lecture 23. Moebius transformations, continued.

References for this lecture are Ahlfors, Chapter 3, Section 3.2 and Rodríguez, Kra and Gilman, Section 8.1.

Let $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. For every $\zeta \in \mathbb{C}$, $\zeta \neq 0$,

$$\frac{az+b}{cz+d} = \frac{\zeta az + \zeta b}{\zeta cz + \zeta d}$$

as meromorphic functions on $\mathbb{C}$. Hence we may assume that $ad - bc = 1$. In other words, every Moebius transformation can be represented by an element of $\mathrm{SL}_2(\mathbb{C})$. Hereafter, we will make this assumption.

Recall that a Moebius transformation is typically only a meromorphic function on $\mathbb{C}$, but a well-defined function on $\widehat{\mathbb{C}}$.

23.1. **Lemma.** *Assume that the Moebius transformation*

$$f(z) = \frac{az+b}{cz+d} \quad (\textit{with } ad - bc = 1)$$

(considered as a function on $\widehat{\mathbb{C}}$) fixes $0, 1$ and ∞, then it is the identity map: $a = d = 1, b = c = 0$.

Proof. Since $f(\infty) = \infty$, and f is bijective, f is holomorphic on $\mathbb{C}$, and the only zero of f is 0. Hence $c = 0$. Without loss of generality, we may assume that $d = 1$, i.e, f is given by a linear polynomial $az + b$. Since we have assumed that $ad - bc = 1$, $a = 1$. Since 0 is the only zero of f, f is the identity map. ☐

23.2. **Proposition.** *Let $\zeta_0, \zeta_1, \zeta_\infty$ be distinct points on $\widehat{\mathbb{C}}$. Then there is a unique Moebius transformation*

$$z \mapsto \frac{az+b}{cz+d}$$

such that $\zeta_0 \mapsto 0$, $\zeta_1 \mapsto 1$ and $\zeta_\infty \mapsto \infty$.

Proof. Let

$$f(z) = \frac{z - \zeta_0}{z - \zeta_\infty} \frac{\zeta_1 - \zeta_\infty}{\zeta_1 - \zeta_0}.$$

It is a Moebius transformation, with $f(\zeta_0) = 0$, $f(\zeta_1) = 1$ and $f(\zeta_\infty) = \infty$. Let $g(z)$ be any Moebius transformation such that $g(\zeta_0) = 0, g(\zeta_1) = 1$ and $g(\zeta_\infty) = \infty$. Then gf^{-1} and fg^{-1} are Moebius transformations, fixing $0, 1$ and ∞. Hence $gf^{-1} = fg^{-1} = \mathrm{id}$. Therefore $g = (f^{-1})^{-1} = f$. ☐

23.3. **Definition.** Let $\zeta, \zeta_0, \zeta_1, \zeta_\infty$ be distinct points on $\widehat{\mathbb{C}}$. Their *cross-ratio* is the image of ζ under the unique Moebius transformation that sends ζ_0 to 0, ζ_1 to 1 and ζ_∞ to ∞. We will denote the cross-ratio by $(\zeta, \zeta_0, \zeta_1, \zeta_\infty)$.

In other words

$$(\zeta, \zeta_0, \zeta_1, \zeta_\infty) = \frac{\zeta - \zeta_0}{\zeta - \zeta_\infty} \frac{\zeta_1 - \zeta_\infty}{\zeta_1 - \zeta_0}.$$

23.4. **Proposition.** *Let $\zeta, \zeta_0, \zeta_1, \zeta_\infty$ be distinct points on $\widehat{\mathbb{C}}$ and f a Moebius transformation. Then*

$$(f(\zeta), f(\zeta_0), f(\zeta_1), f(\zeta_\infty)) = (\zeta, \zeta_0, \zeta_1, \zeta_\infty).$$

Proof. Let g be the unique Moebius transformation that sends ζ_0 to 0, ζ_1 to 1 and ζ_∞ to ∞. Then

$$(\zeta, \zeta_0, \zeta_1, \zeta_\infty) = g(\zeta).$$

Then the Moebius transformation gf^{-1} sends $f(\zeta_0)$ to 0, $f(\zeta_1)$ to 1 and $f(\zeta_\infty)$ to ∞.

$$(f(\zeta), f(\zeta_0), f(\zeta_1), f(\zeta_\infty)) = gf^{-1}(f(\zeta)) = g(\zeta) = (\zeta, \zeta_0, \zeta_1, \zeta_\infty).$$ □

23.5. **Proposition.** *Let $\zeta, \zeta_0, \zeta_1, \zeta_\infty$ be distinct points on $\mathbb{C}$. Then the cross ratio $(\zeta, \zeta_0, \zeta_1, \zeta_\infty)$ is real if and only if the points lie on a circle or a straight line.*

Proof. We start with the following observation. Let $a, b, c \in \mathbb{C}$ be distinct. Then they are collinear if and only if the (non-zero) elements $a - b, a - c \in \mathbb{C}$ are linearly dependent over $\mathbb{R}$ if and only if $\frac{a-b}{a-c}$ is real.

