ASYMPTOTIC QUANTITIES FOR UNDERSTANDING SINGULAR PLANE CURVES

BRIAN HARBOURNE

CHENNAI, INDIA

Abstract. The question underlying these lectures is: what can one say about curves of negative self-intersection on a smooth projective surface X? The focus here will be for surfaces X obtained by blowing up points of the projective plane. There the occurrence of curves of negative self-intersection is closely connected to how singular a plane curve can be. Objects considered will include the nef cone $\mathrm{Nef}(X)$, the effective subsemigroup $\mathrm{Eff}(X)$, the semi-effective cone $\mathrm{Seff}(X)$, and asymptotic quantities defined in terms of these, such as Seshadri constants, Waldschmidt constants and H-constants.

Acknowledgements: I want to thank K. Hanumanthu for the invitation to visit Chennai, and the Chennai Mathematical Institute for providing a wonderful environment for my visit.

1. Background

We will always take our ground field K to be an algebraically closed field, usually of arbitrary characteristic.

1.1. **The commutative algebraic perspective.** Let $p_1, \dots, p_s \in \mathbb{P}^N$ be distinct points and let $m_1, \dots, m_s$ be nonnegative integers. As a notational convenience, we indicate this data as a formal sum $Z = m_1 p_1 + \cdots + m_r p_r$ (i.e., as an element of the free abelian group on the points of $\mathbb{P}^N$), which we will refer to as a *fat point subscheme* of $\mathbb{P}^N$.

The homogeneous coordinate ring of $\mathbb{P}^N$ is $R = K[x_0, \dots, x_N] = K[\mathbb{P}^N]$ (or sometimes, when $N = 2$, $K[x, y, z]$).We regarded it as graded, so $R = \oplus_t R_t$, where R_t is the span of the homogeneous polynomials (i.e., forms) of degree t. The ideal of Z is defined to be $I(Z) = \cap_i I(p_i)^{m_i}$, where $I(p_i)$ is the ideal generated by all forms vanishing at p_i. Note that $I(Z)$ is a homogeneous ideal; i.e., we have $I(Z) = \oplus_{t \geq 0} I(Z)_t$, where $I(Z)_t = I(Z) \cap R_t$ is the K-vector space span of the homogeneous polynomials in $I(Z)$ of degree t. (From the point of view of scheme theory, the ideal $I(Z)$ defines a 0-dimensional subscheme of $\mathbb{P}^N$, which we identify with Z. We will not go into the theory of schemes but this is why we refer to Z as a subscheme of $\mathbb{P}^N$.)

We will regard a plane algebraic curve as being a divisor in the plane. So given $F \in R_t$, factored into irreducible homogeneous factors as $F = F_1^{f_1} \cdots F_r^{f_r}$, the corresponding curve is $C = f_1 C_1 + \cdots + f_r C_r$, where C_i is the zero-locus in $\mathbb{P}^2$ of F_i.

Given a curve C defined by a homogeneous polynomial F and given a point $p \in \mathbb{P}^2$, we say $\mathrm{mult}_{p_i} C = m$ if $F \in I(p)^m$ but $F \notin I(p)^{m+1}$, where $I(p)$ is the ideal generated by all forms vanishing at p (in fact $I(p)$ is generated by any two linear forms vanishing at p which define distinct lines).

The motivating question is: given $t \geq 0$, when is there an algebraic plane curve of degree t with points of multiplicity at least m_i at each point p_i? Equivalently, the question is: when is $I(Z)_t \neq 0$?

Date: April 5, 2017.

We see that the nonzero elements of $I(Z)_t$ (if any) define the curves C of degree t with multiplicity at least m_i at each point p_i. How many such curves there are is of interest, and for this we consider the dimension of $I(Z)_t$ for each t. We begin by noting that $\dim_K R_t = \binom{t+2}{2}$, and we define the Hilbert function $h_{I(Z)}$ as

$$h_{I(Z)}(t) = \dim I(Z)_t.$$

In particular, there actually is such a curve C if and only if $h_{I(Z)}(t) > 0$.

Exercise 1.1.1. Let $Z = mp$ for the point $p = (0, 0, 1)$. Show that $h_{I(Z)}(t) = \max(0, \binom{t+2}{2} - \binom{m+1}{2})$. (Indeed, vanishing with multiplicity at least m at p requires the coefficients of $F \in R_t$ to satisfy $\binom{m+1}{2}$ linear conditions, and these conditions are independent if and only if $t \geq m - 1$; i.e., $h_{I(Z)}(t) = \binom{t+2}{2} - \binom{m+1}{2}$ if and only if $t \geq m - 1$.)

Corollary 1.1.2. *Given a fat point subscheme $Z = m_1p_1 + \cdots + m_sp_s$ of $\mathbb{P}^2$, we have*

$$h_{I(Z)}(t) \geq \max\left(0, \binom{t+2}{2} - \sum_i \binom{m_i+1}{2}\right).$$

Proof. To vanish at each point p_i with multiplicity at least m_i, the coefficients of a form of degree t must satisfy $\binom{m_i+1}{2}$ linear equations on the vector space R_t of dimension $\dim R_t = \binom{t+2}{2}$. The solution space, namely $I(Z)_t$, thus has dimension at least $\dim R_t$ minus the number of equations, i.e., $\dim I(Z)_t \geq \binom{t+2}{2} - \sum_i \binom{m_i+1}{2}$. □

The inequality in the preceding corollary comes from the fact that the $\sum_i \binom{m_i+1}{2}$ conditions imposed on forms $F \in R_t$ to vanish with multiplicity at least m_i at each point p_i need not be independent. For example, to vanish at 3 collinear points p_1, p_2 and p_3, there are 3 conditions, one for each point. But applied to the space R_1 of linear forms we see that a linear form vanishing at p_1 and p_2 is forced to vanish at p_3; the 3 conditions are not independent.

1.2. **The geometric perspective.** We now consider an alternative point of view. Recall that an integral curve E on a smooth projective surface X is called an *exceptional curve* if $E^2 = E{\cdot}K_X = -1$, which implies that E is smooth and rational. What makes an exceptional curve exceptional is the fact that there is a birational morphism $\pi : X \to Y$ contracting E to a smooth point, such that π is the morphism obtained by blowing up Y at this point (see Castelnuovo's Criterion, [32, Theorem 5.7]).

So let $\pi : X \to \mathbb{P}^2$ be the birational morphism obtained by blowing up distinct points $p_1, \ldots, p_s \in \mathbb{P}^2$. We set $\mathrm{Cl}(X)$ to be the divisor class group of X; i.e., the group of divisors modulo linear equivalence. It is a free $\mathbb{Z}$-module on the pullback $\ell = \pi^{-1}L$ of a line and on the exceptional curves $e_i = \pi^{-1}(p_i)$ obtained from the blow ups. An important divisor is the anticanonical divisor $-K_X = 3\ell - e_1 - \cdots - e_s$. Note that each divisor class is uniquely represented by a divisor of the form $d\ell - m_1e_1 - \cdots - m_se_s$. Thus we may think of $d\ell - m_1e_1 - \cdots - m_se_s$ either as a divisor or as a divisor class. Moreover, it will be convenient to refer to the class $d\ell - m_1e_1 - \cdots - m_se_s$ as being *effective* if the divisor $d\ell - m_1e_1 - \cdots - m_se_s$ is linearly equivalent to an effective divisor (i.e., if $h^0(X, \mathcal{O}_X(d\ell - m_1e_1 - \cdots - m_se_s)) > 0$).

We recall the intersection product on $\mathrm{Cl}(X)$. We have $\ell^2 = 1$, $e_i^2 = -1$, $\ell \cdot e_i = e_i \cdot e_j = 0$ for all $i \neq j$. Then for any divisors $C = c\ell - \sum_i c_ie_i$ and $D = d\ell - \sum_i d_ie_i$ we have $C \cdot D = cd - \sum c_id_i$. The basic fact is that if $C \sim C'$ and $D \sim D'$, where $\sim$ denotes linear equivalence, then $C \cdot D = C' \cdot D'$.

It will be helpful to extend the intersection product to $\mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, in the obvious way.

Remark 1.2.1. If C and D are integral and $C \neq D$, then $C \cdot D \geq 0$, since $C \cdot D$ is at least as large as the number of points of intersection of C with D (in fact, $C \cdot D$ is exactly equal to the number of points in $C \cap D$, if we count each point of intersection with an appropriate multiplicity).

Given a divisor C on X, we will denote $h^i(X, \mathcal{O}_X(C))$ more simply just as $h^i(C)$. From the formula of Riemann-Roch we have

$$h^0(C) - h^1(C) + h^2(C) = \frac{C^2 - C \cdot K_X}{2} + 1.$$

Given $Z = m_1p_1 + \cdots + m_sp_s$ and $t \geq 0$, we will use e_Z to denote $\sum_i m_ie_i$ and $F_t(Z)$ to denote $t\ell - e_Z$. An easy check shows that $(F^2 - F \cdot K_X)/2 + 1 = \binom{t+2}{2} - \sum_i \binom{m_i+1}{2}$. Since $h^2(F_t(Z)) = 0$ (because of Serre duality and $t \geq 0$), we have

$$h^0(F_t(Z)) = h^1(F_t(Z)) + \binom{t+2}{2} - \sum_i \binom{m_i+1}{2} \geq \max\left(0, \binom{t+2}{2} - \sum_i \binom{m_i+1}{2}\right).$$

In fact, there are natural identifications $H^0(t\ell) = R_t$ and $H^0(F_t(Z)) = I(Z)_t$, so we see $h^0(F_t(Z)) = h_{I(Z)}(t)$.

Understanding the geometry of a surface X means understanding the curves on X. Various classes of curves on X are of interest. We begin with the set $\mathrm{Eff}(X)$ of classes of effective divisors. This is closed under addition, hence it is a subsemigroup of $\mathrm{Cl}(X)$. We next have the set $\mathrm{Nef}(X) \subset \mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ of all nef elements (i.e., all $F \in \mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ such that $F \cdot C \geq 0$ for all $C \in \mathrm{Eff}(X)$). Then a nef divisor is an element of $\mathrm{Nef}(X) \cap \mathrm{Cl}(X)$ and a nef $\mathbb{R}$-divisor is an element of $\mathrm{Nef}(X)$, but we will not be too fussy about distinguishing between nef divisors and nef $\mathbb{R}$-divisors, leaving it for the most part for the reader to gather from context. Note that $\mathrm{Nef}(X)$ is also closed under addition, hence it is a subsemigroup of $\mathrm{Cl}(X)$, but by the next exercise, if $mF \in \mathrm{Nef}(X)$ for some integer $m > 0$, then $F \in \mathrm{Nef}(X)$. We will refer to a subsemigroup of $\mathrm{Cl}(X)$ with this property as a *cone*.

In addition we define the semi-effective cone $\mathrm{Seff}(X) \subset \mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$; here we have $F \in \mathrm{Seff}(X)$ if and only if $mF \in \mathrm{Eff}(X)$ for some $m > 0$. Then a semi-effective divisor is an element of $\mathrm{Seff}(X) \cap \mathrm{Cl}(X)$ and a semi-effective $\mathbb{Q}$-divisor is an element of $\mathrm{Seff}(X)$, but as with nefness we will not be too fussy about distinguishing between them, leaving it for the most part for the reader to gather from context. Finally we have the set $\mathrm{Neg}(X)$ of integral curves C on X with $C^2 < 0$.

Exercise 1.2.2. Let C_i be integral curves on a surface X (where X is as above).

(a) If $C = \sum_i m_iC_i$ for $m_i \geq 0$, show that $C \cdot C_i \geq 0$ for all i implies $C \in \mathrm{Nef}(X)$.
(b) Show that $mF \in \mathrm{Nef}(X)$ for all $m > 0$ if and only if $mF \in \mathrm{Nef}(X)$ for some $m > 0$ if and only if $F \in \mathrm{Nef}(X)$.
(c) Give an example of a surface X as above and a curve C on X with $C \notin \mathrm{Eff}(X)$ but with $2C \in \mathrm{Eff}(X)$ (so $\mathrm{Eff}(X)$ need not be a cone).

Answer: (a) If C is not nef, then $C \cdot D < 0$ for some integral curve D. But $C \cdot D \geq 0$ unless C and D have a common component, so D must be C_i for some i; i.e., if $C \cdot C_i \geq 0$ for all i, then $C \cdot D \geq 0$ for all integral curves D.

(b) This is clear after noting that if $m > 0$, then $mF \cdot D \geq 0$ for all integral D if and only if $F \cdot D \geq 0$.

(c) Blow up the $r = \binom{n}{2}$ points p_i of pair-wise intersection of n general lines for $n > 2$ even. Take $2C$ to be the proper transform of the lines. Then up to linear equivalence we have $2C = n\ell - 2e_1 - \cdots - 2e_r \in \mathrm{Eff}(X)$, but if D is the proper transform of any of the n lines, then $C \cdot D < 0$ (it is here that we use $n > 2$), thus if $C \in \mathrm{Eff}(X)$ we would have to have that the proper transforms of all of the n lines must be components of C; i.e., $-C = C - 2C \in \mathrm{Eff}(X)$. But $-C \cdot \ell = -n < 0$; since ℓ is nef, we cannot have $-C \in \mathrm{Eff}(X)$.

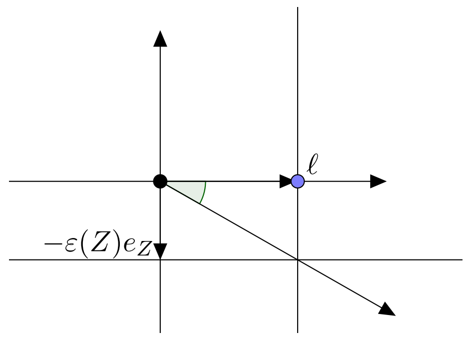


FIGURE 1. The slice of the cone $\mathrm{Nef}(X)$ by the plane spanned by ℓ and e_Z is indicated by the green sector. Let γ be the slope of the bottom ray of $\mathrm{Nef}(X)$, i.e., the slope of $\mathrm{Nef}(X)$ in the $-e_Z$ direction. Then, intuitively, $\varepsilon(Z) = -\gamma$.

2. Seshadri and Waldschmidt constants

It is in general a hard problem to determine any of $\mathrm{Eff}(X)$, $\mathrm{Nef}(X)$, $\mathrm{Seff}(X)$ or $\mathrm{Neg}(X)$. One way to study these is via Seshadri [13, 2] and Waldschmidt [51, 49, 8, 5, 17, 22] constants.

2.1. **Seshadri constants.** Let $Z = m_1p_1 + \dots + m_sp_s$ be a fat point subscheme of the plane, X the surface obtained by blowing up the points p_i. Then the multipoint Seshadri constant for Z is

$$\varepsilon(Z) = \sup\left\{\frac{1}{t} : F_t(Z) \in \mathrm{Nef}(X), t > 0\right\}.$$

If we take the plane in $\mathrm{Cl}(X)$ spanned by ℓ and e_Z, we get Figure 1, showing that $\varepsilon(Z)$ tells us the slope of the cone $\mathrm{Nef}(X)$ in the $-e_Z$ direction. Alternatively, we have

$$\varepsilon(Z) = \sup\left\{\frac{\deg(C)}{\sum_i m_i \,\mathrm{mult}_{p_i}(C)}\right\}$$

where the sup is taken over all curves C containing at least one of the points p_i.

2.2. **Waldschmidt constants.** The Waldschmidt constant is an analogous quantity for $\mathrm{Seff}(X)$:

$$\widehat{\alpha}(Z) = \inf\{t : F_t(Z) \in \mathrm{Seff}(X) \otimes_{\mathbb{Z}} \mathbb{Q}\}.$$

We can also just work over $\mathbb{Z}$:

$$\widehat{\alpha}(Z) = \inf\left\{\frac{n}{m} : F_n(mZ) \in \mathrm{Seff}(X), n, m > 0\right\} = \inf\left\{\frac{n}{m} : F_n(mZ) \in \mathrm{Eff}(X), n, m > 0\right\}.$$

If we take the plane in $\mathrm{Cl}(X)$ spanned by ℓ and e_Z, we get Figure 2, showing that $\widehat{\alpha}(Z)$ tells us the slope of the cone $\mathrm{Seff}(X)$ in the $-e_Z$ direction.

