INTERSECTION THEORY AND AMPLITUDE ON SURFACES

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These is the penultimate version of notes, prepared for the Seshadri Constants Workshop. We closely follow Chapter V.1 of Hartshorne's *Algebraic Geometry*.

1. Intersection number of two line bundles on a surface

By a surface, we will mean (unless otherwise stated) a nonsingular projective surface over an algebraically closed field k. By a curve we mean a reduced, irreducible curve. Given a surface X, we let $Pic(X)$ denote the abelian group of (isomorphism classes of) line bundles on X.

For the reader's convenience, whenever possible I adopt the notations from Hartshorne's book.

Theorem 1.1. *Let X be a surface.There is a unique symmetric bilinear map $Pic(X) \times Pic(X) \to \mathbb{Z}$:*

$$(L_1, L_2) \mapsto L_1.L_2$$

such that if C_1 and C_2 are curves intersecting transversally and $L_i = \mathcal{O}_X(C_i)$, then

$$L_1.L_2 = \#(C_1 \cap C_2), \textit{the number of points of } C_1 \cap C_2$$

Proof. Let $VA(X) \subset Pic(X)$ be the semigroup of very ample line bundles. If $L_i \in VA(X)$, there exist, by Bertini, sections σ_i such that the corresponding divisors C_i are nonsingular curves intersecting transversally. Further

$$\#(C_1 \cap C_2) = deg\ L_1|_{C_2} = deg\ L_2|_{C_1}$$

This shows that $\#(C_1 \cap C_2)$ is independent of the choice of the σ_i as long as the C_i intersect transversally. For $L_i \in VA(X)$ define

$$L_1.L_2 = \#(C_1 \cap C_2)$$

with C_i chosen as above. If $L_1, L_1' \in VA(X)$, we have

$$(L_1 \otimes L_1').L_2 = deg\ (L_1 \otimes L_1')|_{C_2} = deg\ L_1|_{C_2} + deg\ L_1'|_{C_2} = L_1.L_2 + L_1'.L_2$$

This shows that $(L_1, L_2) \mapsto L_1.L_2$ is bi-additive on $VA(X)$.

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Let M be an ample line bundle. Given any line bundle L, there exists a power M^m such that both $L \otimes M^m$ and M^m are very ample, so that $L = (L \otimes M^m) \otimes M^{-m}$ is "a difference of two very ample line bundles". The theorem is now a consequence of the next lemma. □

Lemma 1.2. *Let A be an abelian group, $S \subset A$ a sub-semigroup, and $S \times S \to \mathbb{Z}$ a symmetric bi-additive map, $(l_1, l_2) \mapsto l_1.l_2$. Suppose that the map $S \times S \to A$, $(a, b) \mapsto a - b$ is surjective. Then the map $(l_1, l_2) \mapsto l_1.l_2$ extends uniquely to a symmetric bilinear map*

$$A \times A \to \mathbb{Z}$$

Proof. If such an extension exists, given $c_i = a_i - b_i$, $i = 1, 2$, with $a_i, b_i \in S$, we must have

$$c_1.c_2 = a_1.a_2 + b_1.b_2 - a_1.b_2 - b_1.a_2$$

so the extension, if it exists, is unique. It now suffices to prove the claim: the RHS depends only on the c_i. Suppose then that $c_1 = a'_1 - b'_1$ with $a'_1, b'_1 \in S$. Then $a'_1 + b_1 = a_1 + b'_1 = d$ (say), with $d \in S$. Then

$$\begin{aligned} a'_1.a_2 + b'_1.b_2 - a'_1.b_2 - b'_1.a_2 &= d.a_2 - b_1.a_2 - a_1.b_2 + d.b_2 \\ &\quad + b_1.b_2 - d.b_2 + a_1.a_2 - d.a_2 \\ &= a_1.a_2 + b_1.b_2 - a_1.b_2 - b_1.a_2 \end{aligned}$$

so the claim stands proved. □

Notation: Given divisors D_1, D_2, we set

$$D_1.D_2 = L_1.L_2$$

where L_i is the line bundle $\mathcal{O}_X(D_i)$.