We now prove the proposition. 'If': Suppose that $\zeta, \zeta_0, \zeta_1, \zeta_\infty$ lie on a straight line. Then

$$\frac{\zeta - \zeta_0}{\zeta - \zeta_\infty} \text{ and } \frac{\zeta_1 - \zeta_\infty}{\zeta_1 - \zeta_0}$$

are real numbers, so the cross ratio is a real number.

If the four points lie on a circle, the proof involves a calculation with the angle between the line segments $z - \zeta_0$ and $z - \zeta_\infty$, and similarly between $\zeta - \zeta_0$ and $\zeta - \zeta_\infty$. Please see the file `rkg_p204.pdf` uploaded in moodle.

'Only if': First assume that $\zeta_0, \zeta_1, \zeta_\infty$ are collinear, then $\frac{\zeta_1-\zeta_\infty}{\zeta_1-\zeta_0}$ is a real number, and, if, further, the cross-ratio is real, then $\frac{\zeta-\zeta_0}{\zeta-\zeta_\infty}$ is real, i.e., $\zeta, \zeta_0, \zeta_\infty$ are collinear.

If $\zeta_0, \zeta_1, \zeta_\infty$ are not collinear, then we need to consider the circle containing these points and show, using the a calculation of angles, that ζ also lies on the same circle. Please see the file `rkg_p204.pdf` uploaded in moodle. □

23.6. **Corollary.** *A Moebius transformation maps circles and straight lines to circles and straight lines.*

Exercises.

(1) Show that the composite of two Moebius transformations is a Moebius transformation.

Lecture 24. Singularity at infinity

In Lecture 22, we looked at extending rational functions to ∞ and the resulting singularity at ∞.

24.1. **Definition.** Let f be an entire function. We say that f has a *removable singularity* (respectively, a *pole*, an *essential singularity*) at ∞ if the function $f(\frac{1}{z})$ has a removable singularity (respectively, a pole, an essential singularity) at 0.

Note that an entire function has a power series expansion that is convergent everywhere; take $c = 0$ and $R = \infty$ in Corollary 15.5.

24.2. **Proposition.** *An entire function has a removable singularity at ∞ if and only if it is constant.*

Proof. Let f be an entire function. If it is constant, it has a removable singularity at ∞. Conversely assume that it has a removable singularity at ∞. Write $f(z) = \sum_{n\in\mathbb{N}} a_n z^n$. Then $\sum_{n\in\mathbb{N}} a_n z^{-n}$ has a removable singularity at 0. Therefore

$$\lim_{z\to 0} \sum_{n\in\mathbb{N}} a_n z^{-n+1} = 0.$$

Hence $a_n = 0$ if $-n+1 < 0$, i.e., $f(z) = a_0 + a_1 z$. Then $a_1 = \lim_{z\to 0} zf(\frac{1}{z}) = 0$. Hence $a_n = 0$ for each $n > 0$, i.e., f is constant. □

24.3. **Proposition.** *Let f be an entire function. Then the following are equivalent:*

(1) *f has a pole at ∞;*
(2) *For every $M > 0$ there exists $R > 0$ such that $|f(z)| > M$ for all $|z| > R$;*
(3) *f is a non-constant polynomial.*

Proof. (1) $\implies$ (2): Apply Proposition 17.4 (3) to the function $f(\frac{1}{z})$ at its pole 0, to see that for every $M > 0$ there exists $r > 0$ such that $|f(\frac{1}{z})| > M$ for all $|z| < r$. Take $R = \frac{1}{r}$.

(2) $\implies$ (1): By Proposition 17.9, f does not have an essential singularity at ∞. By Proposition 24.2, f does not have a removable singularity at ∞.

(1) $\implies$ (3): Note that f is a non-constant function. We have already established that for every $M > 0$ there exists $R > 0$ such that $|f(z)| > M$ for all $|z| > R$; hence the zeros of f are in a compact subset of $\mathbb{C}$, so f has only finitely many zeros, say, $c_1, \ldots, c_n$ of orders $m_1, \ldots, m_n$ respectively. Therefore we can write $f(z) = \prod_{i=1}^{n}(z - c_i)^{m_i} g(z)$ where $g(z)$ is an entire function without any zeros.

It suffices to show that g is constant. Assume the contrary. By hypothesis, f does not have an essential singularity at ∞. Therefore by Exercise 10 of Lecture 17 and Proposition 24.2, we see that g has a pole at ∞. Therefore for every $M > 0$ there exists $R > 0$ such that $|g(z)| > M$ for all $|z| > R$. Hence $\frac{1}{g(z)}$ which is an entire function is bounded, so it is constant by Liouville's theorem (Proposition 16.2), contradicting the hypothesis that g is not constant.

