RAMIFICATION THEORY AUG-NOV 2018: PROBLEM SETS

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1. 2018-08-21

1.1. (5 marks) A *free presentation* of M is an exact sequence $F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ with F_0 and F_1 free R-modules; a free presentation is said to be a *finite free presentation* if F_1 and F_2 are of finite rank. M is *finitely presented* if it has a finite free presentation. Let M and N be finitely presented R-modules. Show that $M \otimes_R N$ is finitely presented.

1.2. (5 marks) Let $f : R \longrightarrow S$ be a ring map. Let M and N be R-modules, with generating sets $\{x_\lambda \mid \lambda \in \Lambda\}$ and $\{y_i \mid i \in \mathcal{I}\}$ respectively.

(a) Let $\phi : M \longrightarrow N$ be a map of R-modules. Write $\phi(x_\lambda) = \sum_{i \in \mathcal{I}} r_{i,\lambda} y_i$, where the $r_{i,\lambda}$ are elements (not necessarily uniquely determined) of R. Show that $f^*(\phi)(1 \otimes_R x_\lambda) = \sum_{i \in \mathcal{I}} f(r_{i,\lambda})(1 \otimes_R y_i)$.

(b) Let $G \xrightarrow{\phi} F \longrightarrow M \longrightarrow 0$ be a free presentation of M. Show that $f^*G \xrightarrow{f^*(\phi)} f^*F \longrightarrow f^*M \longrightarrow 0$ is a free presentation of f^*M. In particular if M if finitely generated as an R-module then so is f^*M as an S-module. Similarly if M if finitely presented as an R-module then so is f^*M as an S-module.

(c) Suppose that M is finitely presented with a finite free presentation $G \xrightarrow{\phi} F \longrightarrow M \longrightarrow 0$. Let $m = \operatorname{rk}_R G$ and $n = \operatorname{rk}_R F$. Let $\{g_1, \dots, g_m\}$ and $\{f_1, \dots, f_n\}$ be bases for G and F respectively, and $A = [r_{ij}]$ the matrix of ϕ with respect to this pair of bases. Show that $S^m \xrightarrow{f(A)} S^n \longrightarrow f^*M \longrightarrow 0$ is a finite free presentation of f^*M, where $f(A)$ is the matrix $[f(r_{ij})]$.

1.3. (5 marks) Verify the assertions about tensor product of algebras made in the review section on tensor products.

1.4. (15 marks) Let M and N be R-modules and let $M^* := \operatorname{Hom}_R(M, R)$. There is a natural R-module morphism $\tau_{M,N} : M^* \otimes_R N \longrightarrow \operatorname{Hom}_R(M, N)$, $f \otimes y \mapsto [x \mapsto f(x)y]$. Show that this is neither injective nor surjective in general by using the following example: $R = \mathbb{Z}/(4)$, $I = 2R$, $M = N = R/I$. However, prove the following to see some useful situations where it is injective or bijective.

(a) $\tau_{M,R}$ and $\tau_{R,N}$ are bijective.

(b) For finitely generated M, $\tau_{M,N}$ commutes with localization, i.e., for every $\mathfrak{p} \in \operatorname{Spec} R$,

$$\tau_{M,N} \otimes_R R_\mathfrak{p} = \tau_{M_\mathfrak{p}, N_\mathfrak{p}}$$

where in the right side, we consider them as $R_\mathfrak{p}$-modules.

(c) Let $N_\lambda, \lambda \in \Lambda$ be R-modules, and $N = \oplus_{\lambda \in \Lambda} N_\lambda$. If τ_{M,N_λ} is injective for every λ, then $\tau_{M,N}$ is injective. Show that if Λ is a finite set and τ_{M,N_λ} is bijective for every λ, then $\tau_{M,N}$ is bijective.

(d) If N is projective, $\tau_{M,N}$ is injective. If N is finitely generated projective, then $\tau_{M,N}$ is an isomorphism.

(e) Let $M_1, \cdots, M_n$ be R-modules and $M = \oplus_{i=1}^n M_i$. Show that if each $\tau_{M_i,N}$ is bijective, then $\tau_{M,N}$ is bijective.

(f) If M is finitely generated projective, then $\tau_{M,N}$ is bijective.

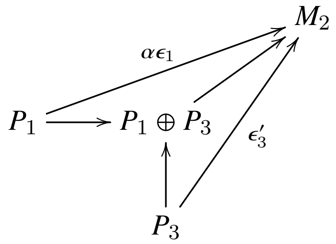
(One can do this without localization; see [Bou98, Chapter II, §4, No. 2].)