Adjunction. Let Ω_X denote cotangent bundle of X. The canonical bundle K_X is the determinant bundle $det\ \Omega_X$; if $K_X = \mathcal{O}_X(K)$ (i.e., K is the divisor defined by a meromorphic 2-form on X), we call K "the" canonical divisor. If C is a nonsingular curve in a surface X, we have an exact sequence of bundles on C:

$$0 \to \mathcal{N}_C^* \to \Omega_X \to K_C \to 0$$

where the conormal bundle $\mathcal{N}_C^*$ in turn is $\mathcal{O}(-C)|_C$, the isomorphism being given by

$$f \mapsto df|_C$$

where f is a section of the ideal sheaf $\mathcal{O}(-C)$. From the above exact sequence, we get

$$K_X|_C = K_C \otimes \mathcal{O}(-C)|_C$$

which in turn yields

$$K.C = -C.C + deg\ K_C = -C.C + 2g_C - 2$$

Equivalently,

$$g_C = \frac{1}{2} C.(C + K) + 1$$

2. Riemann-Roch on surfaces

Given a coherent sheaf $\mathcal{F}$ on X, we set $h^i(X,\mathcal{F}) = dim_k H^i(X,\mathcal{F})$. We define the Euler characteristic to be

$$\chi(\mathcal{F}) = h^0(X,\mathcal{F}) - h^2(X,\mathcal{F}) + h^2(X,\mathcal{F})$$

As before, we let K be the divisor defined by a meromorphic 2-form on X, so that $\mathcal{O}_X(K)$ is the canonical bundle K_X.

Theorem 2.1. *Let D be a divisor, and $L = \mathcal{O}_X(D)$. Then*

$$\chi(L) - \chi(\mathcal{O}_X) = \frac{1}{2}D.(D-K) = \frac{1}{2}(L.L - L.K_X)$$

Proof. Write $D = C - E$, with C, E nonsingular curves of genera g_C and g_E respectively. We have exact sequences:

$$0 \to \mathcal{O}_X(C-E) \to \mathcal{O}_X(C) \to \mathcal{O}_X(C)|_E \to 0$$

and

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_X(C)|_C \to 0$$

which yields

$$\begin{aligned}
\chi(L) - \chi(\mathcal{O}_X) &= \chi(\mathcal{O}_X(C)|_C) - \chi(\mathcal{O}_X(C)|_E) \\
&\underset{a}{=} C.C + 1 - g_c - C.E - 1 + g_E \\
&= C^2 - C.E + g_E - g_C \\
&\underset{b}{=} C^2 - C.E + \frac{1}{2}\{E.(E+K) - C.(C+K)\} \\
&= \frac{1}{2}\{C^2 - 2C.E + E^2 + E.K - C.K\} \\
&= \frac{1}{2}\{(C-E)^2 - (C-E).K\} \\
&= \frac{1}{2}D.(D-K)
\end{aligned}$$

where the equality a uses the Riemann-Roch for curves and the equality b uses Adjunction. □

Lemma 2.2. *Let H be an ample divisor, and D a divisor satisfying $D.H > 0$ and $D^2 > 0$. Then mD is linearly equivalent to an effective divisor for large enough m.*

Proof. Since $D.H$ is strictly positive, $(K - mD).H$ is negative for m large enough so $K - mD$ cannot be effective. So $h^2(\mathcal{O}_X(mD)) = h^0(\mathcal{O}_X(K - mD)) = 0$ for $m \geq m_0$. By Riemann-Roch applied to mD, we have for $m \geq m_0$:

$$h^0(\mathcal{O}_X(mD)) = h^1(\mathcal{O}_X(mD)) + \frac{1}{2}mD(mD-K) + \chi(\mathcal{O}_X)$$

Since $D^2 > 0$, the m^2 term dominates for large m and the RHS is strictly positive. This proves the Lemma. □

The combination $p_a = p_a(X) \equiv \chi(\mathcal{O}_X) - 1$ is called the arithmetic genus of X.