(3) $\implies$ (1): Write $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ with $n > 0$ and $a_n \neq 0$. Then $f(\frac{1}{z}) = a_n z^{-n} + \cdots + a_0$ has a pole of order n at 0. ◻

24.4. **Corollary.** *Let $f(z) = \sum_{n\in\mathbb{N}} a_n z^n$ be an entire function with $a_n \neq 0$ for infinitely many n. Then for every $A \in \mathbb{C}$, every $R > 0$, every $\delta > 0$, there exists $\zeta \notin \overline{B_{0,R}}$ such that $|f(\zeta) - A| < \delta$.*

Proof. Using the above propositions, we see that $f(\frac{1}{z})$ has an essential singularity at 0. Now apply Proposition 17.9 to $f(\frac{1}{z})$ at 0. ◻

Lecture 25. Automorphisms of the complex plane

This lecture is based on Rodríguez, Kra and Gilman, Sections 8.1, 8.2.

25.1. **Definition.** Let $U \subseteq \mathbb{C}$ be a domain. By an *automorphism* of U, we mean a holomorphic function $f : U \longrightarrow U$ such that there exists $g : U \longrightarrow U$ such that $fg = gf = \mathrm{id}_U$.

25.2. **Proposition.** *Let $U \subseteq \mathbb{C}$ be a domain. Let $f : U \longrightarrow U$ be a bijective function. Then the following are equivalent:*

(1) *f is holomorphic;*
(2) *f is biholomorphic, i.e, f and f^{-1} are holomorphic;*
(3) *f and f^{-1} are conformal.*

Proof. If f and f^{-1} are conformal, then they are holomorphic. Conversely, if f is bijective and holomorphic, then it is conformal, since if $f'(c) = 0$ for some c, then f would not be injective in an neighbourhood of c. Therefore it remains to show that if f is holomorphic, then f^{-1} is holomorphic. We will think of U as a subset of $\mathbb{R}^2$ and show that f^{-1} is differentiable as a function of two real variables and that the Cauchy-Riemann equations are satisfied (Theorem 2.8)

We first show that f^{-1} is differentiable as a function of two real variables. Let $p \in U$ and write $q = f(p)$. Note that $f'(p) \neq 0$, for, otherwise, f would not be injective on $B_{p,\delta} \smallsetminus \{p\}$ for some $0 < \delta \ll 1$. Write $f = u + \imath v$. Hence the derivative of (u, v) at p is the jacobian matrix

$$J(f,p) := \begin{bmatrix} \frac{\partial u}{\partial x}(p) & \frac{\partial v}{\partial x}(p) \\ \frac{\partial u}{\partial y}(p) & \frac{\partial v}{\partial y}(p) \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x}(p) & -\frac{\partial u}{\partial y}(p) \\ \frac{\partial u}{\partial y}(p) & \frac{\partial u}{\partial x}(p) \end{bmatrix}.$$

We have used the Cauchy-Riemann equations for f. From Theorem 2.8, we see that

$$|f'(p)|^2 = \left(\frac{\partial u}{\partial x}(p)\right)^2 + \left(\frac{\partial v}{\partial x}(p)\right)^2 = \det J(f,p).$$

Therefore $J(f,p)$ is invertible since $f'(p) \neq 0$. Note that (u,v) is continuously differentiable, since f is complex-analytic. Now using the inverse function theorem (e.g., Rudin, Principles of Mathematical Analysis, Chapter 9), we see that f^{-1} is differentiable in a neighbourhood of q, as a function of two real variables.

Now to show that f^{-1} satisfies the Cauchy-Riemann equations, observe that

$$J(f^{-1},q)J(f,p) = I \text{ i.e. } J(f^{-1},q) = (J(f,p))^{-1}.$$

Since $J(f,p)$ is a non-zero real matrix of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

(such matrices are invertible) its inverse $J(f^{-1},q)$ too is of the same form (Exercise). Hence f^{-1} satisfies the Cauchy-Riemann equations. □

25.3. **Proposition.** *The map $\mathbb{C} \longrightarrow \mathbb{C}$, $z \mapsto az+b$, where $a,b \in \mathbb{C}$, $a \neq 0$ is an automorphism, with inverse $z \mapsto \frac{1}{a}(z-b)$. Conversely, if $f : \mathbb{C} \longrightarrow \mathbb{C}$ is an automorphism, then there exist $a,b \in \mathbb{C}$, $a \neq 0$ such that $f(z) = az+b$ for every $z \in \mathbb{C}$.*

Proof. The map $z \mapsto az+b$ (with $a \neq 0$) is bijective and holomorphic, i.e., an automorphism. Its inverse is the map $z \mapsto \frac{1}{a}(z-b)$. Now assume that $f : \mathbb{C} \longrightarrow \mathbb{C}$ is an automorphism. Since f is entire, it has a convergent power series expansion, valid everywhere on $\mathbb{C}$. (In Corollary 15.5, we can take $c = 0$ and $R = \infty$.) Write $f(z) = \sum_{n\in\mathbb{N}} a_n z^n$. Suppose that $a_n \neq 0$ for infinitely many n. Then by Corollary 24.4 we see that for each $R > 0$, the set $f(\mathbb{C} \smallsetminus \overline{B_{0,R}})$ is dense in $\mathbb{C}$. However, since f is injective, $f(B_{0,1})$ is non-empty but

$$f(\mathbb{C} \smallsetminus \overline{B_{0,1}}) \cap f(B_{0,1}) = \varnothing.$$

Hence $a_n = 0$ for all $n \gg 0$. Write $f(z) = a_0 + a_1 z + \cdots + a_m z^m$ with $m \geq 0$ and $a_m \neq 0$. Since f is injective, it is not constant, so $m \geq 1$.