For an alternative description of $\widehat{\alpha}(Z)$, for a nonzero homogeneous ideal $(0) \neq I \subseteq R$, define $\alpha(I)$ to be the least degree t with $I_t \neq (0)$.

Exercise 2.2.1. Show that

$$\widehat{\alpha}(Z) = \inf_m\left\{\frac{\alpha(I(mZ))}{m}\right\}.$$

Answer: Let $b(Z) = \inf_m\{\alpha(I(mZ))/m\}$. Then $F_{\alpha(I(mZ))}(mZ) \in \mathrm{Eff}(X)$ so $F_{\alpha(I(mZ))/m}(Z) \in \mathrm{Seff}(X)$ so $b(Z) \geq \widehat{\alpha}(Z)$. But there is a decreasing sequence of rationals $t > \widehat{\alpha}(Z)$ with limit $\widehat{\alpha}(Z)$ and $F_t(Z) \in \mathrm{Seff}(X)$ for each t. For each t there is an m with $F_{mt}(mZ) \in \mathrm{Eff}(X)$, hence $\widehat{\alpha}(Z) \leq \alpha(I(mZ))/m \leq mt/m = t$, so we must have $\widehat{\alpha}(Z) = \inf_m\left\{\frac{\alpha(I(mZ))}{m}\right\}$.

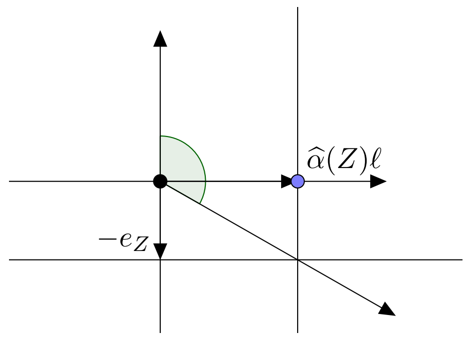


FIGURE 2. The slice of the cone $\mathrm{Seff}(X)$ by the plane spanned by ℓ and e_Z is indicated by the green sector. Let γ be the slope of the bottom ray of $\mathrm{Seff}(X)$, i.e., the slope of $\mathrm{Seff}(X)$ in the $-e_Z$ direction. Then, intuitively, $\widehat{\alpha}(Z) = -1/\gamma$.

Remark 2.2.2. As an aside, given a fat point subscheme $Z \subset \mathbb{P}^N$, we also mention that in fact

$$\widehat{\alpha}(Z) = \lim_{m\to\infty} \frac{\alpha(I(mZ))}{m}.$$

(This is because of Fekete's Subadditivity Lemma [23]. To see this, let m, n be positive integers. Let $F \in I(mZ)$ have degree $\alpha(I(mZ))$ and $G \in I(nZ)$ have degree $\alpha(I(nZ))$. Then $FG \in I((m+n)Z)$, so $\alpha(I((m+n)Z)) \leq \alpha(I(mZ)) + \alpha(I(nZ))$. Fekete's Subadditivity Lemma now says for each n that $\widehat{\alpha}(I(Z)) = \lim_{m\to\infty} \frac{\alpha(I(mZ))}{m} \leq \frac{\alpha(I(nZ))}{n}$.)

It is a hard problem in general even to compute the values of $\varepsilon(Z)$ and $\widehat{\alpha}(Z)$, but it's conceptually easy to at least get upper bounds.

Exercise 2.2.3.
(a) If $F = F_t(Z)$ has $F^2 > 0$, show that $F \in \mathrm{Seff}(X)$.
(b) Let $t \geq 0$ be such that $F_t(Z)^2 = 0$. Show $\widehat{\alpha}(Z) \leq t$ and $\varepsilon(Z) \leq 1/t$.

Answer: (a) Use Riemann-Roch.

(b) Any divisor $F = F_{t'}(Z)$ with $F^2 > 0$ is semi-effective. Thus $\widehat{\alpha}(Z) \leq t'$ for all such t' and since t is the limit of all such t' we have $\widehat{\alpha}(Z) \leq t'$. And any nef divisor F has $F^2 \geq 0$. So the values of t' such that $F = F_{t'}(Z)$ is nef satisfy $t' \geq t$. Thus $\varepsilon(Z) \leq 1/t' \leq 1/t$ for each such t'.

Exercise 2.2.4. Let $Z = \sum_i m_i p_i$. Show $\widehat{\alpha}(Z)/\varepsilon(Z) \geq \sum_i m_i^2$. Give an example to show that equality can fail.

Answer: Note that $\widehat{\alpha}(Z)/\varepsilon(Z) \geq \sum_i m_i^2$ is equivalent to $F_{\widehat{\alpha}(Z)} \cdot F_{1/\varepsilon(Z)}(Z) \geq 0$ but this is clear, since the intersection of a nef divisor with an effective divisor is always nonnegative. Now take 3 collinear points p_1, p_2, p_3 and one point p_4 off that line. Let $Z = p_1 + p_2 + p_3 + p_4$. Then the class of $F_3(Z)$ is in $\mathrm{Nef}(X)$ and $\mathrm{Eff}(X)$ and the class of $F_2(Z)$ is in $\mathrm{Eff}(X)$ and we have $F_3(Z) \cdot F_2(Z) = 2$. Also, $(F_3(Z)).(\ell - e_1 - e_2 - e_3) = 0$, so $\varepsilon(Z) = 1/3$. Next, take $F = (\ell - e_1 - e_2 - e_3) + 2(\ell - e_4)$; this is nef. Note that the class of $G = 5\ell - 3(e_1 + \cdots + e_4) = 2(\ell - e_1 - e_2 - e_3) + (\ell - e_1 - e_4) + (\ell - e_2 - e_4) + (\ell - e_3 - e_4)$ is in $\mathrm{Eff}(X)$, but $F \cdot G = 0$, so $\widehat{\alpha}(Z) = 5/3$. Thus $\widehat{\alpha}(Z)/\varepsilon(Z) = 5 > 4$.

Using complex analysis over the complexes, Waldschmidt and Skoda [51, 49] gave the following lower bound for $\widehat{\alpha}(Z)$ when Z is a reduced scheme of points in $\mathbb{P}^n$:

$$\frac{\alpha(I(Z))}{N} \leq \widehat{\alpha}(I(Z)).$$

See [37] for a proof using multiplier ideals. The simplest proof, discovered by the author and J. Roé (see [30, p. 2], but be sure to look at version 1 of this posting), uses the containment results of [19, 34] and gives a somewhat stronger result than that obtained by Waldschmidt and Skoda, and in fact the following theorem holds more generally in terms of "symbolic powers" of ideals, appropriately defined.

Theorem 2.2.5 ([19, 34])**.** *For any fat point subscheme $Z \subset \mathbb{P}^n$, we have $I((N+m-1)rZ) \subsetneq I(mZ)^r$ for all $m, r > 0$.*

It follows that

$$\alpha(I((N+m-1)rZ)) \geq \alpha(I(mZ)^r) = r\,\alpha(I(mZ)),$$

hence

$$\frac{\alpha(I((N+m-1)rZ))}{r(N+m-1)} \geq \frac{r\,\alpha(I(mZ))}{r(N+m-1)}.$$

As in Remark 2.2.2, taking limits as $r \to \infty$ now gives

$$\frac{\alpha(I(mZ))}{N+m-1} \leq \widehat{\alpha}(Z) \leq \frac{\alpha(I(mZ))}{m}.$$

2.3. **Chudnovksy's Conjecture and Bound.** In [8], Chudnovsky states the following result.

Theorem 2.3.1. *Let $Z = p_1 + \cdots + p_r$ for distinct points $p_1, \ldots, p_r \in \mathbb{P}^2$. Then*

$$\frac{\alpha(Z)+1}{2} \leq \widehat{\alpha}(Z).$$

Proof. No explicit proof is given in [8], but from context it is pretty clear that Chudnovsky's proof is as follows. (See [29] for an alternate proof.)

Let $R = K[\mathbb{P}^2]$ and let $s = \binom{\alpha(I(Z))+2}{2} - \dim I(Z)_{\alpha(I(Z))}$ (hence $s < \binom{\alpha(I(Z))+2}{2}$). Then we can pick a sequence of distinct points p_{i_j} giving a subscheme $Z_s = p_{i_1} + p_{i_2} + \cdots + p_{i_s}$ of Z such that

$$I(Z)_{\alpha(I(Z))} = I(Z_s)_{\alpha(I(Z))} \subsetneq I(Z_{s-1})_{\alpha(I(Z))} \subsetneq \cdots \subsetneq I(Z_1)_{\alpha(I(Z))} \subsetneq R_{\alpha(I(Z))},$$

where $Z_j = p_{i_1} + \cdots + p_{i_j}\}$. We just need to keep picking points which are not base points for the linear system of forms of degree $\alpha(I(Z))$ vanishing on the previously picked points. (There is no reason to expect that $Z_s = p_{i_1} + p_{i_2} + \cdots + p_{i_s}\}$ is unique.)

Note that we cannot have $s < \binom{\alpha(I(Z))+1}{2}$, since vanishing at s points imposes at most s conditions, hence $s < \binom{\alpha(I(Z))+1}{2}$ implies that $\alpha(I(Z_s)) < \alpha(I(Z))$, so there would be a nonzero form F of degree $\alpha(I(Z))-1$ vanishing on Z_s. But this means, for every linear form L, that FL has degree $\alpha(I(Z))$ and vanishes on Z_s, so is in $I(Z)_{\alpha(I(Z))} = I(Z_s)_{\alpha(I(Z))}$. Therefore F must vanish on all of Z, contradicting $\alpha(I(Z))$ being the least degree for which there is a nonzero form vanishing on Z. (We also see that $\alpha(I(Z_s)) = \alpha(I(Z))$.)

Thus we have

$$\binom{\alpha(I(Z))+1}{2} \leq s < \binom{\alpha(I(Z))+2}{2}.$$

Let $Y = Z_s$ and let $t = \binom{\alpha(I(Z))+1}{2}$. Since $\alpha(I(Z_s)) = \alpha(I(Z))$, no nonzero form of degree $\alpha(Z)-1$ vanishes on Y. Thus, in the same way as before, we can pick a subscheme $Y_t = q_1 + \cdots + q_t \subseteq Y$ such that

$$0 = I(Y)_{\alpha(I(Z))-1} = I(Y_t)_{\alpha(I(Z))-1} \subsetneq I(Y_{t-1})_{\alpha(I(Z))-1} \subsetneq \cdots \subsetneq I(Y_1)_{\alpha(I(Z))-1} \subsetneq R_{\alpha(I(Z))-1},$$

where $Y_j = q_1 + \cdots + q_j$.

Let $U = Y_t$. Then $\alpha(I(U)) = \alpha(I(Z))$ and $|U| = \binom{\alpha(I(Z))+1}{2}$. Note that $I(U)_{\alpha(I(Z))}$ is fixed component free. (To see that the elements of $I(U)_{\alpha(I(Z))}$ are not all divisible by some fixed nonconstant form, let $U_i = U - q_i = q_1 + \cdots + q_{i-1} + q_{i+1} + \cdots + q_t$. Then $\dim I(U_i)_{\alpha(I(Z))-1} = 1$, since adding one point to U_i, namely q_i, gives U and $\dim I(U)_{\alpha(I(Z))-1} = 0$. Let F_i be a nonzero form spanning $I(U_i)_{\alpha(I(Z))-1}$. Thus $F_i(q_i) \neq 0$. Then $F_1, \ldots, F_t$ have no nonconstant common factor, for if H were such a common factor, then $F_i(q_i) \neq 0$ for all i implies $H(q_i) \neq 0$ for all i, thus $G_t = F_t/H$ vanishes on U_t, hence for any linear form L vanishing at q_t we see G_tL vanishes on U yet has degree at most $\alpha(I(Z)) - 1$, so $\alpha(I(U)) < \alpha(I(Z))$, contrary to what we proved above. Thus picking forms L_i that vanish on q_i but on no other q_j we get elements $F_1L_1, \ldots, F_tL_t \in I(U)_{\alpha(I(Z))}$ with no common component.)

Now let $m > 0$ and let $F \in I(mZ)_{\alpha(I(mZ))}$ be nonzero. The we can pick $G \in I(U)_{\alpha(I(Z))}$ with no component in common with F. By Bezout's Theorem we have

$$(\alpha(I(mZ)))(\alpha(I(Z))) = \deg(F)\deg(G) \geq m|U| = m\binom{\alpha(I(Z))+1}{2},$$

hence

$$\frac{\alpha(I(mZ))}{m} \geq \frac{\alpha(I(Z))+1}{2}.$$

This holds for all m, so

$$\frac{\alpha(I(Z))+1}{2} \leq \widehat{\alpha}(Z).$$

□

Presumably based on this and on examples, Chudnovsky proposed the following conjecture [8]:

Conjecture 2.3.2. *Let $Z = p_1 + \cdots + p_r$ for distinct points $p_i \in \mathbb{P}^N$. Then*

$$\frac{\alpha(I(Z)) + N - 1}{N} \leq \widehat{\alpha}(Z).$$

In an attempt both to understand why this might be true, based on the proof of the bound $\frac{\alpha(I(mZ))}{N+m-1} \leq \widehat{\alpha}(Z)$ given above, and to explore possible improvements to the containment $I(NrZ) \subseteq I(Z)^r$ discussed above, an ideal containment conjecture was proposed in [29]:

Conjecture 2.3.3. *Let $Z = m_1p_1 + \cdots + m_rp_r$ for distinct points $p_i \in \mathbb{P}^N$ and let $M = (x_0, \ldots, x_N)$ be the irrelevant ideal. Then*

$$I(NrZ) \subseteq M^{r(N-1)}I(Z)^r.$$

Exercise 2.3.4. Let $Z = m_1p_1 + \cdots + m_rp_r$ for distinct points $p_i \in \mathbb{P}^N$ and let $M = (x_0, \ldots, x_N)$. Show that Conjecture 2.3.3 implies Conjecture 2.3.2 (not only for reduced Z but for all fat point subschemes Z).

Answer: Given $I(NrZ) \subseteq M^{r(N-1)}I(Z)^r$, we get $\alpha(I(NrZ)) \geq \alpha(M^{r(N-1)}I(Z)^r) = r\alpha(I(Z)) + r(N-1)$. Thus for all $r > 0$ we get

$$\frac{\alpha(I(NrZ))}{rN} \geq \frac{\alpha(I(Z)) + (N-1)}{N}.$$

Taking the limit as $r \to \infty$ gives the result.

For another paper addressing Chudnovsky's conjecture, see [20].

2.4. **Submaximality.** By Remark 1.2.1, it follows that curves $C, D \subset X$ with no common components have $C \cdot D \geq 0$.

Let $Z = \sum_i m_i p_i$. Since we always have $F_{1/\varepsilon(Z)}(Z)^2 \geq 0$ (i.e., $\varepsilon(Z) \leq 1/\sqrt{\sum_i m_i^2}$) we say $\varepsilon(Z)$ is *suboptimal* if $F_{1/\varepsilon(Z)}(Z)^2 > 0$ (i.e., if $\varepsilon(Z) < 1/\sqrt{\sum_i m_i^2}$). In that case there is a rational t with $\sqrt{\sum_i m_i^2} < t < 1/\varepsilon(Z)$ and for all such t we have $F_t(Z) \in \mathrm{Seff}(X)$ but $F_t(Z) \notin \mathrm{Nef}(X)$. Fix such a t; then for some integer $m > 0$ we have $m(F_t(Z)) = F_{mt}(mZ) \in \mathrm{Eff}(X)$, so there are integral curves C_i and positive integers a_i such that $m(F_t(Z)) = \sum_i a_i C_i$, and for each rational t' with $t \leq t' < 1/\varepsilon(Z)$ we have for some integer $m_{t'} > 0$ that $m_{t'} m(F_{t'}(Z)) = m_{t'} m(t' - t)\ell + m' \sum_i a_i C_i$ is an effective divisor but not nef. Thus for each such t' there is some C_i such that $C_i \cdot F_{t'}(Z) < 0$ and we see moreover for this C_i that $C_i \cdot F_{t''}(Z) < 0$ for all t'' in the range $t \leq t'' \leq t'$. Since $\sum_i a_i C_i$ is a finite sum, there must be an index j such that $C_j \cdot F_{t'}(Z) < 0$ for all t' in the range $t \leq t' < 1/\varepsilon(Z)$. Since $F_{1/\varepsilon(Z)}(Z)$ is nef, we have $\lim_{t' \to 1/\varepsilon(Z)} C_j \cdot F_{t'}(Z) = C_j \cdot F_{1/\varepsilon(Z)}(Z) \geq 0$ even though $C_j \cdot F_{t'}(Z) < 0$, so we conclude that $C_j \cdot F_{1/\varepsilon(Z)}(Z) = 0$.

