GRADUATE TOPOLOGY I, AUG-NOV 2016. PROBLEM SETS

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1. Set 1: For the quiz on 2016-Aug-12.

(1) Consider $\mathbb{R}$ with the standard topology, and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a map. Show that f is continuous if and only if for every $x \in \mathbb{R}$ and every $\epsilon \in \mathbb{R}_+$, there exists $\delta \in \mathbb{R}_+$ such that $f(B_{x,\delta}) \subseteq B_{f(x),\epsilon}$. Formulate and prove a similar statement for a map $f : X \longrightarrow Y$ where X and Y are metric spaces (Munkres, Theorem 21.1).

(2) The *Zariski topology* on $\mathbb{C}^n$ is the topology in which a subset $A \subseteq \mathbb{C}^n$ is closed if and only if there exists a set $I \subseteq \mathbb{C}[X_1, \ldots, X_n]$ (polynomial ring in the indeterminates $X_1, \ldots, X_n$) such that

$$A = \{\mathbf{z} \in \mathbb{C}^n \mid f(\mathbf{z}) = 0 \text{ for every } f \in I\}.$$

  (a) Check that this indeed is a topology. (Hint: For $I_1, \ldots, I_t \subseteq \mathbb{C}[X_1, \ldots, X_n]$, you might need to consider the set $J = \{f_1 \cdots f_t \mid f_i \in I_i, i = 1, \ldots, t\}$.)
  (b) This topology is strictly coarser than the standard topology on $\mathbb{C}^n$, thought of as $\mathbb{R}^{2n}$. In fact, if $n = 1$, this is the co-finite topology.
  (c) For a non-zero $f \in \mathbb{C}[X_1, \ldots, X_n]$, define $D_f := \{\mathbf{z} \in \mathbb{C}^n \mid f(\mathbf{z}) \neq 0\}$. Show that $D_f, f \in \mathbb{C}[X_1, \ldots, X_n], f \neq 0$ is a basis for the Zariski topology on $\mathbb{C}^n$.

(3) Show that the lower-limit topology on $\mathbb{R}$ is strictly finer than the standard topology on $\mathbb{R}$.

(4) Let X and Y be topological spaces and $f : X \longrightarrow Y$ be a map. Then the following are equivalent:
  (a) f is continuous;
  (b) For every basis $\mathcal{B}$ of Y, $f^{-1}(B)$ is open for every $B \in \mathcal{B}$;
  (c) For every subbasis $\mathcal{B}$ of Y, $f^{-1}(B)$ is open for every $B \in \mathcal{B}$.

(5) §13 of Munkres: Lemma 13.3; Examples 1, 2 and 4;

(6) Let X, Y and Z be topological spaces with functions $X \xrightarrow{f} Y \xrightarrow{g} Z$. Show that if f and gf are continuous and Y has the finest topology that keeps f continuous, then g is continuous. (Hint: show that this topology is $\{U \subseteq Y \mid f^{-1}(U) \text{ is open in } Y\}$.)

(7) Show that the product topology on $\mathbb{R}^n$ is finer than the metric topology given by the metric d_2 defined in class.

2. Set 2: For the quiz on 2016-Aug-26.

(1) Munkres §18: Exercises (pp. 127ff.): 11,12,13. §19: Exercises (p. 134): 10. §20: Exercises (pp. 142ff.): 1, 3, 4 (Look at Theorem 20.4 and the definitions) §22: Examples 1, 2, 3, 5, 7.

(2) Show that a continuous surjective closed map is a quotient map.

(3) Show that the map $t \mapsto (\cos 2\pi\imath t, \sin 2\pi\imath t)$ is a open map from $\mathbb{R}$ to $\mathbb{S}^1$, with the standard topologies. Hence it is a quotient map.

(4) *Product of maps*. Let $X \xrightarrow{f} Y$ and $Z \xrightarrow{g} Y$ be continuous maps. We define the *product of f and g* (also called the *fibre product of X and Z over Y*, and denoted $X \times_Y Z$) as

$$\{(x, z) \in X \times Z \mid f(x) = g(z)\}.$$

Write $f' : X \times_Y Z \longrightarrow Z$, $(x,z) \mapsto z$ and $g' : X \times_Y Z \longrightarrow X$, $(x,z) \mapsto x$. Give $X \times_Y Z$ the coarsest topology that makes f' and g' continuous.