We need to show that $m = 1$. By way of contradiction, assume that $m > 1$. Then $\deg f' > 0$, so there exists $\zeta \in \mathbb{C}$ such that $f'(\zeta) \neq 0$. Hence there exists a neighbourhood of ζ on which f is not injective, a contradiction. □

25.4. **Example.** The map

$$z \mapsto \frac{z-\imath}{z+\imath}$$

the upper half plane $\{z \in \mathbb{C} \mid \Im(z) > 0\}$ to the open unit disc. If $z = x + y\imath$, with $y > 0$, then

$$\left|\frac{x+(y-1)\imath}{x+(y+1)\imath}\right| < 1$$

since $|y-1| < |y+1|$. It is a Moebius transformation, and holomorphic on the upper half plane, so the map is conformal.

(The Riemann mapping theorem says that every simply connected domain $U \subsetneq \mathbb{C}$ is biholomorphic to the open unit disc. Above, we have given a specific map that works for the upper half plane.) □

We now prove the Schwarz lemma, which describes maps from $B_{0,1}$ to itself.

25.5. **Proposition.** *Let $f : B_{0,1} \longrightarrow B_{0,1}$ is holomorphic with $f(0) = 0$, then $|f(z)| \leq |z|$ for every $z \in B_{0,1}$ and $|f'(0)| \leq 1$. If $|f(z)| = |z|$ for some $z \in B_{0,1}$, $z \neq 0$ or if $|f'(0)| = 1$, then $f(z) = cz$ for some constant c with $|c| = 1$.*

Proof. Write $f(z) = \sum_{n=1}^{\infty} a_n z^n$. Let

$$g(z) = \begin{cases} \frac{f(z)}{z}, & \text{if } z \neq 0, \\ a_1, & \text{if } z = 0. \end{cases}$$

(Note that $f'(0) = g(0)$.) Then, for every $0 < r < 1$, and every z with $|z| = r$,

$$|g(z)| = \left|\frac{f(z)}{z}\right| = \frac{|f(z)|}{|z|} \leq \frac{1}{r}.$$

Hence by the maximum principle (Proposition 19.1) $|g(z)| \leq \frac{1}{r}$ for every $z \in \overline{B_{0,r}}$ for every $0 < r < 1$. Hence $|g(z)| \leq 1$ for every $z \in B_{0,1}$. Hence $|f(z)| \leq |z|$ for each $z \in B_{0,1}$ and $f'(0) = g(0) \leq 1$.

Now suppose that $|f(z)| = |z|$ for some $z \in B_{0,1}$, $z \neq 0$ or that $|f'(0)| = 1$. Equivalently, $|g(z)| = 1$ for some $z \in B_{0,1}$, then g is a constant function, again by the maximum principle. □

As an immediate corollary, we get a property of holomorphic automorphisms of the unit disc. There is a more precise statement, characterising holomorphic automorphisms of the unit disc. The proof of the general statement is not difficult, but we will skip it. If you are interested, you can look at Rodríguez, Kra and Gilman, Section 8.2, Theorem 8.18, or Lang, Complex Analysis, Chapter VII, Section 2.

25.6. **Corollary.** *Let $f : B_{0,1} \longrightarrow B_{0,1}$ be a bijective holomorphic map. Then it is a Moebius transformation.*

Proof. Let $c = f(0)$. Then check that the Moebius transformation

$$g : z \mapsto \frac{z - c}{1 - \bar{c}z}$$

is an holomorphic automorphism of $B_{0,1}$. Hence gf a holomorphic automorphism of $B_{0,1}$, with $gf(0) = 0$. Therefore it suffices to show that gf is a Moebius transformation. Replacing f by gf, we may assume that $f(0) = 0$. Write $w = f(z)$. Then

$$|z| = |f^{-1}(w)| \leq |w| = |f(z)| \leq |z|$$

by the use of the the Schwarz Lemma (Proposition 25.5) once for f^{-1} and then for f. Hence $|f(z)| = |z|$ for every $z \in B_{0,1}$. Again by Proposition 25.5, there exists a with $|a| = 1$ such that $f(z) = az$. Hence f is a Moebius transformation. □

Exercises.

(1) Show that the inverse of a non-zero real matrix of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

is a matrix of the above form. (Hint: after appropriate scaling, this matrix representation rotation in $\mathbb{R}^2$.)

(2) Determine the images of horizontal and vertical lines in the upper half plane under the map in Example 25.4.

(3) Find the inverse of the map in Example 25.4.

(4) Show that the map
$$z \mapsto -\imath \frac{z-1}{z+1}$$
conformally maps the upper half disc $(\{|z| < 1, \Im z > 0\})$ to the first quadrant.

(5) Find a conformal map from the upper half unit disc to the unit disc.

(6) Show that $\mathbb{C}$ and $B_{0,1}$ are not biholomorphic to each other.