Thus when $\varepsilon(Z)$ is suboptimal, there is an integral curve C such that $C \cdot F_{1/\varepsilon(Z)}(Z) = 0$. Such a curve is called a Seshadri curve for Z. It need not be unique, hence there can also be effective divisors D which are not integral such that $D \cdot F_{1/\varepsilon(Z)}(Z) = 0$. Such divisors were called abnormal by Nagata, and submaximal by some subsequent authors. For any submaximal curve $D \sim d\ell - \sum_i m_i e_i$, we have $d - \varepsilon(Z)(\sum_i m_i) = 0$, hence $\varepsilon(Z) = \frac{d}{\sum_i m_i}$; i.e., D computes the value of $\varepsilon(Z)$.

Moreover, by the Hodge Index Theorem we have $D^2 < 0$. I.e., $C \in \mathrm{Neg}(X)$ for every Seshadri curve C. Thus we see to understand suboptimal Seshari constants it is helpful to understand the occurrence of integral curves C with $C^2 < 0$.

Exercise 2.4.1. Show that each Z has at most finitely many Seshadri curves and that they are linearly independent in $\mathrm{Cl}(X)$.

Answer: Seshadri curves are components of an effective divisor, so there can be at most finitely many. They live in a negative definite subspace of $\mathrm{Cl}(X)$ but meet each other nonnegatively. If they were not independent, then some nonnegative linear combination A of some of them would (up to linear equivalence) equal some nonnegative linear combination B of the rest; i.e., $A \sim B$. Since $A \sim 0$ for an effective divisor A if and only if $A = 0$, to show linear independence we must show $A = 0$ and $B = 0$, so say $A \neq 0$. Then $A \not\sim 0$ so $0 > A^2$ by negative definiteness, hence $0 > A^2 = A \cdot B \geq 0$. The contradiction shows that the Seshadri curves cannot fail to be independent.

The next proposition gives a way of computing Seshadri and Waldschmidt constants in certain cases and of finding Seshadri curves.

Proposition 2.4.2. *Let $Y = m_1 p_1 + \cdots + m_r p_r$, $Z = n_1 p_1 + \cdots + n_r p_r \subset \mathbb{P}^2$ be fat point subschemes where Y is nontrivial (i.e., some $m_i > 0$), let X be the blow up of the points p_i, and let $F_s(Y) \in \mathrm{Seff}(X)$ and $F_t(Z) \in \mathrm{Nef}(X)$. If $F_s(Y) \cdot F_t(Z) = 0$, then $\varepsilon(Z) = 1/t$ and $\widehat{\alpha}(Y) = s$.*

Proof. Since $F_t(Z)$ is nef, we have $\varepsilon(Z) \leq 1/t$. Since $F_s(Y) \in \mathrm{Seff}(X)$, we have $aF_s(Y)$ is effective for some $a > 0$. Since $\ell - e_i$ is nef for all i, we have $aF_s(Y) \cdot (\ell - e_i) \geq 0$, and since $m_i > 0$ for some i we therefore have $aF_s(Y) \cdot \ell \geq m_i > 0$. Thus $aF_s(Y) \cdot F_t(Z) = 0$ implies $aF_s(Y) \cdot F_u(Z) < 0$ for all $u < t$, so we also have $\varepsilon(Z) \geq 1/t$ and hence $\varepsilon(Z) = 1/t$. Since $F_t(Z)$ is nef, having $F_s(Y) \cdot F_t(Z) = 0$ also means $\alpha(I(aY)) \geq as$ for all $a > 0$, so $\widehat{\alpha}(Y) \geq s$, but $F_s(Y) \in \mathrm{Seff}(X)$ means $\widehat{\alpha}(Y) \leq s$, so we obtain $\widehat{\alpha}(Y) = s$. $\square$

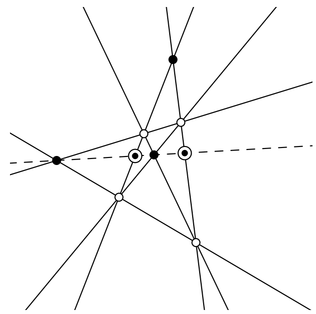


FIGURE 3. A configuration of points giving rise to a Seshadri curve C with $C^2 < -1$.

Example 2.4.3. Let X be the blow up of $\mathbb{P}^2$ at two points p_1 and p_2. Let $Y = Z = p_1 + p_2$. Then $F_s(Y) \in \mathrm{Eff}(X)$ for $s = 1$ (since $F_s(Y)$ is linearly equivalent to the proper transform C of the line through p_1 and p_2) and $F_t(Z) \in \mathrm{Nef}(X)$ for $t = 2$ (since $\ell - e_1, \ell - e_2$ linearly equivalent to prime divisors of nonnegative self-intersection, hence $2\ell - e_1 - e_2 \in \mathrm{Nef}(X)$), and we have $F_s(Y) \cdot F_t(Z) = 0$, so Proposition 2.4.2 we have $\varepsilon(Z) = 1/2$ and $\widehat{\alpha}(Z) = \widehat{\alpha}(Y) = 1$. Since $(F_2(Z))^2 > 0$, we see that C is a Seshadri curve for Z (and so we expect $C^2 < 0$ and indeed $C^2 = -1$), and in fact S is the unique Seshadri curve for Z.

Example 2.4.4. Let X be the blow up of $\mathbb{P}^2$ at three noncollinear points p_1, p_2 and p_3. Let $Y = p_1 + p_2$, $Z = p_1 + p_2 + p_3$ and let C_{ij} be the proper transform of the line through p_i and p_j for $i \neq j$. Then $F_1(Y) \in \mathrm{Eff}(X)$ and $F_2(Z) \in \mathrm{Nef}(X)$, and $F_1(Y) \cdot F_2(Z) = 0$. Thus again $\varepsilon(Z) = 1/2$ and $\widehat{\alpha}(Y) = 1$ and C_{12} is a Seshadri curve for Z. But C_{ij} is also a Seshadri curve for Y whenever $i \neq j$, so here Y does not have a unique Seshadri curve. Note that $C_{12} + C_{13} + C_{23} = F_3(2Z) = 2F_{\frac{3}{2}}(Z)$ is effective and has $F_3(2Z) \cdot F_2(Z) = 0$. Since $F_{\frac{3}{2}}(Z) \in \mathrm{Seff}(X)$, we get $\widehat{\alpha}(Z) = 3/2$.

Examples 2.4.3 and 2.4.4 exhibit Seshadri curves of self-intersection -1. If X is a blow up of $\mathbb{P}^2$ at r points p_i and if $d\ell - m_1 e_1 - \cdots - m_r e_r$ is linearly equivalent to an integral divisor C of $C^2 < 0$, then it is easy to see that C is a Seshadri curve for $Z = m_1 p_1 + \cdots + m_r p_r$. Thus it is easy to exhibit Seshadri curves C with $C^2 < -1$ (take the proper transform of the line through $r > 2$ collinear points, for example). However, no examples are known of a Seshadri curve C with $C^2 < -1$ when the points p_i are general. Indeed, it is an open problem to show that none exist. We now give an example (originally from [14] but considered from the present point of view in [10]) of a Z where one of the points with nonzero coefficient is general which does have a Seshadri curve C with $C^2 < -1$.

Example 2.4.5. Let X be obtained by blowing up 10 points of $\mathbb{P}^2$ arranged as follows. Start with 4 points (say $p_1, \ldots, p_4$), no three of which are collinear (represented by small open circles in Figure 3. The conics passing through these four points is a pencil (i.e., a 1-dimensional family) with three singular members. The singular points (p_5, p_6, p_7) of the singular members are shown as small black dots. Take the line through any two of the singular points of the 3 singular members (this is the dotted line in the figure); it intersects the third singular conic in two points (p_8, p_9), shown as open circles with a small black dot inside. This gives 9 points p_i, to which we add an additional general point p_0 for the 10 points which we blow up to obtain X. Then $4\ell - 3e_0 - e_1 - \cdots - e_9$ is linearly equivalent to an integral curve C with $C^2 = -2$ (see [10]). It is a Seshadri curve for $Z = 3p_0 + p_1 + \cdots + p_9$. It is not unique since the proper transform for any of the three lines through four of the points $p_1, \ldots, p_9$ is also a Seshadri curve for Z.

In the next proposition we assume the ground field is the complex numbers. We say that a statement holds for *very general* points $p_1, \ldots, p_r \in \mathbb{P}^2$ if the set of points in $(\mathbb{P}^2)^r$ fort which the claim is not true is not contained in a countable union of proper Zariski closed subsets.

Proposition 2.4.6. *For $Z = p_1 + \cdots + p_r$ for very general points $p_1, \dots, p_r \in \mathbb{P}^2$, we have $\widehat{\alpha}(Z) = r\varepsilon(Z)$.*

Proof. By Exercise 2.2.4 we have $\widehat{\alpha}(Z) \geq r\varepsilon(Z)$. By Exercise 2.2.3 we have $\varepsilon(Z) \leq 1/\sqrt{r}$ and $\widehat{\alpha}(Z) \leq \sqrt{r}$.

First suppose that $\varepsilon(Z) = 1/\sqrt{r}$ for some set of distinct points q_i. Since $\varepsilon(Z)$ is maximal in this case, there is no Seshadri curve. Thus no divisor $C = d\ell - m_1e_1 - \cdots - m_re_r$ with $C \cdot F_{\sqrt{r}}(Z) < 0$ is effective. There are at most countably many $C = d\ell - m_1e_1 - \cdots - m_re_r$ with $C \cdot F_{\sqrt{r}}(Z) < 0$, and for each one (by uppersemicontinuity of $h^0(X, \mathcal{O}_X(d\ell - m_1e_1 - \cdots - m_re_r)) > 0$), effectivity of C is a closed condition on the set U of distinct points $(p_1, \dots, p_r) \in (\mathbb{P}^2)^r$. These closed conditions are proper closed subsets since the points $(q_1, \dots, q_r)$ are in the complement of all of them. Since $\varepsilon(Z) = 1/\sqrt{r}$ holds on the complement of the union of these closed conditions, and it is nonempty, we see that $\varepsilon(Z) = 1/\sqrt{r}$ holds for very general points. I.e., $\varepsilon(Z) = 1/\sqrt{r}$ holds somewhere if and only if it holds for a very general set of points. But if $\varepsilon(Z) = 1/\sqrt{r}$ holds somewhere, then $\sqrt{r} = r\varepsilon(Z) \leq \widehat{\alpha}(Z) \leq \sqrt{r}$, hence we get $\widehat{\alpha}(Z) = r\varepsilon(Z) = \sqrt{r}$ for very general points.

Now assume that $\varepsilon(Z) = 1/\sqrt{r}$ does not hold for a very general set of distinct points. Thus it holds for no set of distinct points. This means there is some effective divisor C such that $C \cdot F_{\sqrt{r}}(Z) < 0$ for every choice of points $(p_1, \dots, p_r) \in U$. But C is linearly equivalent to some $d\ell - m_1e_1 - \cdots - m_re_r$. By uppersemicontinuity, effectivity of $d\ell - m_1e_1 - \cdots - m_re_r$ is a closed condition. Since no countable union of proper Zariski closed subsets equals U, this means there is a divisor $C = d\ell - m_1e_1 - \cdots - m_re_r$ with $C \cdot F_{\sqrt{r}}(Z) < 0$ such that the locus of points p_i such that C is effective is not contained in a proper closed set. I.e., there is one specific C which works for every set of distinct points. We may assume we picked such a C for which d is as small as possible, and $\sum_i m_i$ is as large as possible for that d. Then C is integral for a nonempty open subset of V'. (To justify this, it is enough to show for each $(p_1, \dots, p_r) \in V'$, there is no $B = d'\ell - m'_1e_1 - \cdots - m'_re_r$ such that both B and $C - B$ are effective and nontrivial. Assume both B and $C - B$ are effective; for specificity say $d' \geq d - d'$. Then we have $0 \leq d' \leq d$ and $m_i - d \leq m'_i \leq d'$. Thus there are only finitely many such $d'\ell - m'_1e_1 - \cdots - m'_re_r$. Since d is minimal, no such B with $0 < d' < d$ is effective for all points of U. This means there is a nonempty open subset V of U which excludes all sets of points p_i such that there is a B with $0 < d' < d$. Thus for each set of points p_i in V, for any B such that $B \cdot F_{\sqrt{r}}(Z) < 0$ where both B and $C - B$ are effective has $d' = d$, so $C - B = \sum_i a_ie_i$ where necessarily $a_i \geq 0$. But by maximality of $\sum_i m_i$ the loci such that $C - e_i$ is effective is a proper closed subset of V. Hence there is a nonempty open subset V' of V which excludes all points of V such that $C - B$ is nontrivial, and hence $B = C$.)

Reindexing induces an automorphism of U, hence for each permutation π, there is a nonempty open subset of U such that $\pi(C) = d\ell - m_{\pi(1)}e_1 - \cdots - m_{\pi(r)}e_r$ is effective and integral. Thus ther is a nonempty open subset W such that all $\pi(C) = d\ell - m_{\pi(1)}e_1 - \cdots - m_{\pi(r)}e_r$ are effective and integral. Adding them up gives an effective class $D = \delta\ell - \mu e_1 - \cdots - \mu e_r$ with $D \cdot F_{\sqrt{r}}(Z) < 0$. Since $D \cdot F_{\sqrt{r}}(Z) < 0$ implies $D^2 < 0$ (so $\delta^2 < r\mu^2$), and since the only components of D are the curves $\pi(C)$, we have $D \cdot \pi(C) < 0$ for some (and hence every) π.

Now consider $F_{r\mu/\delta}(Z)$. We have $D \cdot F_{r\mu/\delta}(Z) = 0$ and hence $\pi(C) \cdot F_{r\mu/\delta}(Z) = 0$ for all π. On the other hand, $\delta\mu F_{r\mu/\delta}(Z) = \delta D + (r\mu^2 - \delta^2)\ell$ is effective and meets each of its components ℓ and $\pi(C)$ nonnegatively, so is nef. By Proposition 2.4.2 we have $\varepsilon(Z) = \delta/(r\mu)$ and $\widehat{\alpha}(Z) = \delta/\mu$, hence $\widehat{\alpha}(Z) = r\varepsilon(Z)$. □

Remark 2.4.7. In the proof above, we saw that if $Z = p_1 + \cdots + p_r$ are very general, for any Seshadri curve $C = d\ell - m_1e_1 - \cdots - m_re_r$, $\pi(C) = d\ell - m_{\pi(1)}e_1 - \cdots - m_{\pi(r)}e_r$ is also a Seshadri curve for every permutation π. By Exercise 2.4.1, their classes are linearly independent, and since they are all orthogonal to $F_{1/\varepsilon(Z)}(Z)$, there can be at most r of them. Thus either $m_1 = \cdots = m_r$

(in which case we say C is uniform) or all but one of the m_i are equal (in which case we say C is almost uniform).

