

TUTORIAL 5 (2026-OCT-09)

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1. MEETING FOUR LINES

Let us calculate of the number of lines in $\mathbb{P}^3$ meeting four given general lines using M2.

A line $\ell \subseteq \mathbb{P}^3$ is specified by an ideal generated by two linearly independent linear forms in $x_0, \dots, x_3$. Hence we can represent it (non-uniquely) as a full-rank 4×2 matrix A_ℓ over $\mathbb{k}$. Two distinct lines ℓ, ℓ' meet if and only if the 4×4 matrix $\begin{bmatrix} A_\ell & A_{\ell'} \end{bmatrix}$ has rank exactly 3.

1.1. **Grassmannian $G_{2,4}$.** Consider

$$(1) \quad \begin{bmatrix} a_0 & b_0 \\ a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$$

For $0 \leq i < j \leq 3$, write $p_{i,j} = \det \begin{bmatrix} a_i & b_i \\ a_j & b_j \end{bmatrix}$. Define a polynomial ring $S := \mathbb{k}[a_0, \dots, a_3, b_0, \dots, b_3, p_{ij}]$. Find an S -ideal I such that eliminating the a_i and the b_j from I will give the equations defining $G_{2,4}$.

1.2. **Intersecting with a given line.** We think of (1) as representing a line to be determined. Use `random` appropriately to represent four general lines. Impose the condition that each of them intersects the line given by (1). Put all of these conditions together as an S -ideal J .

1.3. **Elimination.** Eliminate the a_i and the b_j from $(I + J)$. Do you see distinct points on the Grassmannian $G_{2,4}$ or one point with high multiplicity?

2. BITANGENTS

A line $\ell \subseteq \mathbb{P}^n$ is a *bitangent* line of a curve $C \subseteq \mathbb{P}^n$ if it is tangent at (at least) two points on C . Find an irreducible $F \in \mathbb{k}[x, y, z]$ such that the line $x = 0$ is a tangent to the curve $F = 0$ at $[0 : 0 : 1]$ and at $[0 : 1 : 1]$.

Hint: do this on $\mathbb{A}^2 = \{z \neq 0\}$ and then homogenize.

Next, by trial and error, find an F that defines a non-singular curve. (For this doing it on $\mathbb{A}^2$ might not help.)