2.1. **Cup product and Intersection form; the Hodge Index Theorem.** If $k = \mathbb{C}$, the surface X is a compact oriented real 4-manifold, and the cup product induces a symmetric bilinear form on $H^2(X, \mathbb{Z})$:

$$H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to H^4(X, \mathbb{Z}) = \mathbb{Z}$$

This induces a perfect (unimodular) pairing:

$$H^2(X, \mathbb{Z})^{tf} \times H^2(X, \mathbb{Z})^{tf} \to H^4(X, \mathbb{Z}) = \mathbb{Z}$$

where $H^2(X, \mathbb{Z})^{tf}$ is $H^2(X, \mathbb{Z})$ modulo torsion. Let $\mathcal{Q}$ denote the corresponding quadratic form. If X is a projective surface, any ample line bundle L defines a class $c_1(L)$ in $H^2(X, \mathbb{Z})$ such that $\mathcal{Q}(c_1(L)) > 0$; the Hodge index Theorem of Kähler geometry states that the above intersection form is *negative-definite on the orthogonal complement*. As we noted above, $Amp(X) \subset Num(X) \subset H^2(X, \mathbb{Z})^{tf}$. The intersection form constructed *via* algebraic geometry agrees with the form constructed *via* cup-product. So the Hodge Index Theorem implies the following statement over $\mathbb{C}$, which is in fact true for arbitrary closed k. This proof is due to Grothendieck.

Theorem 2.3. *Let H be an ample divisor, and D a divisor, with $[D] \neq 0$ in $Num(X)$ and $D.H = 0$. Then $D^2 < 0$.*

Proof. Suppose, by contradiction, that $D^2 \geq 0$. We consider two cases:

If $D^2 > 0$, consider $H' = D + nH$. This is ample for n large enough, and $D.H' > 0$. By Lemma 2.2, mD is effective for large enough m. But then $(mD).H > 0$, which contradicts the hypothesis of the Theorem.

If $D^2 = 0$, choose a divisor E such that $D.E \neq 0$ and $E.H = 0$ – such a divisor exists since $[D]$ is assumed nonzero in $Num(X)$ – and consider $D' = nD + E$. Then $D'.H = 0$ and $D'^2 = 2nD.E + E^2$, so that for suitable $n \in \mathbb{Z}$, $D'^2 > 0$. This yields a contradiction as above. □

3. Nakai-Moishezon Criterion for Amplitude

Suppose $X \subset \mathbb{P}(V)$ is a surface embedded in a projective space, and $C \subset X$ a curve. By Bertini the generic hyperplane cuts X in a nonsingular curve H, intersecting C transversally, and we can choose a second hyperplane again cutting X in a nonsingular curve H', intersecting H transversally. Clearly $\mathcal{O}(H) = \mathcal{O}(1)|_X$, $H.C = deg\ \mathcal{O}(1)|_C = \#(H \cap C) > 0$ and $H^2 = \#(H \cap H') > 0$.

Suppose now that H is an ample divisor on a surface X and $C \subset X$ a curve as before. By definition, there is an embedding $X \subset \mathbb{P}(V)$ such that $\mathcal{O}(1)|_X$ is isomorphic to $\mathcal{O}(mH)$ for some $m \geq 1$. Then mH is linearly equivalent to a hyperplane section, so

$$H.C > 0 \textit{ and } H^2 > 0$$

The Nakai-Moishezon criterion is a converse to this:

Theorem 3.1. *A divisor D on a surface is ample iff $D^2 > 0$ and $D.C > 0$ for all curves $C \subset X$.*

Proof. Let H be an ample divisor. Since some multiple of H is effective (and in fact a sufficient large multiple is represented by a nonsingular curve), $H.D > 0$. So Lemma 2.2 applies and mD is linearly equivalent to an effective divisor for m large enough. After relabelling we can henceforth assume D itself is effective – this has the consequence that the sheaf $\mathcal{O}_X(-D)$ is a sheaf of ideals.

Let $L \equiv \mathcal{O}_X(D)$, and let σ denote the regular section (unique upto nonzero scalar) of L such that $(\sigma) = D$.

We need to prove that L is ample. By assumption, L has positive degree when restricted to any curve and is therefore ample on it. By Lemma ? it is ample on any one-dimensional subscheme.