Remark 2.4.8. It will be helpful, for the next example, to recall the quadratic transform centered at three noncollinear points, p_1, p_2 and p_3 (also called a quadratic Cremona transformation). Geometrically, it is a birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by blowing up the points and blowing down the lines through pairs of the points. If we choose coordinates such that the points are the coordinate vertices $(0:0:1),(0:1:0),(1:0:0)$, then a point $(a:b:c)$ such that $abc \neq 0$ maps to $(\frac{1}{a}:\frac{1}{b}:\frac{1}{b}) = (bc:ac:ab)$. In the form $(a:b:c) \mapsto (bc:ac:ab)$ we see that the map is defined away from the coordinate vertices. Given a finite set of points, no three collinear, the quadratic transform centered at any three of them map the others to distinct points. Given such a set of points $p_1,\dots,p_r$, $r \geq 3$, if we apply the quadratic transform centered at p_1, p_2 and p_3, we have the images $p_4 \mapsto p'_4,\dots,p_r \mapsto p'_r$, and it is convenient to regard p'_1 as the image of the line through p_2 and p_3, p'_2 as the image of the line through p_1 and p_3, and p'_3 as the image of the line through p_1 and p_2. With this convention, applying the transform twice takes p_i to p_i for all i.

Note however, even if no three of the points $p_1,\dots,p_r$ are collinear, that three of the points $p'_1,\dots,p'_r$ could be collinear after we apply the quadratic transform centered at three of them. Suppose no three of the points are collinear, and we apply the quadratic transform centered at p_1, p_2 and p_3. Then the points p'_i are distinct, and three of them, say p'_i, p'_j, p'_k, are collinear if and only if none of i, j, k are among $1, 2, 3$ but $p_1, p_2, p_3, p_i, p_j, p_k$ lie on a conic. And assuming no three of the points p_i are collinear and no six lie on a conic, in order for no six of the image points to lie on a conic there can be no cubic through 7 of the points p_i singular at one of the seven. And in order for this to hold for the image points, similar higher order conditions must hold, and in order for those to hold for the image points, even higher order conditions must hold on p_i, etc. But if one successively applies a finite sequence of quadratic transforms with centers in a finite set of points (i.e., starting with a quadratic transform centered at three points of a set of points p_i, and then a second quadratic transform centered at three of the image points, and then a third quadratic transform centered at three points in the images of the image points, etc.), one may assume at each step that no three of the points is collinear, if the original points p_i are general, since after finitely many steps, the condition for there to be three among the resulting points which are collinear is a finite union of proper closed conditions on the points.

Example 2.4.9. Consider $Z = p_1 + \cdots + p_r$ general points $p_i \in \mathbb{P}^2$.

For $r = 4$, $2\ell - e_1 - \cdots - e_4$ is both nef and effective, so $\varepsilon(Z) = 1/2$ and $\widehat{\alpha}(Z) = 2$. Since $\varepsilon(Z)$ is maximal, there are no Seshadri curves.

For $r = 5$, $C = 2\ell - e_1 - \cdots - e_5$ is a Seshadri curve and $F = (5/2)\ell - (e_1 + \cdots + e_5) = (1/2)(2C + \ell)$ is nef, so $\varepsilon(Z) = 2/5$ and $\widehat{\alpha}(Z) = 2$. For an explanation of why C is irreducible, see Figure 4. Alternatively, apply the quadratic transform centered at p_1, p_2, p_3. Then as explained in Remark 2.4.8, we have the image points p'_i, no three of which are collinear. The image of the line L through p'_4 and p'_5 under the inverse transform is a conic Q through all five points p_i, and the image of Q under the original transform is L (hence Q is both unique and irreducible since L is).

For $r = 6$, $C_i = 2\ell - e_1 - \cdots - e_6 + e_i$ is a Seshadri curve for each i, and $F = (5/2)\ell - (e_1 + \cdots + e_6) = (1/10)(2C_1 + \cdots + 2C_6 + \ell)$ is nef, so $\varepsilon(Z) = 2/5$ and $\widehat{\alpha}(Z) = 2$. As in the case of $r = 5$, each C_i is irreducible.

For $r = 7$, $C_i = 3\ell - e_1 - \cdots - e_7 - e_i$ is a Seshadri curve for each i, and $F = (8/3)\ell - (e_1 + \cdots + e_7) = (1/9)((C_i + e_i) + C_1 + \cdots + C_7)$ is nef, so $\varepsilon(Z) = 3/8$ and $\widehat{\alpha}(Z) = 21/8$. (To see that C_i is irreducible, note that if it were not, then there would either be a line through 3 or more of the points, or a conic through 6 or more, but neither happens for general points. Alternatively, apply the quadratic transform centered at three of the points, one of which is p_i, so say p_i, p_j, p_k. The cubic C_i becomes the conic Q through p'_i and through the images of the four remaining points, and C_i is the image of this conic under the inverse transform. Thus C_i is unique and irreducible. Moreover, the singular

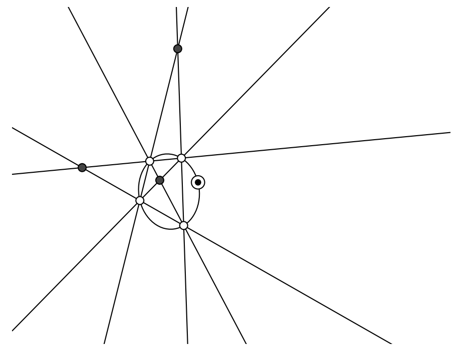


FIGURE 4. A pencil of conics through four general points $p_1, \dots, p_4$ (shown as white dots) has exactly three singular members (see the three pairs of lines crossing at the three black dots). Every other member of the pencil is irreducible (for example, the oval going through a fifth general point p_5, where p_5 is shown as a dotted circle).

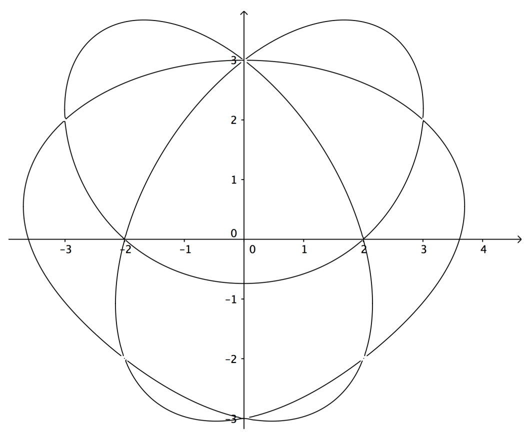


FIGURE 5. An irreducible sextic with a triple point and seven nodes.

point of each C_i is a node rather than a cusp. This is because if C_i were cuspidal, then Q would be tangent to the line through p'_j, p'_k. This is a proper closed condition on the points p_i, which general points can be assumed to avoid.)

For $r = 8$, $C_i = 6\ell - 2e_1 - \cdots - 2e_8 - e_i$ is a Seshadri curve for each i, and $F = (17/6)\ell - (e_1 + \cdots + e_8) = (1/36)((C_i + e_i) + 2C_1 + \cdots + 2C_8)$ is nef, so $\varepsilon(Z) = 6/17$ and $\widehat{\alpha}(Z) = 48/17$. (This sextic is also unique and irreducible for general points p_i, as can be seen by a more involved but basically similar argument involving quadratic transforms which can be used to obtain the sextic as the image of a unique irreducible curve of lower degree. See Figure 5 for a graph of an example irreducible sextic having a triple point at $p_1 = (0, 3, 1)$ and double points at $p_2 = (-3, 2, 1)$, $p_3 = (3, 2, 1)$, $p_4 = (-2, 0, 1)$, $p_5 = (2, 0, 1)$, $p_6 = (-2, -2, 1)$, $p_7 = (2, -2, 1)$ and $p_8 = (0, -3, 1)$.)

For $r = 9$, $C = 3\ell - e_1 - \cdots - e_9$ is both nef and effective, so $\varepsilon(Z) = 1/3$ and $\widehat{\alpha}(Z) = 3$. Since $\varepsilon(Z)$ is maximal, there are no Seshadri curves. (Here one cannot reduce a cubic through 9 general points to a curve of lower degree using quadratic transforms. But such a cubic is irreducible since otherwise it would have a component of a line through at least 3 of the points or a conic through at least 7 of them which we may assume never happens for general points. It is unique since each point imposes one condition on the 10 dimensional vector space of cubic forms; i.e., there is exactly one cubic vanishing at 9 general points. in fact, this cubic is smooth. (To see this, consider the 3-uple Veronese $v : \mathbb{P}^2 \to \mathbb{P}^9$. This is an embedding of $\mathbb{P}^2$ into $\mathbb{P}^9$, hence by Bertini's Theorem (which holds in every characteristic; see [32, Theorem 8.18]) the general hyperplane section is smooth. But the general hyperplane section is the cubic through 9 general points, hence there is a unique cubic through 9 general points, it is smooth and hence irreducible, since a plane curve which is not irreducible has points where the components meet, which must be singular points.)

Examples 2.4.3, 2.4.4 and 2.4.9 show for $r = 2, 3, 5, 6, 7, 8$ general points that Seshadri curves C exist for $Z = p_1 + \cdots + p_r$, and in each case satisfies $C^2 = C \cdot K_X = -1$. For $r > 9$ very general points, no examples are known of a Seshadri curve for $Z = p_1 + \cdots + p_r$, but it is an open problem to prove this and thus to compute $\varepsilon(Z)$ or $\widehat{\alpha}(Z)$, except when r is a square, in which case Nagata showed $\varepsilon(Z) = 1/\sqrt{r}$ [40].

Conjecture 2.4.10 (Nagata, [40])**.** *For $r > 9$ very general points of the complex plane $\mathbb{P}^2$, $\varepsilon(Z) = 1/\sqrt{r}$.*

It is also an open conjecture for $r > 9$ generic points over any algebraically closed field with sufficient transcendence degree over the prime field (where the prime field is the minimal subfield and *generic* means points whose coordinates are algebraically independent over the prime field.)

3. Negative curves, SHGH, bounded negativity and H-constants

Let X be obtained by blowing up points of $\mathbb{P}^2$. What kinds of integral or even reduced curves $C \subset X$ with $C^2 < 0$ can we get and how negative can C^2 be?

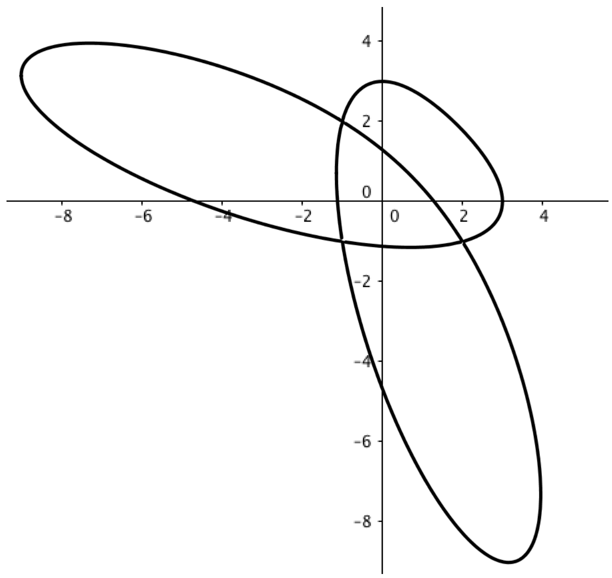
3.1. **Exceptional Curves.** We found examples above of irreducible curves C with $C^2 = -1$ on a blow up X of $\mathbb{P}^2$ at r general points. In each case C also satisfied $C \cdot K_X = -1$. An integral curve C on a smooth projective surface X with $C^2 = C \cdot K_X = -1$ is called an *exceptional curve*. By the adjunction formula for effective divisors on a smooth projective surface X, we have $C^2 + C \cdot K_X = 2p_C - 2$, where p_C is the arithmetic genus of C. It follows that an exceptional curve has $p_C = 0$, so by the next exercise, we see exceptional curves are smooth and rational.

Exercise 3.1.1. Let C be an integral curve on a smooth projective surface X. If $p_C = 0$, show that C is smooth and rational.

Answer: If C were not smooth, we could blow up a singular point and take its proper transform C' on the blown up surface X'. There we would have $(C')^2 < C^2$ and $C' \cdot K_{X'} < C \cdot K_X$, so $p_{C'} < p_C$, but for an integral curve the arithmetic genus is always nonnegative, so this is impossible if $p_C = 0$. Thus C is smooth, in which case the arithmetic genus is the geometric genus, which is thus 0 hence C is rational.

Example 3.1.2. Two cases of exceptional curves we did not see before are $C = 4\ell - 2e_1 - 2e_2 - 2e_3 - e_4 - \cdots - e_8$ and $D = 5\ell - 2e_1 - \cdots - 2e_6 - e_7 - e_8$, $\ell - e_1 - e_2$. In both cases, when the points p_i blown up are sufficiently general, C and D are the classes of effective integral divisors. This can be seen using the same argument based on quadratic transforms used above: the quadratic transform centered at p_1, p_2, p_3 applied to the quartic gives a conic through the image points $p'_4, \cdots, p'_8$, hence both the conic and the quartic are integral. See Figure 6 for a quartic through the points $p_1 = (-1, 2, 1)$, $p_2 = (-1, -1, 1)$, $p_3 = (2, -1, 1)$, $p_4 = (3, -3, 1)$, $p_5 = (-3, 3, 1)$, $p_6 = (3, -9, 1)$, $p_7 = (-9, 3, 1)$ and $p_8 = (-87, 117, 79)$, with nodes at the first three points.

And the quadratic transform centered at p_4, p_5, p_6 applied to the quintic gives a quartic through the image points $p'_1, \cdots, p'_8$ singular at p'_1, p'_2, p'_3; we just saw this quartic is integral, hence so is the quintic. See Figure 6 for a quintic through the points $p_1 = (-3, 0, 1)$, $p_2 = (-2, -2, 1)$, $p_3 = (2, -45, 20)$, $p_4 = (2, -2, 1)$, $p_5 = (1, 0, 1)$, $p_6 = (-2, 2, 1)$, $p_7 = (0, 1, 1)$ and $p_8 = (2, 2, 1)$, with nodes at the first six points.



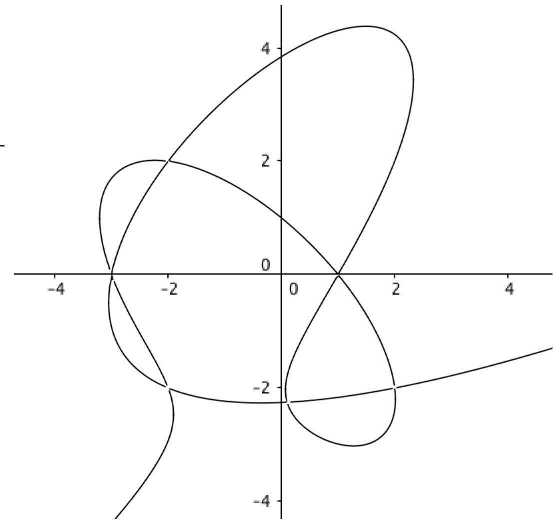


FIGURE 6. An irreducible quartic with three nodes and an irreducible quintic with six nodes.

Exercise 3.1.3. Given a blow up X of $\mathbb{P}^2$ at r distinct points $p_1, \dots, p_r$, we have the classes $\ell, e_1, \dots, e_r$. Recall that $-K_X = 3\ell - e_1 - \cdots - e_r$. Show that the subspace $K_X^\perp \subset \mathrm{Cl}(X)$ of classes C with $C \cdot K_X = 0$ is even (i.e., if $C \cdot K_X = 0$, then C^2 is even). Then show that $K_X^\perp$ is negative definite if and only if $r \leq 8$ (i.e., where negative definite means that if $C \cdot K_X = 0$, then $C^2 \leq= 0$ with equality if and only if $C = 0$). Finally, find all solutions $C = d\ell - m_1e_1 - \cdots - m_re_r \in K_X^\perp$ with $C^2 = -2$ when $r \leq 8$ (and, to keep down the number of trivial variations, assume $m_1 \geq m_2 \geq m_3 \geq \cdots \geq m_8$ and $d \geq 0$).

Answer: It is easy to see that $v_0 = \ell - e_1 - e_2 - e_3$, $v_1 = e_1 - e_2, \dots, v_7 = e_7 - e_8$ gives a basis for $K_X^\perp$ over the integers. Since $v_i^2 = -2$ for each i, it is clear that $(\sum_i a_iv_i)^2$ is even, since expanding the product gives a sum $\sum_i a_i^2v_i^2$ which is even, plus the cross terms $\sum_{i<j} 2a_ia_jv_i \cdot v_j$, which is also even.

