GRADUATE ALGEBRA II, JAN-APR 2018. PROBLEM SETS

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1. SET 1: DUE 2018-JAN-16

1.1. Using the distributive property, show the following, for every $x, y \in R$: $0x = x0 = 0$; $x(-y) = (-y)x = -(xy)$; $(-x)(-y) = xy$.

1.2. For $x \in R$, the *left homothety* λ_x (respectively, *right homothety* ρ_x) is the map $R \longrightarrow R$, $y \mapsto xy$ (respectively, $y \mapsto yx$). Show that these are endomorphisms of the additive group of R.

1.3. Show that $|R| = 1$ if and only if $0 = 1$, in which case $R = \{0\}$. This is the *zero ring*.

1.4. Let X be a subset of R. Show that the centralizer of X in R is a subring of R. The centre of R is a commutative subring.

1.5. Show that the endomorphism ring of the additive group $\mathbb{Z}$ is isomorphic to the ring $\mathbb{Z}$.

1.6. Let X be a subset of R. The *left annihilator* of X in R is the set $\{y \in R \mid yx = 0 \text{ for every } x \in X\}$. Show that it is a left ideal.

1.7. Let $f : R \longrightarrow S$ be a ring homomorphism. Write $\pi : R \longrightarrow R/\ker(f)$ and $\iota : \mathrm{Im}(f) \longrightarrow S$. Show that there is a ring homomorphism $\overline{f}$ such that $f = \iota\overline{f}\pi$. Show that $\overline{f}$ is an isomorphism.

1.8. Say that $x \in R$ is *left-invertible* (respectively, *right-invertible*) if there exists $y \in R$ such that $yx = 1$ (respectively, $xy = 1$). Show that x is left-invertible (respectively, right-invertible) if and only if the right homothety (respectively, left homothety)is surjective. Show that x is invertible if and only if it is left- and right-invertible. Show that in this case, the inverse of x is unique, and that this element is also the unique left- and right-inverses.

1.9. An *integral domain* is a commutative ring that is non-zero and that does not have any zero-divisors. Let R be a commutative ring and I an R-ideal. Show that the following are equivalent: (a) R/I is an integral domain; (b) For every $x, y \in R$, if $xy \in I$ and $x \notin I$, then $y \in I$; (c) I is the kernel of a ring homomorphism from R to an integral domain. A proper ideal satisfying these conditions is called a *prime ideal*. Show that maximal ideals are prime.

1.10. An *idempotent* element in R is an element e such that $e^2 = e$; an idempotent element is *central* if it belongs to the centre of R. Show that if R is a commutative ring and e an idempotent element, then for every prime ideal I of R, $e \in I$ or $1 - e \in I$, and that these conditions are mutually exclusive.

1.11. Show that the set of 2×2 complex matrices of the form

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$

(where $(\bar{\cdot}))$ denotes complex conjugation) forms a subring of $M_2(\mathbb{C})$. This is called the *quaternion ring*. Show that it can also be described as the ring of all $\mathbb{R}$-linear combinations of the following four matrices:

$$I_2, \begin{bmatrix} \imath & 0 \\ 0 & -\imath \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & \imath \\ \imath & 0 \end{bmatrix}.$$

Determine its dimension as a $\mathbb{R}$-vector space.

1.12. Let $q_1, \ldots, q_r$ be pairwise relatively prime integers. Show that the natural map $\mathbb{Z} \longrightarrow \prod_{i=1}^{r} \mathbb{Z}/q_i\mathbb{Z}$ is surjective and that it induces an isomorphism $\mathbb{Z}/(q_1 \cdots q_r)\mathbb{Z} \longrightarrow \prod_{i=1}^{r} \mathbb{Z}/q_i\mathbb{Z}$.

1.13. Let $R_i, 1 \leq i \leq n$ be rings and $R = R_1 \times \cdots \times R_n$. Show that R_i is a quotient ring of R, for each i.