Lecture 26. Automorphisms of the Riemann sphere

We want to understand holomorphic functions from $\widehat{\mathbb{C}}$ to itself. In Lecture 21, we described the following open covering of $\widehat{\mathbb{C}}$: (Here we change our notation a little bit.) $U_0 := \widehat{\mathbb{C}} \smallsetminus \{\infty\}$, $U_\infty := \widehat{\mathbb{C}} \smallsetminus \{0\}$, $\sigma : U_0 \longrightarrow \mathbb{C}$ the stereographic projection map, and $\tau : U_\infty \longrightarrow \mathbb{C}$.

26.1. **Definition.** Let $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ and $p \in \widehat{\mathbb{C}}$ and $q = f(p)$. Say that f is *differentiable* at p if $\tilde{f}$ is differentiable at ζ, where
$$(\tilde{f}, \zeta) = \begin{cases} (\sigma f \sigma^{-1}, \sigma(p)) & \text{if } p \in U_0 \text{ and } q \in U_0; \\ (\tau f \sigma^{-1}, \sigma(p)) & \text{if } p \in U_0 \text{ and } q = \infty; \\ (\sigma f \tau^{-1}, \tau(p)) & \text{if } p = \infty \text{ and } q \in U_0; \\ (\tau f \tau^{-1}, \tau(p)) & \text{if } p = \infty = q. \end{cases}$$
Say that f is holomorphic if it is differentiable at p for each $p \in \widehat{\mathbb{C}}$.

It might appear that we have given preference to U_0 over U_∞ while making the above definition. This is not the case. For example, suppose that $\{p, q\} \subseteq U_0 \cap U_\infty$. Then the following are equivalent:

(1) $\sigma f \sigma^{-1}$ is differentiable at $\sigma(p)$;
(2) $\sigma f \tau^{-1}$ is differentiable at $\tau(p)$;
(3) $\tau f \sigma^{-1}$ is differentiable at $\sigma(p)$;
(4) $\tau f \tau^{-1}$ is differentiable at $\tau(p)$.

Assume that $\sigma f \sigma^{-1}$ is differentiable at $\sigma(p)$; let us show that $\sigma f \tau^{-1}$ is differentiable at $\tau(p)$. Note that, in a neighbourhood of $\tau(p)$,
$$\sigma f \tau^{-1} = (\sigma f \sigma^{-1}) \circ \left(z \mapsto \frac{1}{z}\right)$$
so $\sigma f \tau^{-1}$ is differentiable at $\tau(p)$.

26.2. **Proposition.** *Let $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be a bijective holomorphic map. Then f^{-1} is holomorphic.*

Proof. Let $p \in \widehat{\mathbb{C}}$ and $q = f(p)$. We will assume that $p \in U_0$ and $q \in U_0$; the other cases can be handled similarly. Hence we need to show that $\sigma f^{-1} \sigma^{-1}$ is differentiable at $\sigma(q)$. But note that $\sigma f^{-1} \sigma^{-1}$ is the inverse of the bijective holomorphic map $\sigma f \sigma^{-1}$, so $\sigma f^{-1} \sigma^{-1}$ is differentiable at $\sigma(q)$. □

26.3. **Definition.** By an automorphism of $\widehat{\mathbb{C}}$, we mean a bijective holomorphic map from $\widehat{\mathbb{C}}$ to itself.

The *special linear group* $\mathrm{SL}_2(\mathbb{C})$ is the group of 2×2 complex matrices with determinant 1. It is a group, under usual matrix multiplication. Let f and g be Moebius transformations:
$$f(z) = \frac{az + b}{cz + d} \text{ and } g(z) = \frac{a'z + b'}{c'z + d'}.$$

Then

$$f(g(z)) = \frac{a\frac{a'z+b'}{c'z+d'} + b}{c\frac{a'z+b'}{c'z+d'} + d} = \frac{(aa'+bc')z + (ab'+bd')}{(ca'+dc')z + (cb'+dd')},$$

which is a Moebius transformations. Hence the set of Moebius transformations form a group $\mathbb{M}$ under composition.

Note that the group operation in $\mathrm{SL}_2(\mathbb{C})$ is given by:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{bmatrix}$$

Hence there is a group homomorphism

$$\mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathbb{M}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \frac{az+b}{cz+d}.$$

We have seen in Lecture 23 that every Moebius transformation can be represented by an element of $\mathrm{SL}_2(\mathbb{C})$. Hence the above group homomorphism is surjective, with kernel $\{\pm I\}$. The group

$$\mathrm{SL}_2(\mathbb{C})/\{\pm I\}$$

is usually written $\mathrm{PSL}_2(\mathbb{C})$.

26.4. **Proposition.** *Every element of $\mathbb{M} = \mathrm{PSL}_2(\mathbb{C})$ is a meromorphic conformal automorphism of $\widehat{\mathbb{C}}$. Conversely, every a meromorphic conformal automorphism of $\widehat{\mathbb{C}}$ is given by an element of $\mathbb{M}$.*

Proof. It is easy to check that elements of $\mathbb{M}$ are automorphisms. Conversely, let f be an automorphism of $\widehat{\mathbb{C}}$. If $f(\infty) = \infty$, then $f(z) = az + b$ by Proposition 25.3. If $f(\infty) = c \neq \infty$, then let $g(z) = \frac{1}{z-c}$. Now gf is an automorphism, and fixes ∞, so it is in $\mathbb{M}$. Hence $f \in \mathbb{M}$. □

Exercises.