To show $K_X^\perp$ is negative definite for $r \leq 8$, we may as well let $r = 8$, since $K_X^\perp$ for $r < 8$ is contained in $K_X^\perp$ with $r = 8$ and the intersection products are the same. We now have the $\mathbb{Q}$-basis $u_0 = 8\ell - 3(e_1 + \cdots + e_8)$, $u_1 = v_1$, $u_2 = e_1 + e_2 - 2e_3$, $u_3 = e_1 + e_2 + e_3 - 3e_4$, $\dots$, $u_8 = e_1 + \cdots + e_7 - 7e_8$, which is orthogonal. Since $u_i^2 < 0$ for each i, we see that $K_X^\perp$ is negative definite. For $r \geq 9$, we have $D = (r-9)r\ell - 3K_X \in K_X^\perp$. This is nonzero, but $D^2 = r(r-9) \geq 0$, so $K_X^\perp$ is not negative definite.

Now we find all solutions $C = d\ell - m_1e_1 - \cdots - m_re_r \in K_X^\perp$ with $C^2 = -2$ when $r \leq 8$. Since any solution with $r < 8$ also gives a solution for $r = 8$ by taking $m_i = 0$ for $i > r$, we continue to assume $r = 8$.

Now suppose $C = d\ell - m_1e_1 - \cdots - m_8e_8 \in K_X^\perp$. Let $D = d\ell - m(e_1 + \cdots + e_8)$, where m is the average of the m_i. Then $D \in K_X^\perp$, but $D^2 \geq C^2$ (since $D^2 - C^2 = \sum_i(m_i^2 - m^2) \geq 0$) with $D^2 > C^2$ unless $D = C$. Note that $3d - 8m = 0$, so $m = 3d/8$, hence $D^2 = d^2 - 8m^2 = -d^2/8$. Thus $-2 = C^2 \leq D^2 = -d^2/8$, so $0 \leq d \leq 4$. But if $d = 4$, then $D^2 = -2$ and $m = 3/2$; since m is not an integer we have $C^2 < D^2 = -2$, so $d = 4$ cannot occur. Thus $0 \leq d \leq 3$. If $d = 0$, then we have $\sum_i m_i = 0$ and $\sum_i m_i^2 = 2$, so the only solution is $-e_1 + e_8$ (under the assumption that $m_1 \geq m_2 \geq m_3 \geq \cdots \geq m_8$, or $e_i - e_j$, $i \neq j$ without the assumption). If $d = 1$, then we have $\sum_i m_i = 3$ and $\sum_i m_i^2 = 3$, thus the only solution is $\ell - e_1 - e_2 - e_3$. If $d = 2$, then we have $\sum_i m_i = 6$ and $\sum_i m_i^2 = 6$, thus the only solution is $2\ell - e_1 - \cdots - e_6$. If $d = 3$, then we have $\sum_i m_i = 9$ and $\sum_i m_i^2 = 11$, thus the only solution is $3\ell - 2e_1 - e_2 - \cdots - e_8$.

Exercise 3.1.4. Given a blow up X of $\mathbb{P}^2$ at r distinct points $p_1, \ldots, p_r$, we have the classes $\ell, e_1, \ldots, e_r$. Find all solutions $C = d\ell - m_1e_1 - \cdots - m_re_r$ with $r \leq 8$ to $C^2 = C \cdot K_X = -1$ (and again, to keep down the number of trivial variations, assume $m_1 \geq m_2 \geq m_3 \geq \cdots \geq m_8$).

Answer: We may as well let $r = 8$, since any solution with $r < 8$ also gives a solution for $r = 8$ by taking $m_i = 0$ for $i > r$. Suppose we have $D^2 = -2$ and $D \cdot K_X = 0$. Then $(D - K_X)^2 = (D - K_X) \cdot K_X = -1$, so we get all solutions for C by adding $-K_X$ to all solutions for D. Adding $-K_X$ to $\pm D$ for each D found in Exercise 3.1.3, starting with those D with $d = 0$ and working our way up, we get: $3\ell - 2e_1 - e_2 - \cdots - e_7$; $4\ell - 2e_1 - 2e_2 - 2e_3 - e_4 - \cdots - e_8$, $2\ell - e_1 - e_2 - \cdots - e_5$; $5\ell - 2e_1 - \cdots - 2e_6 - e_7 - e_8$, $\ell - e_1 - e_2$; $6\ell - 3e_1 - 2e_2 - \cdots - 2e_8$, e_9.

Exercise 3.1.5. Let X be the blow up of $\mathbb{P}^2$ at 9 distinct points $p_1, \ldots, p_9$. For each $v \in K_X^\perp$ with $v \cdot e_9 = 0$ let $C_v = e_9 + v + (v^2/2)K_X$. Show that (in spite of the 2 in the denominator) C_v is an integral divisor class, that it satisfies $C^2 = C \cdot K_X = -1$ and that it is the class of an effective divisor. Finally, if $C^2 = C \cdot K_X = -1$, show that $C = C_v$ for some $v \in K_X^\perp$ with $v \cdot e_9 = 0$.

Answer: Since $v \in K_X^\perp$ which is even by Exercise 3.1.3, we know that v^2 is even, hence C_v is integral. We compute that $C_v \cdot K_X = e_9 \cdot K_X = -1$ and that $C_v^2 = e_9^2 = -1$.

Note that $v^2 \leq 0$, with 0 if and only if $v = 0$, since v is in $K_{X'}^\perp \subset \mathrm{Cl}(X')$ for the blow up X' of the first eight points, and we know that $K_{X'}^\perp$ is negative definite. Thus $\ell \cdot C_v = \ell \cdot v - 3v^2/2 = \ell \cdot v + 3|v^2|/2$. Thus this is nonnegative if $\ell \cdot v \geq 0$. To see that this is always nonnegative, write v in the orthogonal basis $u_0 = \ell - 3(e_1 + \cdots + e_8)/8$, $u_1 = v_1$, $u_2 = e_1 + e_2 - 2e_3$, $u_3 = e_1 + e_2 + e_3 - 3e_4$, $\ldots$, $u_8 = e_1 + \cdots + e_7 - 7e_8$. Thus $v = \sum_i a_iu_i$. If $a_0 \geq 0$, then we saw that $\ell \cdot C_v \geq 0$. Say $a_0 < 0$. Then $v^2 \leq (a_0u_0)^2 \leq 0$ and $\ell \cdot v = \ell \cdot a_0u_0 = a_0$, so $\ell \cdot C_v = \ell \cdot v + 3|v^2|/2 \geq \ell \cdot a_0u_0 + 3a_0^2|u_0^2|/2 = 3a_0^2/16 - |a_0|$. This is positive if $a_0 \leq -6$, and since v is integral, we just need to check the cases $0 > a_0 \geq -5$ that the minimum value of $|v^2|$ is enough to ensure that $\ell \cdot v + 3|v^2|/2 \geq 0$.

So let $v = -b_0\ell + b_1e_1 + \cdots + b_8e_8$ with $0 > b_0 \geq -5$. Clearly we may assume that $b_i \geq 0$ for all $i > 0$. For $0 > b_0 \geq -3$, the minimum value of $|v^2|$ is 2 (since the intersection form is negative definite and even); for example, we get $|v^2| = 2$ with $-(\ell - e_1 - e_2 - e_3)$, $-(2\ell - e_1 - \cdots - e_6)$, and $-(3\ell - 2e_1 - e_2 - \cdots - e_8)$. But in these cases we get $\ell \cdot v + 3|v^2|/2 = b_0 + 3 \geq 0$.

We know from the solution to Exercise 3.1.3 that there are no classes v with $v^2 = -2$ when $b_0 < -3$, so the minimum possible value of $|v^2|$ is 4 (and in fact $v^2 = -4$ for $-(4\ell - 2e_1 - \cdots - 2e_4 - e_5 - \cdots - e_8)$ and $-(5\ell - 2e_1 - \cdots - 2e_7 - e_8)$). Thus $\ell \cdot v + 3|v^2|/2 = b_0 + 6 \geq 1$ in both cases.

Now from Riemann-Roch we have $h^2(X, C_v) = 0$ since $C - v \cdot \ell \geq 0$, so $h^0(X, C_v) = h^1(X, C_v) + (C_v^2 - C_v \cdot K_X)/2 + 1 \geq 1$.

Finally, let $C = d\ell - m_1e_1 - \cdots - m_9e_9$ satisfy $C^2 = C \cdot K_X = -1$. Then $v = C - e_9 + (m_9 + 1)K_X$ has $D \cdot e_9 = 0$ and $v \in K_X^\perp$, and we have $C_v = e_9 + (C - e_9 + (m_9 + 1)K_X) + (C - e_9 + (m_9 + 1)K_X)^2/2)K_X = C + (m_9 + 1)K_X) - (m_9 + 1)K_X = C$.

Exercise 3.1.6. Let X be the blow up of $\mathbb{P}^2$ at $r > 9$ distinct points $p_1, \ldots, p_9$. Show that there exist solutions $C = d\ell - m_1e_1 - \cdots - m_re_r$ to $C^2 = C \cdot K_X = -1$ with $C \cdot \ell < 0$, and thus C cannot be the class of an effective divisor.

Answer: Let $C = -3\ell + e_1 + \cdots + e_{10} + m(3\ell - e_1 - \cdots - e_9)$ for any $m \leq 0$.

For general points $p_1, \ldots, p_9$, the points lie on a smooth cubic C', so its proper transform C on the blow up X of the points is a smooth elliptic curve. Since $C = -K_X$ is nef, there is no integral curve D with $-K_X \cdot D < 0$. If the points p_i are sufficiently general over a sufficiently large field K (such as either the complex numbers or K has sufficiently large transcendence degree over the prime field), then the inclusion $C \subset X$ induces an injection $\mathrm{Cl}(X) \to \mathrm{Cl}(C)$. This means the only integral curve D with $-K_X \cdot D = 0$ is $D = C$. Now suppose E is a divisor satisfying $E^2 = E \cdot K_X = -1$. Then for sufficiently general points p_i we must have that E is linearly equivalent to an integral curve. We know that E is linearly equivalent to an effective divisor, and taking E to be that divisor we see that it is reduced with a single component. (This is because $-K_X \cdot E = 1$ so there is one component which meets C, and all other components are orthogonal to C, hence must be C. Thus $E = A + mC$ for some $m \geq 0$ and some integral divisor A with $-K_X \cdot A = 1$. But now $-1 = E^2 = (A + mC)^2 = A^2 + 2mA \cdot C = A^2 + 2m$, so $A^2 = -1 - 2m$, so $-2 - 2m = A^2 + A \cdot K_X = 2p_A - 2$, so $-m = p_A \geq 0$ hence $m = 0$.)

In particular, the surface X obtained by blowing up $r = 9$ (and hence $r \geq 9$ since the curves arising after the first 9 blow ups remain when blowing up additional points) sufficiently general points of a smooth plane cubic C' over a sufficiently large field K, has infinitely many smooth rational curves E with $E^2 = -1$.

3.2. **The Weyl group and exceptional curves.**

For $r > 9$, Exercise 3.1.6 shows that not every solution to $E^2 = E \cdot K_X = -1$ in $\mathrm{Cl}(X)$ comes from an exceptional curve. So it is of interest to know which solutions do.

Remark 3.2.1. Given distinct points $p_1, \ldots, p_r \in \mathbb{P}^2$, let X be the surface obtained by blowing up the points. The birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by the quadratic transform centered at noncollinear points p_1, p_2, p_3 corresponds to blowing up the points p_1, p_2, p_3 and blowing down the proper transforms of the lines through pairs of the points (see Remark 2.4.8). Thus it is the birational map given by $g \circ f^{-1}$, where $f : X \to P^2$ is the morphism give by blowing up the points p_i and $g : X \to \mathbb{P}^2$ is the morphism given by contracting $e'_1 = \ell - e_2 - e_3, e'_2 = \ell - e_1 - e_3, e'_3 = \ell - e_1 - e_2, e'_4 = e_4, \ldots, e'_r = e_r$. The pullback of a line with respect to g is $\ell' = d\ell - m_1 - e_1 - \cdots - m_re_r$, and satisfies $1 = (\ell')^2 = (d\ell - m_1 - e_1 - \cdots - m_re_r)^2$, $0 = \ell' \cdot e'_i = (d\ell - m_1 - e_1 - \cdots - m_re_r) \cdot e'_i$ for all i, hence $m_i = 0$ for $i > 3$, and $d = m_1 + m_2 = m_1 + m_3 = m_2 + m_3$, hence $m_1 = m_2 = m_3 = d/2$, so $1 = 4m^2 - 3m^2$. Thus $m = \pm 1$, but ℓ' is nef, so $m = 1$. Thus $\ell' = 2\ell - e_1 - e_2 - e_3$; i.e., a general line pulls back to a conic through p_1, p_2, p_3 under the quadratic transform.

Note that $\ell \mapsto \ell'$ and $e_i \mapsto e'_i$ for all i can be regarded as a change of basis for $\mathrm{Cl}(X)$. It is given by the "reflection" s_0 by the class $v_0 = \ell - e_1 - e_2 - e_3$; i.e., we define $s_0 : \mathrm{Cl}(X) \to \mathrm{Cl}(X)$ by $s_0(x) = x + (x \cdot v_0)v_0$. Similarly for $1 \leq i < r$ we can define $s_i : \mathrm{Cl}(X) \to \mathrm{Cl}(X)$ by $s_i(x) = x + (x \cdot v_i)v_i$, where $v_i = e_i - e_{i+1}$. For each $0 \leq i < r$, note that s_i^2 is the identity, and that $s_i(v_i) = -v_i$. When $r = 0$ or 1, let W_r be the identity group; when $r = 2$, let W_r be generated by s_1 (so W_r is the group of permutations of e_1 and e_2), and when $r \geq 3$, let W_r be generated by $s_0, \ldots, s_{r-1}$.

Every element $w \in W_r$ can be regarded as a change of basis; given $F = d\ell - m_1e_1 - \cdots - m_re_r \in \mathrm{Cl}(X)$, let $w(F) = d'\ell - m'_1e_1 - \cdots - m'_re_r$ and let $\ell' = w^{-1}(\ell), e'_1 = w^{-1}(e_1), \ldots, e'_r = w^{-1}(e_r)$.

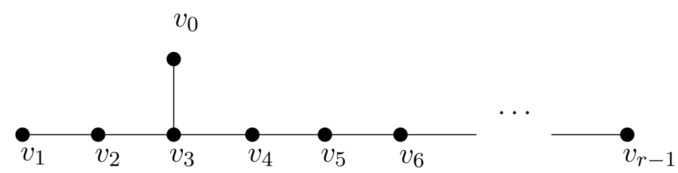


FIGURE 7. The Dynkin diagram associated to the Weyl group W_r.

Then $w(F) = d'w(\ell') - m_1'w(e_1') - \cdots - m_r'w(e_r') = d'\ell - m_1'e_1 - \cdots - m_r'e_r$, hence applying w^{-1} gives $F = d'\ell' - m_1'e_1' - \cdots - m_r'e_r'$.

The group W_r is the Weyl group associated to a Dynkin diagram. (For $r \geq 4$, the Dynkin diagram is given in Figure 7. The dots represent the elements v_i and the number of edges between dots v_i and v_j give the value of $v_i \cdot v_j$.) It is infinite exactly when $r \geq 9$. Note that the subgroup of W_r generated by $s_1, \ldots, s_{r-1}$ is just the group of permutations on $e_1, \ldots, e_r$. It is also useful to note that $w(K_X) = K_X$ for all $w \in W_r$, and that W_r preserves the intersection form; that is, $w(x) \cdot w(y) = x \cdot y$ for all $w \in W_r$ and all $x, y \in \mathrm{Cl}(X)$.