(a) The following diagram commutes:

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{g'} & X \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & Y \end{array}$$

(b) the fibre $(f')^{-1}(z)$ over $z \in Z$ is homeomorphic to $f^{-1}(g(z))$.

(c) When Y is a point, $X \times_Y Z = X \times Z$.

(d) This topology on $X \times_Y Z$ is the subspace topology as the subspace of the product topology on $X \times Z$.

(e) For $A \subseteq X$, $f'(g'^{-1}(A)) = g^{-1}(f(A))$.

(f) The following properties of f are transferred to f': injectivity, surjectivity, openness, closedness (Feel free to add more properties!)

(g) If Z is a subspace of Y and g is the inclusion map, then $X \times_Y Z = f^{-1}(Z)$ with the subspace topology. If X and Z are subspaces of Y and f and g the inclusion maps, then $X \times_Y Z = X \cap Z$ with the subspace topology.

(5) Let a topological group G act on a topological space X; write $\pi : X \longrightarrow X/G$ for the quotient map. For every $U \subseteq X$, show that $\pi^{-1}(\pi(U)) = \cup_{g \in G} g(U)$, so $\pi(U)$ is open. Therefore π is an open map. Is it also a closed map?

(6) Let X be a topological space. The *diagonal map* to $X \times X$ (in the product topology) is the map $\delta : X \longrightarrow X \times X$, $x \mapsto (x,x)$. Show that δ gives a homeomorphism from X to its image with the subspace topology. Show that X is Haudorff if and only if the $\operatorname{Im}(\delta)$ is closed in $X \times X$

3. Set 3: For the quiz on 2016-Sep-09.

(1) Munkres §23: Example 7; Theorem 23.6; Exercises 1–4, 7, 9, 11, 12.

(2) Munkres §24: Theorem 24.1; Examples 3, 4, 5, 7; Exercises 1, 2 (Hint: consider $t \mapsto f(t) - f(-t)$.), 3 (Hint: consider $t \mapsto t - f(t)$.), 4 (Look up the definition of order topology in §14; at least try to prove for $\mathbb{R}$.) 8, 9, 10, 11.

(3) Munkres §25: Theorems 25.3, 25.4; Exercise: 8.

(4) Munkres §26: Exercises 1, 3, 4, 5, 6 (Further conclude that if f is surjective, then it is an open map, and, then, Theorem 26.6.), 7, 8, 11, 12.

4. Set 4: For the quiz on 2016-Oct-07.

(1) Munkres §29: Exercises 1, 2, 3, 5, 6, 8, 10

(2) Munkres §30: Theorem 30.1, 30.2, 30.3(b) (See the next problem); Exercises 1(a), 2, 3, 4, 5(a).

(3) Munkres §33: Exercises 1,2,3

(4) Let $\mathcal{B}$ be a basis for the topology on X. Show that a subset A of X is dense if and only if $A \cap U \neq 0$ for every $U \in \mathcal{B}$.

5. Set 5: For the quiz on 2016-Oct-21.

(1) Munkres §31: Exercises 1, 2, 4, 5, 6, 7.

(2) Munkres §32: Exercises 1, 2, 3.

(3) Munkres §33: Exercises 8.

(4) Munkres §34: Exercises 3, 4, 5.

(5) Munkres §35: Exercises 3, 4.

6. Set 6: For the quiz on 2016-Nov-11.

(1) Let Y be a Haudorff space and $\alpha, \beta : X \longrightarrow Y$ be continuous maps. Then the set of points $x \in X$ with $\alpha(x) = \beta(x)$ is a closed set in X.

(2) Let a group G act X such that for every $x \in X$ there exists a neighbourhood U such that the orbits $g(U)$ are disjoint, i.e, for every $g \neq g' \in G$, $g(U) \cap g'(U) = \varnothing$. (Definition: G is said to act *properly discontinuously* on X, if this is the case.) Show that the quotient map $X \longrightarrow X/G$ is a covering map.

