

TUTORIAL 2 (2026-OCT-06)

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These exercises illustrate the use of elimination.

1. TWO PROJECTIONS OF THE TWISTED CUBIC

Let $C \subset \mathbb{P}^3$ be parametrized by $[s : t] \mapsto [s^3 : s^2t : st^2 : t^3]$.

1.1. **Secant projection.** Compute the ideal of the image under the projection π_q from $q = [1 : 0 : 0 : 1]$ as follows:

- (1) Show that the map is $[X_0 : X_1 : X_2 : X_3] \mapsto [X_1 : X_2 : X_0 - X_3]$. The map as I stated in the lecture is correct. π_q *should not be* defined at q . An equation for a complementary hyperplane is $X_0 + X_3 = 0$. Hence we can take $[X_1 : X_2 : X_0 - X_3 : X_0 + X_3] = [Y_0 : Y_1 : Y_2 : Y_3]$ as homogeneous coordinates for $\mathbb{P}^3$. Show that projection from the point $[Y_0 : Y_1 : Y_2 : Y_3] = [0 : 0 : 0 : 1]$ to the hyperplane $Y_3 = 0$ is given by $[Y_0 : Y_1 : Y_2 : Y_3] \mapsto [Y_0 : Y_1 : Y_2]$. (E.g., if $p = [a : b : c : d] \neq q$, then find two linear polynomials that define the line $\overline{pq}$; show that $\overline{pq} \cap \{Y_3 = 0\} = [a : b : c : 0]$.)
- (2) Write the induced parametrization in $\mathbb{P}^2$.
- (3) Compute the defining ideal in M2 of the image by elimination.
- (4) Find the singular point using M2.
- (5) Determine its inverse image in C .

1.2. **Tangent projection.** Repeat the above exercise for the projection from $[0 : 0 : 1 : 0]$.

2. SINGULAR MEMBERS IN A TWO-PARAMETER FAMILY

Consider the family of plane cubics $C_{a,b} : F = y^2z - x^3 - axz^2 - bz^3 = 0$, with parameters $(a, b) \in \mathbb{A}^2$. The goal is to determine the locus in parameter space for which $C_{a,b}$ is singular.

- (1) Compute the partial derivatives F_x, F_y, F_z by hand and by M2 (? diff).
- (2) Determine all the points inside $\mathbb{P}^2$ where $F_x = F_y = F_z = 0$. (Recall that this is equivalent to finding such points inside $V(F)$.) Show that these lie in the open affine set $z \neq 0$, and $y = 0$ at these points. (This is independent of the values of a and b .)
- (3) Simplify the $F_x = F_y = F_z = 0$ substituting $z = 1$ and $y = 0$. (? sub in M2).
- (4) Eliminate to get the values of (a, b) such that the projective curve $C_{a,b} \subseteq \mathbb{P}^2$ is singular. (You should get $4a^3 + 27b^2 = 0$.)
- (5) Further curious things: Determine the singularities of the curve $4a^3 + 27b^2 = 0$ inside $\mathbb{A}^2$. I.e., which points are singular, and are they cusps or nodes?
- (6) If $4a^3 + 27b^2 = 0$ and $(a, b) \neq (0, 0)$, then $C_{a,b}$ is nodal; $C_{0,0}$ is cuspidal. (Use M2 to look at the system $(F, 4a^3 + 27b^2)$ on the affine space $z \neq 0$.)

3. PROVING PRIMENESS USING THE JACOBIAN CRITERION

This follows the strategy of Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Theorem 18.15.

Let $S = \mathbf{Q}[x_0, x_1, x_2, x_3]$, $I = (x_0^2 + x_1^2 + x_2^2 + x_3^2, x_1^2 + 2x_2^2 + 3x_3^2)$ and $R = S/I$. The goal is to prove that I is prime without asking M2 directly whether it is prime.

- (1) Show that I is a complete intersection, both by hand and using M2. Hence R is CM.
- (2) Using M2, compute the jacobian matrix J of I and determine the singular locus of R (which equals $V(I + I_2(J))$, since $\text{ht } I = 2$). ? jacobian, ?minors.
- (3) Using Serre's criterion, conclude that R is a product of normal domains.
- (4) Since R is positively graded and R_0 a field, R cannot have non-trivial idempotents.
- (5) Hence I is a prime ideal.

4. DETERMINANTAL VARIETY

We discussed in the lecture that if we naively consider the equations

$$(1) \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

inside $\mathbb{A}^4 \times \mathbb{A}^2$ and project to $\mathbb{A}^4$, we will not get the locus of non-full-rank matrices inside $\mathbb{A}^4$.

- (1) See this explicitly using M2 by eliminating x, y from the system given by (1) using M2.
- (2) Prove or disprove the following statement: if $I \subseteq (x_1, \dots, x_n) \subseteq \mathbb{k}[x_1, \dots, x_n, y_1, \dots, y_m]$ where the x_i and the y_j form disjoint sets of variables, $\text{eliminate}(\{x_1, \dots, x_n\}, I)$ is the zero ideal.
- (3) Preview for tomorrow: In order to use projection correctly in the determinantal situation, we need to get $(ax + by, cx + dy)$ 'outside' (x, y) .