1.14. Let R be a ring and S the ring of 2×2 matrices over R. Relate the centres of R and of S.

1.15. Give an example of ideals $I, J, K \subseteq \mathbb{Z}$ such that $IJ \neq I \cap J$ and $(I + J)(I + K) \neq (I + JK)$.

1.16. Let R be a ring and I the two-sided ideal generated by $\{xy - yx \mid x, y \in I\}$. Show that every ring map $R \longrightarrow S$ with S commutative has I in its kernel. Hence we can think of I as the smallest two-sided ideal such that R/I is commutative.

2. Set 2: Due 2018-Jan-30

2.1. Let $M_i, i \in \mathcal{I}$ and $N_\lambda, \lambda \in \Lambda$ be two families of R-modules. Show that the map

$$\mathrm{Hom}_R(\bigoplus_{i \in \mathcal{I}} M_i, \prod_{\lambda \in \Lambda} N_\lambda) \longrightarrow \prod_{(i,\lambda) \in \mathcal{I} \times \Lambda} \mathrm{Hom}_R(M_i, N_\lambda)$$

given by $g \mapsto \mathrm{pr}_\lambda \circ g \circ \alpha_i$ is an isomorphism of abelian groups.

2.2. Let M and N be two R-modules and suppose that M is the direct sum of submodules $M_1, \ldots, M_m$ and N the direct sum of submodules $N_1, \ldots, N_n$. By the previous exercise, $\mathrm{Hom}_R(M, N)$ can be identified with $\prod \mathrm{Hom}_R(M_i, N_j)$. Show that this identification is as follows: The element $(u_{ji}) \in \prod \mathrm{Hom}_R(M_i, N_j)$ (with $u_{ji} : M_i \longrightarrow N_j$) is determined by the maps $x_i \mapsto \sum_j u_{ji}(x_i)$ for every $x_i \in M_i$ for every i. (First observe that in order to define a map $M \longrightarrow N$, it is enough to define it on each of the M_i.) Now suppose that P is another R-module that is the direct sum of submodules $P_1, \ldots, P_p$. Let $v : N \longrightarrow P$ be an R-linear map, with canonical identification with the family (v_{kj}), with $v_{kj} : N_j \longrightarrow P_k$. Show that the composite map $v \circ u : M \longrightarrow P$ corresponds to the family $(\sum_j v_{kj} \circ u_{ji})$.

2.3. Let $M = M_1 \oplus M_2$. Show that the restriction to M_1 of the canonical surjective map $M \longrightarrow M/M_2$ is an isomorphism.

2.4. Let M_1 be a submodule of M. We say that M_1 is a *direct summand* (or, sometimes, just *summand*) if there is a submodule M_2 of M such that M is the direct sum of M_1 and M_2.

(a) Show that the submodule M_2 in the definition above need not be unique. However, any two are isomorphic to each other.

(b) For a submodule M_1 of M to be a direct summand, it is necessary and sufficient that there exists a projection $\phi \in \mathrm{End}_R(M)$ such that $M_1 = \phi(M)$ which holds if and only if there exists a projection $\phi \in \mathrm{End}_R(M)$ such that $M_1 = \ker \phi$.

2.5. Let $0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$ be an exact sequence of R-modules. Then the following are equivalent:

(a) The submodule $f(M_1)$ of M_2 is a direct summand.

(b) There exists an R-linear map $\alpha : M_2 \longrightarrow M_1$ such that $\alpha f = \mathrm{id}_{M_1}$.

(c) There exists an R-linear map $\beta : M_3 \longrightarrow M_2$ such that $g\beta = \mathrm{id}_{M_3}$.

If these conditions hold, then the map $(f + \beta) : M_1 \oplus M_3 \longrightarrow M_2$ is an isomorphism. (We say that the above exact sequence is a *split* sequence if these conditions hold.)

3. Set 3: Due 2018-Mar-22

3.1. Say that a module M is *free* if there is a subset T of M such that the natural map $R^{(T)} \longrightarrow M$ is an isomorphism; such a subset is called a *basis* of M.