(3) (a) Check that the topology on $\mathbb{RP}^2$ defined in the class makes it into a quotient space of $\mathbb{S}^2$.
(b) Let $\mathbb{Z}/2\mathbb{Z}$ act on $\mathbb{S}^2$ with $\overline{1} \cdot x = -x$. Show that this action is properly discontinous and that the quotient space for this action is $\mathbb{RP}^2$.

(4) Show that covering maps are open maps. In particular, they are quotient maps.

(5) Show that every fibre of a covering map is discrete.

(6) Read Munkres Section 59.

(7) Let Y be a compact space and let $\{U_\alpha\}$ be an open cover of $Y \times I$. We show that there exists a finite cover $\{V_\beta\}$ of Y of connected open sets and real numbers $0 = t_0 < t_1 < \cdots < t_k = 1$ such that for each β and each i, there exists α such that $V_\beta \times [t_{i-1}, t_i] \subseteq U_\alpha$ as follows. (This argument is due to Yash.)
(a) Each U_α is a union of open sets of the form $W_\gamma \times (s_0, s_1)$ where the W_γ are open subsets of Y and $0 \leq s_0 < s_1 \leq 1$. Therefore, without loss of generality, we may assume that every U_α is of the form $W_\gamma \times (s_0, s_1)$ where the W_γ are open subsets of Y and $0 \leq s_0 < s_1 \leq 1$.
(b) Let $y \in Y$. Then there exists a finite subcollection $\mathcal{U}_y$ of $\{U_\alpha\}$ that covers $\{y\} \times I$. Hence, arguing as in the proof of the tube lemma, there is a connected open neighbourhood $V_y \subseteq Y$ of y such that $V_y \subseteq \bigcup_{U \in \mathcal{U}_y} U$. From the cover $\{V_y\}$, pick a finite cover $\{V_\beta\}$ of Y. Write $\mathcal{U}_\beta$ for the corresponding $\mathcal{U}_y$.
(c) Write p_2 for the projection map $Y \times I \longrightarrow I$. For each β, $\{p_2(U) \mid U \in \mathcal{U}_\beta\}$ is an open cover of I. Let ρ_β be a Lebesgue radius for this open cover. Let $\rho = \min_\beta \rho_\beta$. Therefore, for each closed interval of length at most 2ρ and for every V_β, their product is contained in some $U \in \mathcal{U}_\beta$.

(8) This is the proof that $G_1^\alpha|_{V_\alpha \cap V_\beta} = G_1^\beta|_{V_\alpha \cap V_\beta}$. We may assume that $V_\alpha \cap V_\beta \neq \varnothing$. For $\gamma = \alpha, \beta$, there exists l_γ such that $F(V_\gamma \times \{t_0\}) \subseteq U_{l_\gamma}$. Let W_γ be the open set of $\tilde{X}$ that is part of $p^{-1}(U_{l_\gamma})$ and gets mapped homeomorphically to U_{l_γ} by p. Therefore $G_1^\gamma(V_\alpha \cap V_\beta \times \{t_0\}) \subseteq W_\alpha \cap W_\beta$, $\gamma = \alpha, \beta$. For every $y \in V_\alpha \cap V_\beta$ and $s \in [t_0, t_1]$, $\{y\} \times [t_0, s] \subseteq W_\alpha \cap W_\beta$ since it is connected, so $G_1^\gamma(V_\alpha \cap V_\beta \times [t_0, t_1]) \subseteq W_\alpha \cap W_\beta$, $\gamma = \alpha, \beta$. Since $p|_{W_\alpha \cap W_\beta}$ is injective, and $p \circ G_1^\gamma = F|_{V_\gamma \times [t_0, t_1]}$, we see that $G_1^\alpha|_{V_\alpha \cap V_\beta} = G_1^\beta|_{V_\alpha \cap V_\beta}$.

(9) Munkres §52: Exercises 1, 3 (Look up the definition of $\hat{\gamma}$ earlier in this section; this what we denoted (sometimes) by $\gamma_\sharp$), 5, 7.

(10) Munkres §54: Exercise 8.

(11) Munkres §55: Theorem 55.2

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