Example 3.2.2. When $r \geq 3$ and the points p_i are sufficiently general, it turns out that the classes of the exceptional curves form a single orbit under W_r [41]; i.e., if E is the class of an integral curve C with $C^2 = C \cdot K_X = -1$, then $w(E) = e_r$ for some $w \in W_r$, and every class $w(e_r)$ is the class of an integral curve C with $C^2 = C \cdot K_X = -1$. When the points are not general, a class E in the orbit of e_r need not be the class of an exceptional curve. For example, say p_1, p_2, p_3 are collinear. Then the only exceptional curves on X are e_1, e_2 and e_3, even though $\ell - e_1 - e_2$ is in the orbit of e_3. The problem is that $\ell - e_1 - e_2$ has two components, $\ell - e_1 - e_2 - e_3$ and e_3. However, since exceptional curves are stable under generalizing the points p_i, if E is an exceptional curve, then its class is in the orbit of e_r. But note for $w, w' \in W_r$ that $w(e_r) \cdot w'(e_r) \geq 0$ unless $w(e_r) = w'(e_r)$.

As an example, consider $F = 5\ell - 3e_1 - 3e_2 - e_3 - \cdots - e_{10}$ satisfies $F^2 = F \cdot K_X = -1$, but F is never the class of an exceptional curve. To see this, consider $s_0(F) = 3\ell - e_1 - e_2 + e_3 - e_4 \cdots - e_{10}$. Thus $s_0(F) \cdot e_3 < 0$, even though $s_0(F) \neq e_3$. Thus F cannot be in the orbit of e_{10}, hence F is never the class of an exceptional curve.

To avoid special but not very interesting cases, when we consider the action of W_r on $\mathrm{Cl}(X)$ we will always assume that $r \geq 3$. The previous example shows it is useful to be able to tell when classes F and G are in the same orbit of W_r. Define the subsemigroup $\Delta_r \subset \mathrm{Cl}(X)$ by

$$\Delta_r = \{F \in \mathrm{Cl}(X) : F \cdot v_i \geq 0, 0 \leq i < r\}.$$

Then Δ_r is a fundamental domain for $W_r\Delta_r = \cup_{w \in W_r} w\Delta_r$ (known as Tit's cone). In particular, for each element $F \in \Delta_r$, the orbit W_rF meets Δ_r only at F; i.e., $W_rF \cap \Delta_r = \{F\}$ [52, Theorem 4.3(a)]. Another useful fact is that if $F \in \Delta_r$, then for all $w \in W_r$ we have $w(F) = F + v$ for some nonnegative linear combination v of $v_0, \ldots, v_{r-1}$; see [26, Lemma 0.9] or [38, I.3.3].

Another useful fact that can be proved directly from the action of W_r (but which takes some development) is that if $e, e' \in W_re_r$ and $e \neq e'$, then $e \cdot e' \geq 0$. Another way to see this is that for sufficiently general points p_i (as mentioned above), the classes distinct classes in W_re_r are classes of prime divisors. Thus $e, e' \in W_re_r$ with $e \neq e'$ imlplies that $e \cdot e' \geq 0$.

Exercise 3.2.3. Let X be the blow up of $\mathbb{P}^2$ at distinct points $p_1, \ldots, p_r$ for $r \geq 3$. Show that Δ_r is the set of all nonnegative linear combinations of the classes $\nu_{-1} = -3\ell + e_1 + \cdots + e_r$, $\nu_0 = -2\ell + e_1 + \cdots + e_r$, $\nu_1 = -2\ell + e_2 + \cdots + e_r$, $\nu_2 = -\ell + e_3 + \cdots + e_r$, $\nu_3 = e_4 + \cdots + e_r$, $\ldots$, $\nu_{r-1} = e_r$, and $\nu_r = 3\ell - e_1 - e_2 - e_3 - \cdots - e_r$.

Answer: For $0 \leq i < r$ that ν_i is dual to v_i; $\nu_i \cdot v_j$ is 1 if $i = j$ and 0 if $i \neq j$. Thus, given any $F \in \Delta_r$, let $G = \sum_i (F \cdot v_i)\nu_i$. Then $F - G$ is orthogonal to every v_i. But the only classes orthogonal to all v_i are the integer multiples of $-K_X = 3\ell - e_1 - \cdots - e_r$ (here is where we use $r \geq 3$). Thus $F - G$ is a multiple of $-K_X = \nu_r$, hence $\nu_0, \ldots, \nu_{r-1}$ together with ν_r and ν_{-1} generate Δ_r.

For geometrical and algorithmic reasons, subcones $\Delta''_r \subset \Delta_r$ and $\Delta'_r \subset \Delta_r$ are more useful. Define

$$\Delta''_r = \{F \in \Delta_r : F \cdot e_r \geq 0\}$$

and

$$\Delta'_r = \{F \in \Delta_r : F \cdot \ell \geq 0, F \cdot (\ell - e_1) \geq 0\}.$$

Part of the geometrical interest of Δ''_r and Δ'_r comes from the following exercise and subsequent theorem.

Exercise 3.2.4. Let X be the blow up of $\mathbb{P}^2$ at distinct points $p_1, \ldots, p_r$ for $r \geq 3$. Show that $W_r\Delta'_r = \{F \in \mathrm{Cl}(X) : w(F) \cdot \ell \geq 0, w(F) \cdot (\ell - e_1) \geq 0$ for all $w \in W_r\}$.

Answer: For each $w \in W_r$ there is (as noted above) a nonnegative linear combination v of $v_0, \ldots, v_{r-1}$ such that $w(F) = F + v$, so $w(F) \cdot \ell, w(F) \cdot (\ell - e_1) \geq 0$ for all $w \in W_r$ and all $F \in \Delta'_r$. Thus $W_r\Delta'_r \subseteq \{F \in \mathrm{Cl}(X) : w(F) \cdot \ell, w(F) \cdot (\ell - e_1) \geq 0$ for all $w \in W_r\}$. On the other hand, if $G \in \{F \in \mathrm{Cl}(X) : w(F) \cdot \ell, w(F) \cdot (\ell - e_1) \geq 0$ for all $w \in W_r\}$, then there is an element $w \in W_r$ such that $w(G) \cdot \ell$ is as small as possible, which if we write $w(G) = d\ell - m_1e_1 - \cdots - m_re_r$, ensures that $d \geq m_{i_1} + m_{i_2} + m_{i_3}$ for all $i_1 < i_2 < i_3$, and by reordering the coefficients m_i we may assume that $m_1 \geq \cdots \geq m_r$ and $d \geq m_1 + m_2 + m_3$. Thus $w(G) \in \Delta_r$ and by assumption, $w(G) \cdot \ell, w(G) \cdot (\ell - e_1) \geq 0$, so $w(G) \in \Delta'_r$ and hence $G \in W_r\Delta'_r$.

Theorem 3.2.5. *Let X be the blow up of $r \geq 3$ distinct points $p_i \in \mathbb{P}^2$.*

(a) *Then $W_r\Delta''_r = \{F \in \mathrm{Cl}(X) : F \cdot w(e_r) \geq 0$ for all $w \in W_r\}$ and hence $\mathrm{Nef}(X) \subseteq W_r\Delta''_r$.*
(b) *If the points p_i are sufficiently general (very general over the complexes or generic over an arbitrary algebraically closed field K), then $\mathrm{Eff}(X) \subseteq W_r\Delta'_r$.*

Proof. (a) If $F \cdot w(e_r) \geq 0$ for all $w \in W_r$, then also $w'(F) \cdot w'w(e_r)$ for all $w, w' \in W_r$. In particular, $w(F) \cdot \ell \geq 0$ for all $w \in W_r$, since $\ell = (\ell - e_1 - e_2) + e_1 + e_2$, and $(\ell - e_1 - e_2), e_1, e_2$ all are in the orbit of e_r (since $r \geq 3$). Thus there is a minimum value of $w(F) \cdot \ell$. Let $d\ell - m_1e_1 - \cdots - m_re_r = w(F)$ for some w that achieves this minimum, hence $d \geq 0$. Since W_r includes the permutations on $m_1, \ldots, m_r$, we may as well assume that $m_1 \geq m_2 \geq \cdots \geq m_r$ (hence $w(F) \cdot v_i \geq 0$ for $i > 0$). Since $w(F) \cdot e_r \geq 0$ we have $m_r \geq 0$ (hence $w(F) \cdot e_r \geq 0$). If $d < m_1 + m_2 + m_3$, then the reflection s_0 by v_0 applied to $w(F)$ would give $s_0(w(F)) \cdot \ell < w(F) \cdot \ell$, so we must have $d \geq m_1 + m_2 + m_3$, hence $w(F) \cdot v_0 \geq 0$. Thus $w(F) \in \Delta''_r$; i.e., $\{F \in \mathrm{Cl}(X) : F \cdot E \geq 0$ for all $E \in W_re_r\} \subseteq W_r\Delta''_r$.

On the other hand, if $F \in W_r\Delta''_r$, then $w(F) \in \Delta''_r$ for some $w \in W_r$, so $w(F) \cdot e_r \geq 0$, and since $w(F) \in \Delta'_r$, for each $w' \in W_r$ there is (as noted above) a nonnegative linear combination v of $v_0, \ldots, v_{r-1}$ such that $w'(w(F)) = w(F) + v$, so $w'(w(F)) \cdot e_r = (w(F) + v) \cdot e_r \geq 0$ for all $w \in W_r$ and all $F \in \Delta''_r$. Thus $F \cdot w(e_r) \geq 0$ for all $w \in W_r$, so $F \in \{F \in \mathrm{Cl}(X) : F \cdot w(e_r) \geq 0$ for all $w \in W_r\}$.

Since $e_r \in \Delta_r$ and $\ell \cdot e_r = 0$, we have $\ell \cdot (e_r) \geq 0$ for all $w \in W_r$. Thus $h^2(X, w(e_r)) = 0$ for all $w \in W_r$. Therefore $h^0(X, w(e_r)) - h^1(X, w(e_r)) = \frac{(w(e_r))^2 - K_X \cdot w(e_r)}{2} + 1 = \frac{e_r^2 - K_X \cdot e_r}{2} + 1 = 1$, so $h^0(X, w(e_r)) > 0$. It now follows that $\mathrm{Nef}(X) \subseteq W_r\Delta''_r$.

(b) Given that the points p_i are sufficiently general, it follows by [26, Theorem 0.1] that $w(\ell)$ and $w(\ell-e_1)$ are nef for all $w \in W_r$. Let $F = d\ell - m_1e_1 - \cdots - m_re_r$ be the class of an effective divisor. Then $F \cdot w(\ell) \geq 0$ and $F \cdot w(\ell - e_1) \geq 0$ for all $w \in W_r$, so also $w(F)\cdot \ell \geq 0$ and $w(F)\cdot(\ell-e_1) \geq 0$ for all $w \in W_r$, hence $F \in W_r\Delta'_r$, so $\mathrm{Eff}(X) \subseteq W_r\Delta'_r$. □

Remark 3.2.6. Note that $\mathrm{Eff}(X) \subseteq W_r\Delta'_r$ fails in general, since, for one example, $\ell - e_1 - e_2 - e_3$ can be in $\mathrm{Eff}(X)$ but it is never in $W_r\Delta'_r$. Also, equality in $\mathrm{Nef}(X) \subseteq W_r\Delta''_r$ can fail for $r > 9$, since in that case $-K_X = 3\ell - e_1 - \cdots - e_r$ is never in $\mathrm{Nef}(X)$ but is in Δ''_r.

Exercise 3.2.7. Let X be the blow up of $\mathbb{P}^2$ at distinct points $p_1, \ldots, p_r$ for $r \geq 3$. Show that Δ''_r is the set of all nonnegative linear combinations of the classes $h_0 = \nu_r + \nu_0 = \ell$, $h_1 = \nu_r + \nu_1 = \ell - e_1$, $h_2 = \nu_r + \nu_2 = 2\ell - e_1 - e_2$, $h_3 = \nu_r + \nu_3 = 3\ell - e_1 - e_2 - e_3$, …, $h_{r-1} = \nu_r + \nu_{r-1} = 3\ell - e_1 - e_2 - e_3 - \cdots - e_{r-1}$, and $h_r = 3\ell - e_1 - e_2 - e_3 - \cdots - e_r$, and thus that $\Delta''_r \subseteq \Delta'_r$. Moreover, show that $F = d\ell - m_1e_1 - \cdots - m_re_r \in \Delta''_r$ if and only if $d \geq m_1 + m_2 + m_3$ and $m_1 \geq \cdots \geq m_r \geq 0$. And finally show that each element of Δ''_r is a nonnegative linear combination of the h_i in a unique way.

Answer: It is easy to check that nonnegative linear combinations of the classes h_i are in Δ''_r and in Δ'_r. Moreover, for $0 \leq i, j < r$, we have $h_i \cdot v_j = \delta_{ij}$, the Kronecker δ, while $h_r \cdot v_j = 0$ for all j. Thus if $F \in \Delta''_r$, then for $G = \sum_{i=0}^{r-1}(F \cdot v_i)h_i$, the class $F - G$ is orthogonal to all v_i, hence $F = G + m(3\ell - e_1 - \cdots - e_r) = G + mh_r$ for some integer m. But $F \cdot e_r \geq 0$ while $G \cdot e_r = 0$, so $m \geq 0$.

Now assume $F = d\ell - m_1e_1 - \cdots - m_re_r \in \Delta''_r$. Since $\Delta''_r \subset \Delta_r$, we have $d \geq m_1 + m_2 + m_3$ and $m_1 \geq \cdots \geq m_r$. Since $F \cdot e_r \geq 0$ we have $m_r \geq 0$. Conversely, it is clear from the definition of Δ''_r and of Δ_r that $d \geq m_1+m_2+m_3$ and $m_1 \geq \cdots \geq m_r \geq 0$ implies $F = d\ell - m_1e_1 - \cdots - m_re_r \in \Delta''_r$. The uniqueness comes from the fact that the classes h_i are linearly independent.

Remark 3.2.8. Elements of Δ'_r have Zariski-like decompositions, as we now show. Let $F = d\ell - m_1e_1 - \cdots - m_re_r \in \Delta'_r$. Define F_- to be $F_- = -\sum_E (E \cdot F)E$ where the sum is over all $E \in W_re_r$ with $E \cdot F < 0$. We show this is a finite sum, that $F - F_- = F_+ \in \Delta''_r$ and that $(F_+)\cdot(F_-) = 0$.

First assume $F\cdot(\ell - e_1 - e_2) < 0$. Since $F \in \Delta'_r$, we have $d \geq 0$, $d \geq m_1$ (hence $d + m_2 \geq m_1 + m_2$ so $m_2 \geq m_1 + m_2 - d$), $d \geq m_1 + m_2 + m_3$ and $m_1 \geq \cdots \geq m_r$. Since $d - m_1 - m_2 = F \cdot (\ell - e_1 - e_2) < 0$, we have $0 < m_1 + m_2 - d \leq m_2 \leq m_1$. Thus $F' = F - (m_1 + m_2 - d)(\ell - e_1 - e_2) = (2d - m_1 - m_2)\ell - (d - m_2)e_1 - (d - m_1)e_2 - m_3e_3 - \cdots - m_re_r$. Since $d \geq m_1 + m_2 + m_3$, we see $0 > d - m_1 - m_2 \geq m_3 \geq \cdots \geq m_r$, so $m_i < 0$ for $i \geq 3$. Thus $F = ((2d - m_1 - m_2)\ell - (d - m_2)e_1 - (d - m_1)e_2) + (m_1 + m_2 - d)(\ell - e_1 - e_2) + |m_3|e_3 + \cdots + |m_r|e_r$. Let $F_+ = ((2d - m_1 - m_2)\ell - (d - m_2)e_1 - (d - m_1)e_2)$.

Note that $(F_+)\cdot \ell \geq 0, (F_+)\cdot(\ell - e_1) \geq 0, (F_+)\cdot(\ell - e_1 - e_2) = (F_+)\cdot v_0 = 0$ and $(F_+)\cdot v_i \geq 0$ for all $i > 0$ and $(F_+)\cdot e_r \geq 0$ (since $r \geq 3$). Thus $F_+ \in \Delta''_r$. So $(F_+)\cdot w(e_r) \geq 0$ for all $w \in W_r$. Therefore for any element $e \in W_re_r$ other than $\ell - e_1 - e_2, e_3, \ldots, e_r$, we have $e \cdot e' \geq 0$ for all $e' \in \{\ell - e_1 - e_2, e_3, \ldots, e_r\}$, thus $e \cdot F \geq 0$. In particular, $F_- = F - F_+$, so $F_+ = F - F_- \in \Delta''_r$ and we have $(F_+)\cdot(F_-) = 0$.