3.2. Let M be a free R-module with basis $x_t, t \in T$. Let N be an R-module, and $y_t, t \in T$ elements of N. Then there exists a unique R-map $M \longrightarrow N$ such that $x_t \mapsto y_t$ for every $t \in T$.

3.3. Let $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ be an exact sequence of R-modules with M_3 free. Show that it is a split sequence.

3.4. An R-module is *simple* if it is non-zero and has no submodules different from 0 and itself. Show that if R is commutative, then the simple modules are exactly $R/\mathfrak{m}$ where $\mathfrak{m}$ is a two-sided maximal ideal.

3.5. Let $\rho : R \longrightarrow S$ be a ring map. Let $M_i, i \in \mathcal{I}$ be S-modules. Then $\rho_*(\bigoplus_{i\in\mathcal{I}} M_i) = \bigoplus_{i\in\mathcal{I}} \rho_* M_i$ and $\rho_*(\prod_{i\in\mathcal{I}} M_i) = \prod_{i\in\mathcal{I}} \rho_* M_i$.

3.6. An R-module M is *projective* if the functor $\operatorname{Hom}_R(M, -)$ is exact, i.e., takes exact sequences to exact sequences. Show that M is projective if and only if it takes short exact sequences to short exact sequences, or *equivalently*, if and only if it takes surjective R-maps to surjective R-maps. Show that free modules are projective.

3.7. Show that M is projective if and only if it is a direct summand of a free module. (Hint: Apply $\operatorname{Hom}_R(M, -)$ to a surjective map $F \longrightarrow M$ with F free.)

3.8. Let I be a two-sided R-ideal and J a left R-ideal. Show that

(a) the image of $R/I \otimes_R J$ in R/I (for the natural map $R/I \otimes_R (J \hookrightarrow R)$) is the left R/I-ideal $J(R/I)$ which is $I + J/I$;

(b) $R/I \otimes_R R/J$ is the left R- and R/I-module $R/I + J$.

(c) In particular, if $I + J = R$ (as left ideals), then $R/I \otimes_R R/J = 0$.

3.9. CAUTION: Let M and N be left R-modules. There is no canonical R-module structure (left or right) on $\operatorname{Hom}_R(M, N)$. In some sense the underlying issue is that, for R-linear $f : M \longrightarrow N$ and $r \in R$, the map $M \longrightarrow N, x \mapsto f(rx) = rf(x)$ is not necessarily R-linear. It is, if r is central. However if S is another ring, and M is ${}_RM_S$, then the definition $(s \cdot f) := [x \mapsto f(xs)]$ makes $\operatorname{Hom}_R(M, N)$ into a *left* S-module. Note (a) that the two module structures on M need to be compatible with each other; (b) that there need not be a ring morphism $R \longrightarrow S$ or $S \longrightarrow R$ for this to make sense.

3.10. Adjoint functors: Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{D}$ be functors. Say that F is the *left adjoint* to G (and similarly that G is the *right adjoint* to F) if $\operatorname{Hom}_{\mathcal{D}}(FA, B) = \operatorname{Hom}_{\mathcal{C}}(A, GB)$ for every object $A \in \mathcal{C}$ and $B \in \mathcal{D}$. For every $A \in \mathcal{C}$, putting $B = FA$, we get, corresponding to id_{FA}, a morphism $A \longrightarrow GFA$; this gives a natural transformation $\mathrm{id}_{\mathcal{C}} \longrightarrow GF$. Similary we get a natural transformation $FG \longrightarrow \mathrm{id}_{\mathcal{D}}$. For a ring morphism $\rho : R \longrightarrow S$, the constructions ρ_* and ρ^* are functors, and ρ_* is right-adjoint to ρ^*. See (Bourbaki *Algebra* Chapter II, §5, No. 1, Remark 4) for the definition of a right adjoint of ρ_*. The R-linear map $\phi_M : M \longrightarrow \rho_*(\rho^*(M))$ is an instance of the natural transformation $\mathrm{id}_{\mathcal{C}} \longrightarrow GF$. Similarly we get a map $\psi_N : \rho^*(\rho_*(N))$ for S-modules N as an instance of the natural transformation $FG \longrightarrow \mathrm{id}_{\mathcal{D}}$. Now, (Bourbaki *Algebra* Chapter II, §5, No. 2, Proposition 5) can be thought of as an instance of the following property of adjoint functors: $FA \longrightarrow FGFA \longrightarrow FA$ is id_{FA} and $GB \longrightarrow GFGB \longrightarrow GB$ is id_{GB}. (You label the arrows!)

