

TUTORIAL 4 (2026-OCT-08)

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1. INTERSECTION FAILURE AND PROJECTIVE DIMENSION

Verify the claims about the following example from class by computing in M2. In $S = \mathbb{Q}[x_0, \dots, x_4]$, take ideals $I_{L_1} = (x_1, x_2)$, $I_{L_2} = (x_3, x_4)$, $I_Y = I_{L_1} \cap I_{L_2}$, and $I_X = (x_1 + x_3, x_2 + x_4)$.

- (1) Use the Proj command to define ProjectiveVariety objects corresponding to these ideals. Compute their degrees.
- (2) Compute a free resolution of S/I_Y and conclude that it is not Cohen-Macaulay.
- (3) Tensor it with S/I_X and take homology to see the Tor modules. Use degree to compute their multiplicities and take the alternating sum. Check that this agrees with $\deg(X \cap Y)$.

2. EIGHT GENERAL POINTS

See Chapter 3, Part A of Eisenbud, *Geometry of Syzygies*. Over $S = \mathbb{k}[x, y, z]$, define
`Phi = random(S^{-3, -3, -4}, S^{-5, -5});`
`I = minors(2, Phi);`

- (1) Compute the free resolution of I and the degree of $\text{Proj } S/I$. Let J_3 be the saturation of the ideal generated by the cubics inside I . Compute $\deg S/J_3$. Compute $\text{degree } (I/J_3)$. Write $\mathcal{I}$ be the sheaf of ideals inside $\mathcal{O}_{\mathbb{P}^2}$ defined by I . This means that the map

$$\mathcal{O}_{\mathbb{P}^2}(-3)^2 \rightarrow \mathcal{I} \text{ or, equivalently, } \mathcal{O}_{\mathbb{P}^2}^2 \rightarrow \mathcal{I}(3)$$

given by the cubics inside I cannot be surjective, i.e., $\mathcal{I}(3)$ is not generated by global sections.

- (2) Repeat the same with J_4 the saturation of the ideal generated by all the quartics in I . Show that $J_4 = I$. This implies that $\mathcal{I}(4)$ is generated by global sections.
- (3) Compute a minimal free resolution of I and check that $\text{reg}(I) = 4$.

This is an illustration of Mumford's theorem: If a coherent sheaf $\mathcal{F}$ is m -regular, then $\mathcal{F}(m)$ is generated by global sections.

Note: This is only a sufficient condition. In the above computation, you have already seen an example of coherent sheaf $\mathcal{F}$ and an integer m such that $\mathcal{F}$ is not m -regular but $\mathcal{F}(m)$ is generated by global sections.