Consider the exact sequence:

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

which defines the one-dimensional subscheme $D \subset X$. By the above remarks, $L|_D$ is ample, and therefore there exists m_0 s.t. $H^1(D, L^m) = 0$ for $m \geq m_0$. Note that $\mathcal{O}_X(-D) = L^{-1}$, so the above sequence can also be written:

$$0 \to L^{-1} \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

Tensor by L^m and consider the corresponding cohomology sequence. For $m \geq m_0$ we get

$$0 \to H^0(X, L^{m-1}) \to H^0(X, L^m) \to H^0(D, L^m|_D) \to H^1(X, L^{m-1}) \to H^1(X, L^m) \to 0$$

This shows that the maps $H^1(X, L^{m-1}) \to H^1(X, L^m)$ (given by multiplying by the defining section of $L = \mathcal{O}(D)$) are surjective for $m \geq m_0$. Since the k vector spaces $H^1(X, L^{m-1})$ are finite-dimensional, eventually the maps are isomorphisms, and thenceforth the restriction map $H^0(X, L^m) \to H^0(D, L^m_D)$ is eventually onto.

Choose m large enough that

(1) $H^0(X, L^m) \to H^0(D, L^m_D)$ is surjective, and

(2) $L^m|_D$ is very ample on D.

We temporarily reinstate the notational distinction between a line bundle L and the corresponding sheaf of sections $\underline{L}$. Consider the (evaluation) map on X:

$$H^0(X, \underline{L}^m) \otimes_{\mathbb{C}} \mathcal{O}_X \to \underline{L}^m$$

The section σ^m generates $\underline{L}^m$ on the complement of D. Let now $x \in D$, and consider the commutative diagram of maps of stalks:

$$\begin{array}{ccc} H^0(X, \underline{L}^m) \otimes_{\mathbb{C}} (\mathcal{O}_X)_x & \xrightarrow{E_x} & (\underline{L}^m)_x \\ \downarrow R_x & & \downarrow r_x \\ H^0(X, \underline{L}^m|_D) \otimes_{\mathbb{C}} (\mathcal{O}_D)_x & \xrightarrow{e_x} & (\underline{L}^m|_D)_x \end{array}$$

The maps e_x and R_x are surjective by design and r_x is surjective because D is a closed subscheme. This proves surjectivity of E_x.

We conclude that L^m is globally generated. Let $\Phi : X \to \mathbb{P}(H^0(X, L^m))$ be the corresponding map. Under this map $\mathcal{O}(1)$ pulls back to $\underline{L}^m$, so if we prove that Φ is finite, it would follow that L^m is ample and hence L itself. Since Φ is a map of projective varieties, it suffices[1] to show that Φ has finite fibres. If it were otherwise, there would exist a curve $C \in X$ mapping to a point in $\mathbb{P}(H^0(X, L^m))$ and L^m would be trivial on C, which would contradict the hypothesis that L has positive degree on every curve. □

4. Nef line bundles

Definition A divisor D is *nef* if $D.C \geq 0$ for every curve. (Equivalently, a nef line bundle is one whose restriction to every curve has non-negative degree.)

The next result is due to Kleiman:

Theorem 4.1. *If D is nef, we have $D^2 \geq 0$.*

Proof. Let H be a very ample divisor, and consider, for $t \in \mathbb{R}$,

$$P(t) = (tH + D)^2$$

Expanding the expression on the right we get

$$P(t) = t^2 H^2 + 2tD.H + D^2$$

Since H is ample, $H^2 > 0$. By Bertini we can suppose H is a nonsingular curve, so $D.H \geq 0$. So $P'(t) \geq 0$ on $t \geq 0$.

[1]Hartshorne's Exercise III.11.2; this needs Stein factorisation!

If D^2 were strictly negative, we would have $P(0) < 0$, and P would have a real positive root t_0, with $P(t) > 0$ for $t > t_0$. For $t = \frac{a}{b} > t_0$, with a, b positive integers, we have

$$P(t) = \frac{1}{b^2}(aH + bD)^2 > 0$$

which yields $(aH + bD)^2 > 0$. In addition, $(aH + bD).C > 0$ for any curve C, so that by Nakai criterion the divisor $aH + bD$ is ample. This has the consequence, which we will use in a minute, that for t as above

$$D.(tH + D) = \frac{1}{b}D.(aH + bD) \geq 0$$

By continuity, $D.(t_0H + D) \geq 0$. To reach a contradiction, write (for t real from now on)

$$P(t) = R(t) + Q(t)$$

where $R(t) = t^2H.H + tH.D$ and $Q(t) = D.(tH + D)$. Since $H^2 > 0$ and $H.D \geq 0$, we have $R(t_0) > 0$. Since $Q(t_0) \geq 0$ we have the desired contradiction. □

5. Seshadri's criterion

Let $C \subset X$ be a curve and $x \in X$. Since X is nonsingular (in particular locally factorial) and C has codimension one, the ideal $\mathcal{O}_X(-C)$ of functions vanishing on C is locally (at x) generated by a single function f, unique up to a unit. The *multiplicity* $mult_x(C)$ of C at x is the least integer m such that $f \in \mathfrak{m}_x^m$, where $\mathfrak{m}_x$ is the maximal ideal in the local ring $\mathcal{O}_x$.