(1) Let $\zeta, \zeta_0, \zeta_1, \zeta_\infty$ be distinct points of $\widehat{C}$ and σ a permutation of four symbols. Determine the relation between the cross ratio $(\zeta, \zeta_0, \zeta_1, \zeta_\infty)$ and $(\sigma(\zeta, \zeta_0, \zeta_1, \zeta_\infty))$.

Lecture 27. Review of path homotopy

Reference for this section is Munkres, *Topology*, the chapter on covering spaces and fundamental groups.

Let U be an open subset of $\mathbb{C}$. Let $\gamma, \eta : [a, b] \longrightarrow U$ be two paths with $\gamma(a) = \eta(a)$ and $\gamma(b) = \eta(b)$. We say that γ and η are *path-homotopic* to each other if there exists a continuous map $H : [0, 1] \times [a, b] \longrightarrow U$ such that

$$\begin{aligned} H(0,t) &= \gamma(t) \text{ for all } t \in [a,b]; \\ H(1,t) &= \eta(t) \text{ for all } t \in [a,b]; \\ H(s,0) &= \gamma(0) = \eta(0) \text{ for all } s \in [0,1]; \\ H(s,1) &= \gamma(1) = \eta(1) \text{ for all } s \in [0,1]. \end{aligned}$$

We can think of this as a continuously varying family of paths $[a, b] \longrightarrow U$ parameterised by $[0, 1]$ such that for every member of the family is a path from $\gamma(0)$ to $\gamma(1)$. We say that H is a *path-homotopy* between γ and η. We will say that a closed path γ is *null-homotopic* if γ and the constant path $e_{\gamma(0)}$ at $\gamma(0)$ (i.e, the map $[a, b] \longrightarrow U$, $t \mapsto \gamma(0)$) are path-homotopic to each other. Note that being path-homotopic is an equivalence relation; we will refer to the equivalence classes under this relation as *path-homotopy classes*.

We say that U is *simply-connected* if for every closed path in U is null-homotopic. For example, $\mathbb{C}$ is simply connected, but $\mathbb{C} \setminus \{0\}$ is not.

The result we want to prove is that if f is holomorphic on U, then $\int_\gamma f\mathrm{d}z$ depends only on the path-homotopy class of γ. However, there is (at least) one issue that needs to be sorted out: even if γ and η are path-homotopic piecewise-differentiable paths, with path homotopy H, the paths $H(s, -)$ need not be piecewise-differentiable for $s \in (0, 1)$. Hence we need to understand what $\int_\tau f\mathrm{d}z$ is, when τ is merely a continuous path.

Reference for the remainder of this section is Lang, Complex Analysis, Chapter III, Section 4.

27.1. **Lemma.** *Let $U \subseteq \mathbb{C}$ be an open set and $\gamma : [a, b] \longrightarrow U$ a continuous path in U. Then there exists $r > 0$ such that for every $x \in \operatorname{Im}(\gamma)$ and for every $y \in \mathbb{C} \setminus U$, $|x - y| > r$.*

Proof. The function $\delta : U \longrightarrow \mathbb{R}$, $t \mapsto \inf\{|\gamma(t) - y| : y \in \mathbb{C} \setminus U$ is attained by some $y \notin U$, since it suffices to consider y lying inside a closed and bounded subset of $\mathbb{C}$. It is continuous: Let $t_n \longrightarrow t$. let $y, y_n \notin U$ be such that $\delta(t) = |\gamma(t) - y|$ and $\delta(t_n) = |\gamma(t_n) - y_n|$. Let $\epsilon > 0$. Then there exists N such that for every $n > N$, $|\gamma(t_n) - \gamma(t)| < \epsilon$. Hence $\delta(t) < \delta(t_n) + \epsilon$ and $\delta(t_n) < \delta(t) + \epsilon$, for every $n > N$. Hence $\delta(t_n) \longrightarrow \delta(t)$. Therefore there exists $t_0 \in [a, b]$ such that $\delta(t_0) = \inf\{\delta(t) \mid t \in [a, b]\}$. Since U is open, $\delta(t_0) > 0$. □

27.2. **Discussion.** For now, assume that γ is piecewise-differentiable. Let $\epsilon > 0$ be small enough such that $B_{\gamma(t),\epsilon} \subseteq U$ for every $t \in [a, b]$; such an ϵ exists by Lemma 27.1. Since γ is uniformly continuous, there exists $\delta > 0$ such that $\gamma(B_{t,\delta}) \subseteq B_{\gamma(t),\epsilon}$. Then there exist

(1) a partition $a = t_0 < t_1 < \ldots < t_n = b$ such that
   (a) $t_{i+1} - t_i < \delta$;
   (b) γ is differentiable on (t_i, t_{i+1});
(2) a covering of $\operatorname{Im}(\gamma)$ by open discs B_i, $0 \le i \le n - 1$ such that $\gamma([t_i, t_{i+1}]) \subseteq B_i$.