2. DUE 2018-09-14 IN CLASS

2.1. (5 marks) Let

$$0 \longrightarrow M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \longrightarrow 0$$

be an exact sequence of R-modules. Let P_1 and P_3 be projective modules with surjective maps $P_1 \xrightarrow{\epsilon_1} M_1$ and $P_3 \xrightarrow{\epsilon_3} M_3$. Show that there is a map $\epsilon_3' : P_3 \longrightarrow M_2$ such that $\beta\epsilon_3' = \epsilon_3$. Show that there is a commutative diagram



(Label all the unlabelled arrows.) Now show that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_1 & \longrightarrow & P_2 & \longrightarrow & P_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \epsilon_1} & & \downarrow{\scriptstyle \epsilon_2} & & \downarrow{\scriptstyle \epsilon_3} & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \end{array}$$

where the rows are exact, P_2 is projective and ϵ_2 is surjective.

2.2. (10 marks) Let $F_\bullet$ be a flat resolution of M. Show that for every module N, $\operatorname{Tor}_i(M, N) \simeq \operatorname{H}_i(F_\bullet \otimes_R N)$.

2.3. (5 marks) Let $F \subseteq K$ be an algebraic extension of fields. An element $a \in K$ is said to be *separable* (over F) if its minimal polynomial $f[x]$ is separable, i.e., $(f, f') = F[x]$. K/F is said to *purely inseparable* if for every $a \in K \smallsetminus F$, a is not separable over F. Suppose that $F \neq K$. Show that the following are equivalent: K/F is purely inseparable; $\operatorname{char} F = p > 0$ and for every $a \in K$, its minimal polynomial over F is of the form $x^{p^e} - b$ for some $b \in F$.

2.4. (5 marks) Let $a \in \overline{F}$, an algebraic closure of F. Show that if a is the only root (in $\overline{F}$) of its minimal polynomial over F, then $a \in F$ or $\operatorname{char} F = p > 0$ and the minimal polynomial of a over F is of the form $x^{p^e} - b$ for some $b \in F$.

2.5. (5 marks) Let K/F be a normal extension, and $G = \operatorname{Aut}_F(K)$. Show that K^G/F is purely inseparable.

2.6. (5 marks) Say that a morphism $R \longrightarrow S$ *has the going down property* if for every $\mathfrak{p}_1 \subseteq \mathfrak{p}_2 \in \operatorname{Spec} R$ and for every $\mathfrak{q}_2 \in \operatorname{Spec} S$ lying over $\mathfrak{p}_2$, there exists $\mathfrak{q}_1 \in \operatorname{Spec} S, \mathfrak{q}_1 \subseteq \mathfrak{q}_2$ lying over $\mathfrak{p}_1$. Show that $R \longrightarrow S$ has the going down property if and only if the induced map $\operatorname{Spec} S_\mathfrak{q} \longrightarrow \operatorname{Spec} R_\mathfrak{p}$ is surjective for all $\mathfrak{p} \in \operatorname{Spec} R$ and for all $\mathfrak{q} \in \operatorname{Spec} S$ lying over $\mathfrak{p}$.

2.7. (5 marks) Show that if $R \longrightarrow S$ is faithfully flat (i.e., S is a flat R-module and for every non-zero R-module M, $S \otimes_R M$ is non-zero) then the map $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is surjective. Show that the conclusion does not hold for arbitrary flat maps.

2.8. (5 marks) Flat maps have the going down property.

2.9. (10 marks) Let $R := \Bbbk[t^2 - 1, t(t^2 - 1), z] \subseteq \Bbbk[t, z] =: S$. Show that S is the normalization of R (i.e., the integral closure in field of fractions). Show that this map does not have the going down property as follows: Consider this map as the map $\Bbbk^2 \longrightarrow \operatorname{Spec} R$. Let $\mathfrak{p}_1 \in \operatorname{Spec} R$ be the prime ideal corresponding to the image of the line $(t = z)$ inside $\Bbbk^2$. Show that the image of $(-1, 1) \in \Bbbk^2$ is defined by a prime ideal $\mathfrak{p}_2 \supseteq \mathfrak{p}_1$. There is no height one prime ideal $\mathfrak{q}_1$ that contracts to $\mathfrak{p}_1$ and is inside $(t + 1, z - 1)$.

2.10. (5 marks) Let L/K be a purely inseparable extension of fields, R normal domain with fraction field K and S its integral closure of R in L. Show that for every $\mathfrak{p} \in \operatorname{Spec} R$, there exists a unique $\mathfrak{q} \in \operatorname{Spec} S$ lying over $\mathfrak{p}$.

2.11. (5 marks) Let $S = \mathbb{C}[x, y]$ where x, y are variables. Let $n \geq 2$ be an integer and let $\mathbb{Z}/n\mathbb{Z}$ act on $\mathbb{C}(x, y)$ with $1 \in \mathbb{Z}/n\mathbb{Z}$ sending $x \mapsto \exp \frac{2\pi\iota}{n} x$, $y \mapsto \exp \frac{2\pi\iota}{n} y$. Let $R = S \cap \mathbb{C}(x, y)^{\mathbb{Z}/n\mathbb{Z}}$. Find the orbits of this action on $\mathbb{C}^2 = \max\operatorname{Spec} S$. Determine $(x - 1, y - 1) \cap R$ and the the primes of S lying over it.