Let $\mu : X' \to X$ be the blow-up of X at x; denote by E be the exceptional divisor. The multiplicity of a curve C passing through x has the following geometric interpretation: if $\tilde{C}$ is the proper transform of C in X', then

$$mult_x(C) = \tilde{C}.E$$

(In spite of the definition, the multiplicity is intrinsic to C.)

If C and D are two curves intersecting at x and having no other points in common in a neighbourhood, their *intersection multiplicity* at x is

$$(C.D)_x = dim_k\ \mathcal{O}_x/\{f, g\}$$

where $f = 0$ and $g = 0$ are local defining equations for C and D respectively, and $\{f, g\}$ is the ideal generated by f and g in the local ring $\mathcal{O}_x$. We have the following inequality:

$$(C.D)_x \geq mult_x(C) mult_x(D)$$

Theorem 5.1. *Let X be a smooth projective surface, and L a line bundle on X. Then L is ample iff there exists $\epsilon(L) > 0$ such that for every (reduced, irreducible) curve C and every point $x \in C$, we have*

$$deg\ L|_C \geq \epsilon(L)\ mult_x\ C \tag{1}$$

(In other words, the degree of L restricted to any curve C should be bounded below uniformly in terms of the "maximum singularity" of C.)

Proof. Suppose first that L is ample. Then some power L^m is very ample. Given C and a point $x \in C$, there exists (by Bertini) a section σ of L^m such that $E \equiv (\sigma)$ is a nonsingular curve passing though x, but $C \neq E$. (That is, such that $\sigma|_C \neq 0$ but $\sigma(x) = o$). Then

$$m\ deg\ L|_C = deg\ L^m|_C = \sum_{y \in C \cap E} i(E, C, y) \geq i(E, C, x) \geq\ mult_x C$$

Here $i(E, C, y)$ is the intersection multiplicity of E and C at y. So (1) holds with $\epsilon(L) = 1/m$.

Conversely, suppose (1) holds for some positive $\epsilon(L)$. Using Nakai's criterion, it suffices to show that

$$L.L > 0$$

Fix a point $x \in X$, and let $\mu : X' \to X$ be the blow-up of X at x; let E be the exceptional divisor. Note that E is isomorphic to $\mathbb{P}^1$ and $\mathcal{O}(+E)|_E$ is the normal bundle to E, and hence equal to $\mathcal{O}_{\mathbb{P}^1}(-1)$. Hence $\mathcal{O}(-E)|_E$ is the hyperplane bundle. In particular, $E.E = degree\ \mathcal{O}(E)|_E = -1$.

Let $L = \mathcal{O}_X(D)$; then $\mu^* L = \mathcal{O}_{X'}(D')$ where D' is the inverse image of of D by μ. We claim that $\mu^* L^m(-E) = \mathcal{O}_{X'}(mD' - E)$ is nef on X' provided $\epsilon(L) < 1/m$. Granting this, we have by Kleiman's Theorem:

$$(mD' - E)^2 = m^2 D.D + E.E = m^2 D.D - 1 \geq 0$$

which yields the desired inequality $D.D > 0$.

Turning now to the claim, let C' be any (reduced, irreducible) curve in X'. We need to show that

$$deg\ \mu^* L|_{C'} \geq \frac{1}{m} deg\ \mathcal{O}(E)|_{C'}$$

If $C' \subset E$, then L is trivial on C' and $\mathcal{O}(E))$ has negative degree on C'. If $C' \not\subseteq E$, let $C = \mu(C')$, so that C' is the proper transform of C. Now

$$deg\ \mu^* L|_{C'} = deg\ L|_C$$

and

$$deg\ \mathcal{O}(E)|_{C'} = mult_x(C)$$

so that by hypothesis, the claim is proved. □