For $0 \le i \le n - 1$, let g_i be a primitive of f in B_i. Then

$$\int_{t_i}^{t_{i+1}} f(\gamma(t))\gamma'(t)\mathrm{d}t = g_i(\gamma(t_{i+1})) - g_i(t_i), \text{ and hence,}$$

$$\int_\gamma f\mathrm{d}z = \sum_{i=0}^{n-1} g_i(t_{i+1}) - g_i(t_i). \qquad \square$$

In view of the discussion above, we can extend the definition of $\int_\gamma f\mathrm{d}z$ to continuous paths γ as follows. Note that we did not use all the information about the partition, in the above discussion.

27.3. **Definition.** Let U be a domain and $\gamma : [a, b] \longrightarrow U$ a continuous path. Let $a = t_0 < t_1 < \ldots < t_n = b$ be a partition and $B_0, \ldots, B_{n-1}$ be open discs in U such that $\gamma([t_i, t_{i+1}]) \subseteq B_i$ for every $0 \le i \le n - 1$. Let g_i be a primitive of f on B_i. Define

$$\int_\gamma f\mathrm{d}z = \sum_{i=0}^{n-1} g_i(\gamma(t_{i+1})) - g_i(\gamma(t_i)).$$

27.4. **Proposition.** *The definition of $\int_\gamma f\mathrm{d}z$ is independent of the choice of the partition $a = t_0 < t_1 < \ldots < t_n = b$, the open discs B_i, $0 \le i \le n - 1$ and the primitives g_i, $0 \le i \le n - 1$ of f.*

Proof. For the sake of clarity, we will write $I(\{t_i\}, \{B_i\}, \{g_i\})$ to denote $\sum_{i=0}^{n-1} g_i(\gamma(t_{i+1})) - g_i(\gamma(t_i))$ in Definition 27.3. Let $\{\tilde{t}_i\}$, $\{\tilde{B}_i\}$ and $\{\tilde{g}_i\}$ another set of choices.

Step 1: Assume that $\{\tilde{t}_i\} = \{t_i\}$. Then $\gamma([t_i, t_{i+1}]) \subseteq B_i \cap \tilde{B}_i$, on which g_i and $\tilde{g}_i$ are primitives of f. Then there exists $c_i \in \mathbb{C}$ such that $g_i(z) - \tilde{g}_i(z) = c_i$ for every $z \in B_i \cap \tilde{B}_i$. Hence

$$g_i(\gamma(t_{i+1})) - g_i(\gamma(t_i)) = \tilde{g}_i(\gamma(t_{i+1})) - \tilde{g}_i(\gamma(t_i))$$

so $I(\{t_i\}, \{B_i\}, \{g_i\}) = I(\{t_i\}, \{\tilde{B}_i\}, \{\tilde{g}_i\})$.

Step 2: Assume that $\{\tilde{t}_i\}$ is a refinement of $\{t_i\}$. Then the covering $\{B_i\}$ and the primitives $\{g_i\}$ (which were defined for $\{t_i\}$) induced a covering and primitives with respect to $\{\tilde{t}_i\}$. More precisely, if $\tilde{t}_{j_i} = t_i$ and $\tilde{t}_{j_{i+1}} = t_{i+1}$, then use B_i and g_i for the intervals $[\tilde{t}_j, \tilde{t}_{j+1}]$, $j_i \leq j < t_{j_{i+1}}$. We abuse notation and continue to use $\{B_i\}$ and $\{g_i\}$ for the induced covering and primitives. It is easy to see that $I(\{t_i\}, \{B_i\}, \{g_i\}) = I(\{\tilde{t}_i\}, \{B_i\}, \{g_i\})$. By the earlier case, $I(\{\tilde{t}_i\}, \{B_i\}, \{g_i\}) = I(\{\tilde{t}_i\}, \{\tilde{B}_i\}, \{\tilde{g}_i\})$.

Step 3: Consider the general case. Let $\{\hat{t}_i\}$ be a common refinement of the $\{t_i\}$ and the $\{\tilde{t}_i\}$. As in Step 2, the covering $\{B_i\}$ and the primitives $\{g_i\}$ induce a covering and primitives on $\{\hat{t}_i\}$, which we abuse notation and denote by $\{B_i\}$ and $\{g_i\}$. Similarly, we get $\{\tilde{B}_i\}$ and $\{\tilde{g}_i\}$ from the partition $\{\tilde{t}_i\}$. Then

$$\begin{aligned} I(\{t_i\}, \{B_i\}, \{g_i\}) &= I(\{\hat{t}_i\}, \{B_i\}, \{g_i\}) \\ &= I(\{\hat{t}_i\}, \{\tilde{B}_i\}, \{\tilde{g}_i\}) \\ &= I(\{\tilde{t}_i\}, \{\tilde{B}_i\}, \{\tilde{g}_i\}) \end{aligned}$$

where the first and the third equalities follow from Step 2 and the second one from Step 1. □

We emphasise that order to make sense of Definition 27.3, we need to know that holomorphic functions on discs have primitives (Theorem 11.1).

