COMMUTATIVE ALGEBRA II, JAN-APR 2018: PROBLEM SETS

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Notation:
$\Bbbk$, a field
R, S, commutative rings, with multiplicative identity.
M, N, etc., sometimes with subscripts: modules.

1. 2019-01-21, IN CLASS

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1.1. (10 marks) Let $R = \Bbbk[u, v, x, y]$ and $I = (ux, vy, uy + vx)$. Determine $\dim R/I$, a system of parameters for R/I, a chain in $\operatorname{Spec} R/I$ of maximum length and a primary decomposition of R/I.

1.2. (10 marks) Recall that the M/M_λ have discrete topology, $\prod_\lambda M/M_\lambda$ the product topology and $\widehat{M}$ the subspace topology. Show that this topology on $\widehat{M}$ agrees with the linear topology given by the family $M_\lambda^*, \lambda \in \Lambda$, where $M_\lambda^* = \ker\left(\widehat{M} \longrightarrow M/M_\lambda\right)$. Show that $\widehat{M}$ is complete with respect to this topology.

1.3. (5 marks) Show that the natural map $\iota : M \longrightarrow \widehat{M}$ is R-linear and continuous.

1.4. (10 marks) Let R be a noetherian ring and $I \subseteq J$ R-ideals. Show that $I = J$ if and only if $I_\mathfrak{p} = J_\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Ass} R/I$. Give an example to show that the hypothesis that $I \subseteq J$ is necessary.

1.5. (10 marks) Let R be a standard graded ring, i.e., $R = \oplus_{i \in \mathbb{N}} R_i$ with $R = R_0[R_1]$). Let I be the ideal $\oplus_{i \geq 1} R_i$. Show that the I-adic completion of R is $\prod_{i \in \mathbb{N}} R_i$. Now drop the assumption that R is standard graded (but still graded by $\mathbb{N}$). Then describe a topology on R such that the completion is $\prod_{i \in \mathbb{N}} R_i$.

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