3.11. Let $M \subseteq N \subseteq P$ be R-modules, each being a submodule of the next. Suppose that N is a direct summand of P. Then N/M is a direct summand of P/M and, if further M is a direct summand of N, then it is a direct summand of P. Now suppose that M is a direct summand of P; then it is a direct summand of N, and if additionally, N/M is a direct summand of P/M, then N is direct summand of P.

3.12. Let M and N be left R-modules and let $M^* := \operatorname{Hom}_R(M, {}_RR)$, endowed with the canonical right R-module structure. There is a natural map $\tau_{M,N} : M^* \otimes_R N \longrightarrow \operatorname{Hom}_R(M, N)$,

$f \otimes y \mapsto [x \mapsto f(x)y]$. Show that this is neither injective nor surjective in general by using the following example: $R = \mathbb{Z}/(4)$, $I = 2R$, $M = N = R/I$.

3.13. If S is an R-algebra and M and N S-modules, then the natural map $\operatorname{Hom}_S(M, N) \longrightarrow \operatorname{Hom}_R(M, N)$ is injective.

3.14. Let M be an R-module. Let C be the centre of R; then there is a natural map $C \longrightarrow \operatorname{End}_R(M)$ (but not necessarily $R \longrightarrow \operatorname{End}_R(M)$) and the C-module structure (induced from the R-module structure) is also induced from the $\operatorname{End}_R(M)$-module structure on M. Hence $\operatorname{End}_{\operatorname{End}_R(M)}(M) \subseteq \operatorname{End}_C(M)$. Now suppose that $\Bbbk$ is a field and M a finite-dimensional $\Bbbk$-vector-space. Let $R = \operatorname{End}_{\Bbbk}(M)$. Then every R-endomorphism of M is given by multiplication by an element of $\Bbbk$.

3.15. Let R be a division ring and M an R-module. Show that M is free.

4. Set 4: Due 2017-Apr-10, Preliminary version

4.1. Let E be a ring and B a subset of E. Write B' and B'' for its commutant and bicommutant respectively. Show that $B \subseteq B''$ and that B' equals its bicommutant. Suppose that B is a commutative subring of E. Then B is a central subring of B' and B'' is the centre of B'.

4.2. Let M be an R-module and N a subset of M. The *annihilator* of N is the set $\{r \in R \mid rx = 0 \text{ for every } x \in N\}$, denoted by $\operatorname{Ann}(N)$. Show that $\operatorname{Ann}(N)$ is a left ideal of R. If N is a submodule of N, then $\operatorname{Ann}(N)$ is a two-sided ideal of R.

4.3. Let R and S be rings and M and N a semisimple R-module and a semisimple S-module respectively. Show that $M \oplus N$ is a semisimple $(R \times S)$-module.

4.4. Let R be a ring and M a semisimple R-module. Let N be a simple R-module. Let M' be a submodule of M. Then the following are equivalent:

(a) M' is the largest isotypic submodule of M of type N, i.e., M' is isotypic of type N and if N' is a simple submodule of M isomorphic to N, then $N' \subseteq M'$.

(b) M' is the (direct) sum of all the simple submodules of M that are isomorphic to N.

(c) $M' = \operatorname{Hom}_R(N, M)$.

Let $N_\lambda, \lambda \in \Lambda$ be all the distinct (up to isomorphism) simple R-modules. Then $M = \bigoplus_{\lambda \in \Lambda} \operatorname{Hom}_R(N_\lambda, M)$. This is called the *isotypic decomposition* of M.

4.5. direct proof of Wedderburn : TBD

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