Now let $F = d\ell - m_1e_1 - \cdots - m_re_r \in \Delta'_r$, but assume $F \cdot (\ell - e_1 - e_2) \geq 0$. Thus $d \geq 0, d \geq m_1, d \geq m_1 + m_2, d \geq m_1 + m_2 + m_3, m_1 \geq \cdots \geq m_r$. If $m_r \geq 0$, then $F \in \Delta''_r$, so $F \cdot w(e_r) \geq 0$ for all $w \in W_r$. Thus $F_- = 0$, $F_+ = F - F_- \in \Delta''_r$ and $(F_+)\cdot(F_-) = 0$. If $m_r < 0$, let i be the least index such that $m_i < 0$. Then $F = (d\ell - \sum_{j<i} m_je_j) + (\sum_{j\geq i}|m_j|e_j)$ and we have $d\ell - \sum_{j<i} m_je_j \in \Delta''_r$, so $F \cdot e \geq 0$ for all $e \in W_re_r$ except $e_i, \cdots, e_r$, so we have $F_- = \sum_{j \geq i}|m_j|e_j$, $F_+ = F - F_-$ and again $(F_+)\cdot(F_-) = 0$.

Thus every element $F = W_r\Delta'_r$ has a unique decomposition $F = (F_+) + (F_-)$, where $F_- = -\sum_E (E\cdot F)E$ where the sum is over all $E \in W_re_r$ with $E\cdot F < 0$ and $F_+ \in W_r\Delta''_r$ and we have $(F_+)\cdot(F_-) = 0$. To find the decomposition, find $w \in W_r$ (as discussed below) such that $w(F) \in \Delta'_r$, and get the decomposition $w(F) = (w(F)_+) + (w(F)_-)$ as done above for elements in Δ'_r; then $F_+ = w^{-1}(w(F)_+)$ and $F_- = w^{-1}(w(F)_-)$.

Part of the interest in Δ''_r and Δ'_r is algorithmic. Given $F \in \mathrm{Cl}(X)$, it is easy to tell if W_rF intersects Δ''_r (resp., Δ'_r), and if so to find the intersection and thus to find the decomposition $F = (F_+) + (F_-)$. To see how, recall, as noted above, that $w(F) = F + v$ for some nonnegative linear combination v of the v_i if $F \in \Delta_r$.

This implies that W_rF does not intersect Δ''_r if $w(F)\cdot\ell < 0$ or $w(F)\cdot e_r < 0$ for some $w \in W_r$. This is because $h_i\cdot\ell, h_i\cdot e_r, v_i\cdot\ell, v_i\cdot e_r \geq 0$ for all i, so we see that $w(F)\cdot\ell \geq 0$ and $w(F)\cdot e_r \geq 0$ for all $w \in W_r$ and all $F \in \Delta''_r$.

By Exercise 3.2.4 we have $w(F)\cdot\ell \geq 0$ and $w(F)\cdot(\ell - e_1) \geq 0$ for all $w \in W_r$ and all $F \in \Delta'_r$, so we see that W_rF does not intersect Δ'_r if $w(F)\cdot\ell < 0$ or $w(F)\cdot(\ell - e_1) < 0$ for some $w \in W_r$.

Here's how to apply these facts, using a greedy reduction algorithm. We first do the case of Δ''_r, then we do the similar but easier case of Δ'_r.

Consider the case of Δ''_r. Given any $F = d\ell - m_1e_1 - \cdots - m_re_r \in \mathrm{Cl}(X)$, permute the m_i (i.e., apply s_i for $i > 0$ as needed) to get coefficients m'_i with $m'_1 \geq m'_2 \geq \cdots \geq m'_r$, and take $d' = d$. Call the resulting class $F_1 = d'\ell - m'_1e_1 - \cdots - m'_re_r$. If $d' \geq m'_1 + m'_2 + m'_3$, then $F_1 \in \Delta_r$, hence $F_1 \in \Delta''_r$ if and only if $m'_r \geq 0$. If $d' < m'_1 + m'_2 + m'_3$, applying s_0 gives $s_0(F_1)\cdot\ell < F_1\cdot\ell$. Now $s_0(F_1) = d^*\ell - m^*_1e_1 - \cdots - m^*_re_r$ and after reordering we get coefficients m''_i with $m''_1 \geq m''_2 \geq \cdots \geq m''_r$ and $d'' = d^*$. Call the resulting class $F_2 = d''\ell - m''_1e_1 - \cdots - m''_re_r$. If $d'' \geq m''_1 + m''_2 + m''_3$, then $F_2 \in \Delta_r$, hence $F_2 \in \Delta''_r$ if and only if $m''_r \geq 0$. If $d'' < m''_1 + m''_2 + m''_3$, we reorder and apply s_0 again. Continuing in this way, we get a sequence of classes $F_1, F_2, \ldots$ with $F_1\cdot\ell > F_2\cdot\ell > \cdots$. Eventually we either get that some iterate $F_i = d^{(i)}\ell - m^{(i)}_1e_1 - \cdots - m^{(i)}_re_r$ is in Δ_r and thus $F_i \in \Delta''_r$ if and only if $m^{(i)}_r \geq 0$, or we get $F_i\cdot\ell = d^{(i)} < 0$ and hence W_rF does not intersect Δ''_r.

Consider the case of Δ'_r. Given any $F = d\ell - m_1e_1 - \cdots - m_re_r \in \mathrm{Cl}(X)$, permute the m_i (i.e., apply s_i for $i > 0$ as needed) to get coefficients m'_i with $m'_1 \geq m'_2 \geq \cdots \geq m'_r$, and take $d' = d$. Call the resulting class $F_1 = d'\ell - m'_1e_1 - \cdots - m'_re_r$. If $d' \geq m'_1 + m'_2 + m'_3$, then $F_1 \in \Delta_r$, hence $F_1 \in \Delta'_r$ if and only if $d' \geq 0$ and $d' - m'_1 \geq 0$. If $d' < m'_1 + m'_2 + m'_3$, applying s_0 gives $s_0(F_1)\cdot\ell < F_1\cdot\ell$. Now $s_0(F_1) = d^*\ell - m^*_1e_1 - \cdots - m^*_re_r$ and after reordering we get coefficients m''_i with $m''_1 \geq m''_2 \geq \cdots \geq m''_r$ and $d'' = d^*$. Call the resulting class $F_2 = d''\ell - m''_1e_1 - \cdots - m''_re_r$. If $d'' \geq m''_1 + m''_2 + m''_3$, then $F_2 \in \Delta_r$, hence $F_2 \in \Delta'_r$ if and only if $d'' \geq 0$ and $d'' - m''_1 \geq 0$. If $d'' < m''_1 + m''_2 + m''_3$, we reorder and apply s_0 again. Continuing in this way, we get a sequence of classes $F_1, F_2, \ldots$ with $F_1\cdot\ell > F_2\cdot\ell > \cdots$. Eventually we either get that some iterate $F_i = d^{(i)}\ell - m^{(i)}_1e_1 - \cdots - m^{(i)}_re_r$ is in Δ_r and thus $F_i \in \Delta'_r$ if and only if $d^{(i)} \geq 0$ and $d^{(i)} - m^{(i)}_1 \geq 0$, or we get $F_i\cdot\ell = d^{(i)} < 0$ and hence W_rF does not intersect Δ'_r.

Example 3.2.9. Consider $F = 13\ell - 3e_1 - 4e_2 - 6e_3 - 6e_4 - 6e_5 - 6e_6$. Then the sequence of classes we get using the algorithm above is: $F_1 = 13\ell - 6e_1 - 6e_2 - 6e_3 - 6e_4 - 4e_5 - 3e_6$, $F_2 = 8\ell - 6e_1 - 4e_2 - 3e_3 - e_4 - e_5 - e_6$, and $F_3 = 3\ell - e_1 - e_2 - e_3 - e_4 + e_5 + 2e_6$. We see that F_3 is in Δ_6 and Δ'_6 and not in Δ''_6, and we can write $F_3 = (F_3)_+ + (F_3)_-$, where $(F_3)_+ = 3\ell - e_1 - e_2 - e_3 - e_4 \in \Delta''_6$ and $(F_3)_- = e_5 + 2e_6$. Also note that $F^2 = F_i^2$ and $F\cdot K_X = F_i\cdot K_X$ for all i (this helps one to check that the F_i's are being computed correctly). On the other hand, the sequence for $G = 12\ell - 3e_1 - 4e_2 - 6e_3 - 6e_4 - 6e_5 - 6e_6$ is $G_1 = 12\ell - 6e_1 - 6e_2 - 6e_3 - 6e_4 - 4e_5 - 3e_6$, $G_2 = 6\ell - 6e_1 - 4e_2 - 3e_3$, and $G_3 = -1\ell + e_1 + 3e_2 + 4e_3$, so $G \notin \Delta_6$.

3.3. **The SHGH Conjecture.** Theorem 3.2.5 shows that $W_r\Delta'_r$ and $W_r\Delta''_r$ are related to $\mathrm{Eff}(X)$ and $\mathrm{Nef}(X)$. A more precise connection is given by the SHGH Conjecture [48, 27, 24, 33]. The restriction to $r \geq 3$ is just to simplify the statement. If X_r is the blow up of $r \geq 0$ distinct points of $\mathbb{P}^2$, then $\mathrm{Cl}(X_0) \subset \mathrm{Cl}(X_1) \subset \mathrm{Cl}(X_2) \subset \cdots$ and we have $\mathrm{Eff}(X_r) = \mathrm{Eff}(X_{r+1}) \cap \mathrm{Cl}(X_r)$ and $\mathrm{Nef}(X_r) = \mathrm{Nef}(X_{r+1}) \cap \mathrm{Cl}(X_r)$ for all r, if we know $\mathrm{Eff}(X_{r+1})$ then we know $\mathrm{Eff}(X_r)$, and if we know $\mathrm{Nef}(X_{r+1})$ then we know $\mathrm{Nef}(X_r)$. Thus nothing in the conjecture is lost by assuming $r \geq 3$. Also, under the assumption of generality in the conjecture, we have $F \in \mathrm{Eff}(X)$ if and only if $w(F) \in \mathrm{Eff}(X)$ for all $w \in W_r$ (and indeed $h^i(X,F) = h^i(X, w(F))$ for all $w \in W_r$; see section 8 of [11]), and $F \in \mathrm{Nef}(X)$ if and only if $w(F) \in \mathrm{Nef}(X)$ for all $w \in W_r$. Thus to determine whether F is effective or nef, check as in the algorithm described above whether W_rF intersects Δ'_r (if not, then F is not effective) or Δ''_r (if not, then F is not nef); if it does intersect, test the reduced class for effectivity or nefness. I.e., it is enough to consider classes F in Δ'_r or in Δ''_r. Moreover, since $F \in \Delta'_r$ implies $F \cdot \ell \geq 0$, we have $h^2(X,F) = 0$ whenever F is in Δ'_r or in Δ''_r.

Conjecture 3.3.1 (SHGH Conjecture)**.** *Let X be the blow up of $r \geq 3$ sufficiently general points (very general over the complexes or generic over an arbitrary algebraically closed field K).*

(a) *Let $F \in \Delta'_r$. Then*

$$h^0(X,F) = \max\left(0, \frac{F_+^2 - F_+ \cdot K_X}{2} + 1\right).$$

(b) *Let $F \in \Delta''_r$. Then F is nef if and only if $F^2 \geq 0$.*

Conjecture 3.3.1 is known to be true when $s \leq 9$ (see the results of [28] for example) or when s is a square [21, 9, 45].

Exercise 3.3.2. Assume the SHGH Conjecture, and that X satisfies the hypotheses of the SHGH Conjecture.

(a) Show that $F \in \Delta'_r$ is effective if and only if $\frac{F_+^2 - F_+ \cdot K_X}{2} + 1 > 0$.
(b) If $F \in \Delta'_r$ is effective, then

$$h^1(X,F) = \max\left(0, -\frac{F_-^2 - F_- \cdot K_X}{2}\right)$$

and, moreover, $h^1(X,F) > 0$ if and only if $F \cdot e_r < -1$.
(c) Show that $h^0(X,F)h^1(X,F) = 0$ if F is nef.
(d) Show $C \in W_r e_r$ if C is an integral effective curve with $C^2 < 0$; i.e., $\mathrm{Neg}(X) = W_r e_r$, so $C^2 < 0$ implies $C^2 = -1$.

Answer: (a) The fact that F is effective if and only if $\frac{F_+^2 - F_+ \cdot K_X}{2} + 1 > 0$ is immediate form the statement of part (a) of the conjecture.

(b) Assume F is effective. Then $h^0(X,F) = \frac{F_+^2 - F_+ \cdot K_X}{2} + 1 > 0$, but $F_+ \cdot F_- = 0$, so

$$h^0(X,F) - h^1(X,F) = \frac{F^2 - F \cdot K_X}{2} + 1 = \frac{F_+^2 - F_+ \cdot K_X}{2} + 1 + \frac{F_-^2 - F_- \cdot K_X}{2} = h^0(X,F) + \frac{F_-^2 - F_- \cdot K_X}{2}.$$

Thus $h^1(X,F) = -\frac{F_-^2 - F_- \cdot K_X}{2}$. If $h^1(X,F) > 0$, then we must have $F_-^2 - F_- \cdot K_X < 0$, but $F_- = m_i e_i + \cdots + m_r e_r$ with $0 \leq m_i \leq \cdots \leq m_r$, so $-\frac{F_-^2 - F_- \cdot K_X}{2} = \frac{-1}{2} \sum_{j \geq i} m_j(m_j - 1)$ is negative if and only if $F \cdot e_r = -m_r < -1$.

(c) Assume F is nef. If $F \notin \mathrm{Eff}(X)$, then $h^0(X,F) = 0$ so $h^0(X,F)h^1(X,F) = 0$. If $F \in \mathrm{Eff}(X)$, then pick $w \in W_r$ such that $w(F) \in \Delta'_r$. Since $w(F)$ is also nef, we have $w(F) \cdot e_r \geq 0$, so $h^1(X,F) = h^1(X, w(F)) = 0$, and again $h^0(X,F)h^1(X,F) = 0$.

(d) Assume C is an integral effective curve with $C^2 < 0$ but $C \notin W_r e_r$. Since $w(e_r)$ is an integral curve for every $w \in W_r$ (see Example 3.2.2), then $C \cdot w(e_r) \geq 0$ for all $w \in W_r$, so

$w(C) \in \Delta''_r$ for some $w \in W_r$ by Theorem 3.2.5(a). Since $w(C) \cdot e_r \geq 0$, we see that $(w(C))_- = 0$, so $w(C) = (w(C))_+$, hence $0 < h^0(X, C) = h^0(X, w(C)) = h^0(X, (w(C))_+) = \frac{(w(C))_+^2 - (w(C))_+ \cdot K_X}{2} + 1 = \frac{C^2 - C \cdot K_X}{2} + 1$. But $C^2 < 0$, and either $p_a(C) > 0$ or $p_a(C) = 0$ and $C^2 < -1$ (otherwise C is an exceptional curve, hence $C \in W_r e_r$, contrary to assumption), so $C^2 + C \cdot K_X = 2p_a(C) - 2$ gives $-C \cdot K_X = C^2 - 2p_a(C) + 2 \leq 0$. Thus $0 < h^0(X, C) = \frac{C^2 - C \cdot K_X}{2} + 1 \leq \frac{C^2}{2} + 1 < 1$, which is impossible.

Exercise 3.3.3. Show that SHGH implies Nagata (i.e., Conjecture 3.3.1 implies Conjecture 2.4.10).