Exercises.

Lecture 28. General version of Cauchy integral theorem.

Let U be a domain.

The following lemma is Lang, Complex Analysis, Chapter III, Section 4, Lemma 4.3.

28.1. **Lemma.** *Let γ and η be two continuous paths $[a, b] \longrightarrow U$. Assume that there exist a partition $a = t_0 < t_1 < \ldots < t_n = b$ and open discs $B_0, \ldots, B_{n-1}$ in U such that $\gamma([t_i, t_{i+1}]) \subseteq B_i$ and $\eta([t_i, t_{i+1}]) \subseteq B_i$ for every $0 \leq i \leq n-1$. Then*

$$\int_\gamma f \mathrm{d}z = \int_\eta f \mathrm{d}z.$$

Proof. Write $z_i = \gamma(t_i)$ and $w_i = \eta(t_i)$. Let g_i be a primitive of f on B_i, $0 \leq i \leq n-1$. Since g_{i+1} and g_i are primitives of f on $B_{i+1} \cap B_i$, the function $g_{i+1} - g_i$ is constant on $B_{i+1} \cap B_i$. It follows that

$$g_{i+1}(z_{i+1}) - g_i(z_{i+1}) = g_{i+1}(w_{i+1}) - g_i(w_{i+1})$$

for all $0 \leq i \leq n-1$. Hence

$$\begin{aligned}\int_\gamma f\mathrm{d}z - \int_\eta f\mathrm{d}z &= \sum_{i=0}^{n-1} [g_i(z_{i+1}) - g_i(z_i))] - [g_i(w_{i+1}) - g_i(w_i))] \\ &= \sum_{i=0}^{n-1} [g_{i+1}(z_{i+1}) - g_i(z_i))] - [g_{i+1}(w_{i+1}) - g_i(w_i))] \\ &= [g_n(z_n) - g_0(z_0)] - [g_n(w_n) - g_0(w_0)] \\ &= 0\end{aligned}$$

since $z_0 = w_0$ and $z_n = w_n$. □

28.2. **Theorem.** *Let γ and η be path-homotopic continuous paths $[a,b] \longrightarrow U$. Let f be holomorphic on U. Then*

$$\int_\gamma f\mathrm{d}z = \int_\eta f\mathrm{d}z.$$

Proof. Let $H = [0,1] \times [a,b] \longrightarrow U$ be a path homotopy. Since $\mathrm{Im}(H)$ is compact, there exists $r > 0$ such that for every $x \in \mathrm{Im}(H)$ and for every $y \in \mathbb{C} \smallsetminus U$, $|x-y| > r$, as in Lemma 27.1. Hence there exists ϵ such that $B_{x,\epsilon} \subseteq U$ for every $x \in \mathrm{Im}(H)$. Since H is uniformly continuous, there exists $\delta > 0$ such that for every $p \in [0,1] \times [a,b]$, $H(B_{p,\delta}) \subseteq B_{H(p),\epsilon}$. Hence there exist partitions $0 = s_0 < s_1 < \ldots < s_m = 1$ and $a = t_0 < t_1 < \ldots < t_n = b$ such that for each i, j, there exists an open disk $B_{i,j}$ such that $H([s_i, s_{i+1}] \times [t_j, t_{j+1}]) \subseteq B_{i,j} \subseteq U$. (For example, choose the s_i and the t_j such that the diagonal of the rectangle $[s_i, s_{i+1}] \times [t_j, t_{j+1}]$ has length at most 2δ.) For $i = 0, \ldots, m$, define paths $\gamma_i : [a,b] \longrightarrow U$ by $\gamma_i(t) = H(s_i, t)$. Note that $\gamma_0 = \gamma$ and $\gamma_m = \eta$. We now induct on i and apply the lemma to conclude that $\int_{\gamma_i} f\mathrm{d}z = \int_\gamma f\mathrm{d}z$ for every $1 \leq i \leq m$. □

28.3. **Corollary.** *If U is simply connected, then $\int_\gamma f\mathrm{d}z = 0$ for every holomorphic f on U and every closed path γ.*

Exercises.

(1) Let $U \subseteq \mathbb{C}$ be a bounded domain. Show that it is simply connected if and only if $\mathbb{C} \smallsetminus U$ is connected.

(2) Let γ be a closed path in $\mathbb{C}$, not passing through 0. Assume further that there exists a ray through the origin $\{r\zeta \mid r \in \mathbb{R}, r > 0, \zeta \in \mathbb{C}, \zeta \neq 0\}$ that does not intersect $\mathrm{Im}(\gamma)$. Find a simply connected domain U containing γ that admits a branch of the logarithm. Conclude that $\int_\gamma \frac{1}{z}\mathrm{d}z = 0$.

(3) Let U be a domain. Show that U is simply connected if and only if for every closed path γ in U and every $\zeta \notin U$, $n(\zeta, \gamma) = 0$.

Exercises

References

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