2.12. Read [Lan02, Chapter VI, Section 5] about trace.

2.13. Read [HS06, Section 3.1] (available online at Swanson's home-page) about separable extensions and integral closure.

3. 2018-10-08 IN CLASS

3.1. (5 marks) Let $\mu : R \otimes_{\Bbbk} R \longrightarrow R$ be the map given by $\mu(a \otimes b) = ab$. Show that $\ker \mu$ is the $R \otimes_{\Bbbk} R$-ideal generated by $\{a \otimes 1 - 1 \otimes a \mid a \in R\}$.

3.2. (15 marks) Let $\mathrm{d} \in \operatorname{Der}_{\Bbbk}(R, M)$. Show the following:

(a) $\mathrm{d}(1) = 0$; for every $a \in \Bbbk$, $\mathrm{d}a = 0$,

(b) Show that $\ker \mathrm{d}$ is a subring A of R and that $\mathrm{d} \in \operatorname{Der}_A(R, M)$.

(c) Show that $\mathrm{d}(x^n) = nx^n \mathrm{d}x$. Suppose that $\operatorname{char} \Bbbk = n > 0$. Then $r^n \in \ker \mathrm{d}$ for every $r \in R$.

(d) Suppose that $M = R$ and that $\Bbbk$ is of prime characteristic $p > 0$. Show that $\mathrm{d}^p := \underbrace{\mathrm{d} \circ \mathrm{d} \circ \cdots \circ \mathrm{d}}_{p \text{ times}} \in \operatorname{Der}_{\Bbbk}(R)$.

(e) Show that if s is invertible in R, then $\mathrm{d}rs^{-1} = s^{-2}(s\mathrm{d}r - r\mathrm{d}s)$.

(f) Let $W \subseteq \Bbbk$ be a multiplicatively closed set such the map $\Bbbk \longrightarrow R$ factors through the map $\Bbbk \longrightarrow W^{-1}\Bbbk$. Show that $\mathrm{d} \in \operatorname{Der}_{W^{-1}\Bbbk}(R, M)$.

3.3. (5 marks) Show that the maps i and π between R and $R \ltimes M$ defined in class give isomorphisms between $\operatorname{Spec} R$ and $\operatorname{Spec}(R \ltimes N)$.

3.4. (5 marks) Write $\mathcal{H} = \{h \in \operatorname{Hom}_{\Bbbk\text{-alg}}(R, R \ltimes M) \mid \pi \circ h = \mathrm{id}_R\}$. Show that the map $\operatorname{Der}_{\Bbbk}(R, M) \longrightarrow \mathcal{H}$, $f \mapsto \hat{f}$ is a bijective correspondence.

3.5. (5 marks) Show that if R is generated as a $\Bbbk$-algebra by a subset $A \subseteq R$, then $\Omega_{R/\Bbbk}$ is generated by $\{\mathrm{d}r \mid r \in A\}$ as an R-module.

3.6. (5 marks) Let $\Bbbk$ be a field of characteristic $p > 0$, $R = \Bbbk[x^p]$ and $S = \Bbbk[x]$. Determine the modules and maps in the first fundamental exact sequence for $\Bbbk \longrightarrow R \longrightarrow S$.

3.7. (10 points) Determine $\Omega_{(R \ltimes M)/R}$ for the map $\iota : R \longrightarrow R \ltimes M, r \mapsto (r, 0)$ and for the map $\tilde{d} : R \longrightarrow R \ltimes M, r \mapsto (r, dr)$ where $d : R \longrightarrow M$ is a derivation.

3.8. (10 points) Prove the following statement using [Mat89, Theorem 26.5] or [Eis95, Theorem 16.14]: Let $L/\Bbbk$ be an extension of fields, finitely generated if $\operatorname{char} \Bbbk > 0$; if $\Omega_{L/\Bbbk} = 0$, then $L/\Bbbk$ is algebraic and separable.

3.9. (5 marks) Let S/R be an unramified extension. Show the following:

(a) For every R-ideal I, $R/I \longrightarrow S/IS$ is unramified.

(b) For every multiplicatively closed $U \subseteq R$, $U^{-1}R \longrightarrow U^{-1}S$ is unramified.

3.10. (5 marks) Let S be an R-algebra. Show that S/R is unramified if and only if for every $\mathfrak{p} \in \operatorname{Spec} R$, $S \otimes_R \kappa(\mathfrak{p})$ is a separable $\kappa(\mathfrak{p})$-algebra.

References

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