

TUTORIAL 3 (2026-OCT-07)

MANOJ KUMMINI

1. RANK-ONE 3×3 MATRICES

Define $R = \mathbb{K}[x_{0,0}, \dots, x_{2,2}, y_{0,0}, \dots, y_{2,1}]$ in M2. Using `apply`, define X to be the 3×3 matrix x_{ij} and Y to be the 3×2 matrix y_{ij} . We want to emulate what we did for non-full-rank matrices to find the locus rank-one 3×3 matrices.

- (1) Define J to be the ideal generated by entries of XY . Do this in at least two different ways in M2. For $j = 1, 2$, define I_j to be the ideal defined by the $j \times j$ minors of Y .
- (2) Run each of the commands and explain the answer you get in each case:
`eliminate(Ys, J)`
`eliminate(Ys, saturate(J, I1))`
`eliminate(Ys, saturate(J, I2))`
- (3) Now compute the associated primes of J and understand where they come from. What did each of the saturation commands above achieve?

2. FLAT LIMIT OF TWO SKEW LINES

Define the following family of lines inside $\mathbb{A}^3$ parametrized by t . (Eisenbud-Harris, *Geometry of Schemes*, II-25.)

```
R = QQ[t,x..z];
L_t = ideal "y,z-t";
M = ideal "x,z";
Xt = saturate(intersect(L_t, M), t)
```

`saturate` is to ensure that the ring R/X_t is flat over $\mathbf{Q}[t]$, but it is superfluous in this example. Show that for $a \neq 0$, X_a defines two skew lines inside $\mathbb{A}^3$.

```
X1 = trim sub(Xt, t=>1)
primaryDecomposition X1
X0 = trim sub(Xt, t=>0)
primaryDecomposition X0
```

Bonus question! Explain by looking at the output of
`reduceHilbert hilbertSeries X1`

why X_0 has an embedded component associated to (x, y, z) . You have to use the fact that the Hilbert series is constant in flat families and something about the Hilbert series of Cohen-Macaulay rings.

3. GROBNER DEGENERATION

Check the calculation done in the lecture about the weight order

```
R = QQ[t,x,y];
f1 = poly "x2-y";
f2 = poly "xy-1";
wts = {1,3,1};
{f1, f2} / (f -> homogenize(f, t, wts))
```

```

It = ideal oo;
Jt = saturate(It, t)
trim sub(It, t=>1)
trim sub(It, t=>0)
trim sub(Jt, t=>1)
trim sub(Jt, t=>0)

```

One of the ideals above does not even have the correct dimension. Show that R/J_t is free of rank 3 over $\mathbf{Q}[t]$. (Constancy of dimension is related to flatness.)

4. GOOD INTERSECTIONS

Let $C = V(x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_2^2 - x_1x_3) \subset \mathbb{P}^3$ (over $\mathbf{Q}$ or a field with many elements). Compute the Hilbert polynomial, Hilbert series, degree and dimension of C .

Using `random`, find a sufficiently general hyperplane subspace L .

Compute the Hilbert polynomial, Hilbert series, degree and dimension of $C \cap L$.

Do the same for $H = V(x_3)$. Numerical invariants are the same as with L , but check using M2 that

- (1) $C \cap L$ consists of three reduced points.
- (2) $C \cap H$ consists one point, but with local multiplicity 3.

Now prove these statements suggested by M2 calculation. (Hint: For the second part, look at the affine space $x_0 \neq 0$.)

5. BAD INTERSECTIONS

Let $I \subseteq \mathbb{K}[a, b, c, d]$ be the defining ideal of the curve $[s^4 : s^3t : st^3 : t^4]$. Check the following

```

degree I
degree (I + ideal "a")
degree (I + ideal "a,d")
reduceHilbert hilbertSeries I
reduceHilbert hilbertSeries (I + ideal "a")
reduceHilbert hilbertSeries (I + ideal "a, d")

```

and explain the answers you get. (Hint: look also at `betti res I`.)