Answer: Say C' is a plane curve through $r > 9$ very general points p_i of degree d with multiplicity at least m at each point p_i. Let $Z = p_1 + \cdots + p_r$. Let X be the blow up of the points. Then $C = d\ell - m(e_1 + \cdots + e_r)$ is effective. But $D = 3m\ell - m(e_1 + \cdots + e_r) \in \Delta''_r$ and $D \cdot e_r \geq 0$, so $D = D_+$, hence $h^0(X, D) = \max(0, \frac{D^2 - D \cdot K_X}{2} + 1) = \max(0, \frac{(m^2+m)(9-r)}{2} + 1) \leq \max(0, 10 - r) = 0$, so $d \geq 3m$, hence $C \in \Delta''_r$, but $C \cdot e_r \geq 0$ so $w(C) \cdot e_r \geq 0$ for all $w \in W_r$. Therefore $C \cdot w(e_r) \geq 0$ for all $w \in W_r$, so C does not, by Exercise 3.3.2, meet any negative curve negatively, hence C is nef, so $C^2 \geq 0$. Thus $d^2 \geq rm^2$, so $\widehat{\alpha}(Z) \geq \sqrt{r}$, but $\widehat{\alpha}(Z) \leq \sqrt{r}$ by Exercise 2.2.3, so $\widehat{\alpha}(Z) = \sqrt{r}$, and hence $\varepsilon(Z) = 1/\sqrt{r}$ by Proposition 2.4.6.

3.4. **Bounded Negativity.** We have seen that a surface X can have infinitely many integral curves C with $C^2 < 0$. In positive characteristic, taking graphs of Frobenius morphisms $C \to C$, where C is a smooth curve of genus $g \geq 2$, one can obtain integral curves on $C \times C$ of arbitrarily negative self-intersection (see [32, Exercise V.1.10]). No other examples of this sort of behavior seem to be known. Thus this gave rise to the following conjecture, known as the Bounded Negativity Conjecture:

Conjecture 3.4.1. *Let X be a smooth rational or complex projective surface. Then there is an integer b_X such that $C^2 \geq b_X$ for all integral curves C on X.*

When $-K_X$ is semi-effective, the conjecture is true. In that case we have $C^2 = 2p_C - 2 - C \cdot K_X$ and $-K_X \cdot C \geq 0$ except possibly when C is a component of $-mK_X$ for some $m > 0$ for which $-mK_X$ is effective, but $-mK_X$ has only finitely many components. Thus except for at most finitely many C we have $C^2 \geq -2$.

The conjecture can also be stated for effective reduced divisors:

Conjecture 3.4.2. *Let X be a smooth rational or complex projective surface. Then there is an integer B_X such that $C^2 \geq B_X$ for all reduced curves C on X.*

Certainly, Conjecture 3.4.2 implies Conjecture 3.4.1. Conversely, consider an effective curve $C = C_1 + \cdots + C_r$ with $C^2 < 0$, where the curves C_i are integral and distinct. Suppose $C_i \cdot C \geq 0$ for some i (which we may assume to be r). Then $C^2 = (C_1 + \cdots + C_{r-1}) \cdot C + C_r \cdot C \geq (C_1 + \cdots + C_{r-1}) \cdot C = (C_1 + \cdots + C_{r-1})^2 + (C_1 + \cdots + C_{r-1}) \cdot C_r \geq (C_1 + \cdots + C_{r-1})^2$, since $C_i \cdot C_r \geq 0$ for $i < r$.

Thus we can reduce to the case that $C \cdot C_i < 0$ for all i. Suppose now there were integers a_i, not all 0, such that $\sum_i a_i C_i$ were numerically trivial. Then we would have disjoint sums such that $\sum_j a_{i_j} C_{i_j} = \sum_k -a_{i_k} C_{i_k}$ with $a_{i_j} \geq 0$ and $a_{i_k} \leq 0$, and hence neither sum numerically trivial. But $(\sum_j a_{i_j} C_{i_j})^2 < (\sum_j a_{i_j} C_{i_j}) \cdot C < 0$, while $(\sum_j a_{i_j} C_{i_j}) \cdot (\sum_k -a_{i_k} C_{i_k}) \geq 0$. Hence the classes of the C_i are linearly independent up to numerical equivalence. Thus r is at most the Picard number s_X of X, so $C^2 \geq \sum_i C_i^2 \geq s_X b_X$; i.e., $B_X \geq s_X b_X$, so Conjecture 3.4.1 implies Conjecture 3.4.2.

3.5. ***H*-constants.** The Bounded Negativity Conjecture says that reduced curves on a surface can't be too negative, but that raises the question of how negative can they be. Fixing the surface is very constraining, so to gain understanding one can try to construct curves which are as negative as possible by changing the surface to accommodate the curve, and then devising a measure of negativity that is independent of the surface and which allows for comparison of the negativity. That's the idea of H-constants.

Example 3.5.1. Take any reduced curve $C' \subset \mathbb{P}^2$ of some degree d. Pick points $p_1, \ldots, p_r \in C'$. Let $X \to \mathbb{P}^2$ be the blow up of the points p_i and take C to be the proper transform of C'. Then $C^2 \leq d^2 - r$, so for r large enough we obtain a reduced curve C with $C^2 \leq d - r$ arbitrarily negative.

In order to compare negativity of C^2 for different choices of C' and of the points p_i, we can introduce H-constants [18, 36, 42, 43, 44, 50]. Given $C' \subset \mathbb{P}^2$ and distinct points $p_1, \ldots, p_r \in \mathbb{P}^2$ (not necessarily points of C'), define

$$H(C', p_1, \ldots, p_r) = \frac{\deg(C') - \sum_i (\mathrm{mult}_{p_i}(C'))^2}{r}.$$

If we blow up the points p_i to get X and take C to be the proper transform of C', note that $H(C', p_1, \ldots, p_r) = C^2/r$. Clearly, if $C^2 < 0$, then $H(C', p_1, \ldots, p_r)$ is more negative if we only count points p_i which are on C'. Moreover, if the points p_i all are smooth points of C', then we get $H(C', p_1, \ldots, p_r) = -1 + (\deg(C')/r)$.

Exercise 3.5.2. Let $C' \subset \mathbb{P}^2$ be a reduced plane curve with points $p_1, \ldots, p_r \in \mathbb{P}^2$. Assume $H(C', p_1, \ldots, p_r) \leq -1$. Show that some of the points p_i are singular points of C' and that

$$H(C', q_1, \ldots, q_s) \leq H(C', p_1, \ldots, p_r),$$

where $\{q_1, \ldots, q_s\}$ is the subset of $\{p_1, \ldots, p_r\}$ of those points p_i which are singular points of C'. Moreover show that $H(C', q_1, \ldots, q_s) < H(C', p_1, \ldots, p_r)$ unless either $H(C', q_1, \ldots, q_s) = H(C', p_1, \ldots, p_r) = -1$ or every point p_i is a singular point of C'.

Answer: Let $d = \deg(C')$. If none of the points p_i are singular points, then $H(C', p_1, \ldots, p_r) \geq (d^2 - r)/r > -1$, so we see that some of the points must be singular points of C'. Also, let $\{a_1, \ldots, a_t\}$ be the subset of $\{p_1, \ldots, p_r\}$ of those points p_i which are points of C'. Then $0 > d^2 - \sum_i (\mathrm{mult}_{p_i}(C'))^2 = d^2 - \sum_j (\mathrm{mult}_{a_j}(C'))^2$, and since $t \leq r$ we get

$$H(C', a_1, \ldots, a_t) \leq H(C', p_1, \ldots, p_r) \leq -1.$$

Moreover, if $t < r$, then $H(C', a_1, \ldots, a_t) < H(C', p_1, \ldots, p_r) \leq -1$. So we reduce to the case that every point p_i is a point of C'.

Now assume $H(C', q_1, \ldots, q_s) = H(C', p_1, \ldots, p_r)$ and $\{q_1, \ldots, q_s\} \subsetneq \{p_1, \ldots, p_r\}$ and so $s < r$. Let $u = r - s$. We have

$$H(C', q_1, \ldots, q_r) = H(C', p_1, \ldots, p_r) = \frac{sH(C', q_1, \ldots, q_s) - u}{s + u},$$

which simplifies to $H(C', q_1, \ldots, q_r) = -1$ as claimed.

By Exercise 3.5.2, in order to get reduced curves C' with $H(C', p_1, \ldots, p_r) < -1$ and as negative as possible, it is best if each point p_i is a singular point of C'. So for simplicity we define $H(C')$ for a reduced singular curve C' to be

$$H(C') = \frac{(\deg(C'))^2 - \sum_i (\mathrm{mult}_{p_i}(C'))^2}{r}$$

where the sum is over *all* of the singular points p_i of C'. No examples are yet known where taking a proper subset of the singular points gives a more negative value, but it seems possible that this could happen.

It is an open problem to determine how negative $H(C')$ can be. Let us define

$$H(\mathbb{P}^2) = \inf\{H(C')\}$$

where the infimum is over all reduced singular curves C' and

$$H_{irr}(\mathbb{P}^2) = \inf\{H(C')\}$$

where the infimum is over all integral singular curves C'.

Fact 3.5.3 ([3]). $H_{irr}(\mathbb{P}^2) \leq -2$.

There are curves C'_d of degree d with $\binom{d-1}{2}$ points of multiplicity 2 arising as images of $\mathbb{P}^1$ in $\mathbb{P}^2$. Then $H(C'_d) > -2$ but $\lim_{d\to\infty} H(C'_d) = -2$. No example is currently known of an integral curve C' with $H(C') \leq -2$ (even if $\mathrm{char}(K) = p > 0$).

Fact 3.5.4 ([46]). $H(\mathbb{P}^2) \leq -4$.

In positive characteristics, $H(C)$ can be made arbitrarily negative when C is a union of lines (take all of the lines defined over a large finite subfield). Thus the case of unions of lines is of interest mostly in characteristic 0. If C' is a union of lines in the complex projective plane, then $H(C') > -4$ [3], but the most negative example currently known where C' is a union of lines in characteristic 0 is $H(C') = 225/67 \approx -3.36$. In this case C' is a union of 45 lines, where there are 201 singular points, 36 of multiplicity 5, 45 of multiplicity 4 and 120 of multiplicity 3 [53].

When looking at examples of C when C is a union of lines, some simplifications arise. Given a plane curve C which is a union of d lines, for $2 \leq k \leqslant d$, let t_k be the number of points of C of multiplicity exactly k (i.e., the number of points where exactly k lines cross).

Exercise 3.5.5. Show that $\binom{d}{2} = \sum_{k=2}^d t_k\binom{k}{2}$. Conclude that

$$H(C) = \frac{d - \sum_{k=2}^d kt_k}{\sum_{k=2}^d t_k}.$$

Answer: The number of pairs of d lines is $\binom{d}{2}$, but at each in $\mathbb{P}^2$ each pair meets at a unique point, and if the point is a point of multiplicity k, then $\binom{k}{2}$ of the pairs occur there. So counting the pairs at each singular points gives $\sum_{k=2}^d t_k\binom{k}{2}$ pairs, hence $\binom{d}{2} = \sum_{k=2}^d t_k\binom{k}{2}$.

The number of singular points is $\sum_{k=2}^d t_k$. And from $\binom{d}{2} = \sum_{k=2}^d t_k\binom{k}{2}$ we get $d^2-d = \sum_{k=2}^d (k^2-k)t_k$ or $d^2 - \sum_k k^2t_k = d - \sum_k kt_k$. Thus

$$H(C) = \frac{d^2 - \sum_{k=2}^d k^2t_k}{\sum_{k=2}^d t_k} = \frac{d - \sum_k kt_k}{\sum_{k=2}^d t_k}.$$

The example above where C is a union of 45 lines with $H(C) = 225/67$ suggests it might be fruitful to look at unions of lines such that t_2 is small. The smallest possibility is $t_2 = 0$. Such line arrangements are easy to find in positive characteristics (and so not so interesting), but there are no nontrivial examples over the reals [39]:

Theorem 3.5.6. *Given a real line arrangement of s lines with $t_s = 0$ (i.e., the lines are not concurrent), we have*

$$t_2 \geq 3 + \sum_{k>2} t_k(k-3).$$

Using this one can show that $H(C) > -3$ for real line arrangements C, and easy examples show that one can achieve -3 in the limit. It is currently an open problem to determine how negative $H(C)$ can be for rational real line arrangements. The known real arrangements whose H-values approach -3 have singular points with irrational coordinates.

Remark 3.5.7. Only four complex examples seem to be known:

- Any set of $s \geq 3$ concurrent lines.
- The Fermat arrangement of $3n$ lines for $n \geq 3$: The lines of this arrangement are defined by the factors of $(x^n - y^n)(x^n - z^n)(y^n - z^n)$, shown for $n = 3$ in Figure 8 [25]. Each line contains $n + 1$ of the points, and we have $t_k = 0$ except for $t_3 = n^2$ and $t_n = 3$ when $n > 3$ or $t_3 = 12$ when $n = 3$. This gives H-constants greater than -3 but with limit -3 as $n \to \infty$.

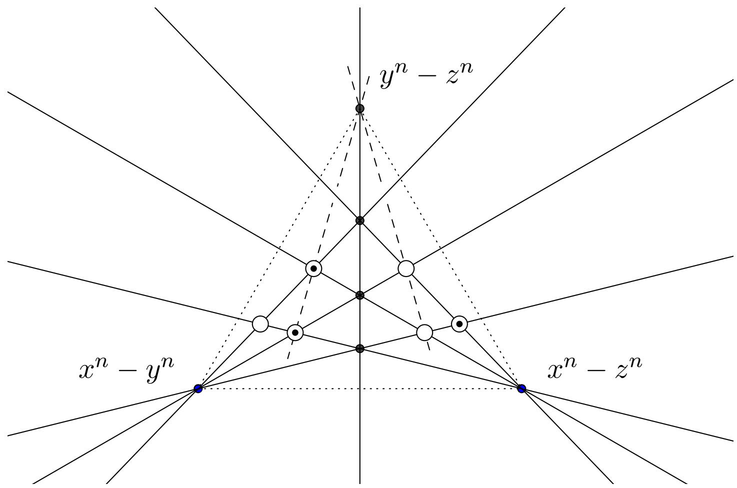


FIGURE 8. The Fermat arrangement of $3n$ complex lines (defined by the factors of $(x^n - y^n)(x^n - z^n)(y^n - z^n)$) and their $n^2 + 3$ points of intersection (indicated by circles and dots) for $n = 3$. (The coordinate axes are represented by dotted lines. At each coordinate vertex there occur n of the $3n$ lines, defined by the forms shown; the $n^2 + 3$ points consist of a complete intersection of n^2 points plus the 3 coordinate vertices. No combinatorially equivalent arrangement exists over the reals: one must regard the three empty circles as representing collinear points on one of the dashed lines, and likewise the three circles with central black dots as also representing collinear points on the other dashed line.)

- The Klein arrangement of 21 lines [35]: here $t_k = 0$ except for $t_4 = 21$ and $t_3 = 28$. For this arrangement, each line contains 4 points where 3 lines cross and 4 points where 3 lines cross. This gives an H-constant of exactly -3.
- The Wiman arrangement of 45 lines [53] mentioned above: here $t_k = 0$ except for $t_5 = 36$, $t_4 = 45$ and $t_3 = 120$. For this arrangement, each line contains 4 points where 5 lines cross, 4 points where 4 lines cross and 8 points where 3 lines cross. This gives the H-constant $-225/67$ noted above.

It seems to be an open problem to find additional examples of complex line arrangements with $t_2 = 0$, or to show that there are no others.

If one considers arrangements of curves other than lines, then one can find additional examples where there are no singular points of multiplicity 2. One way to generate examples is to take two

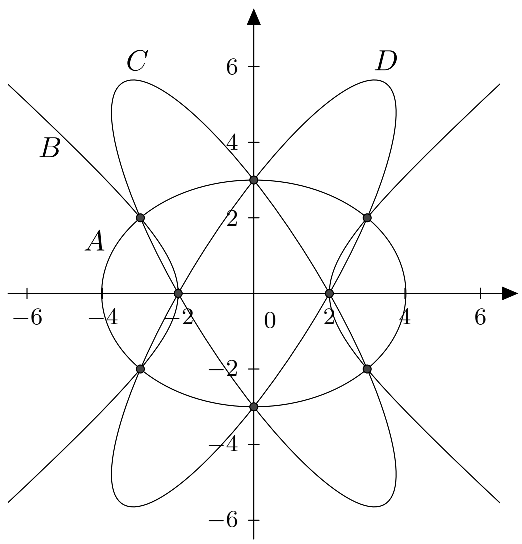


FIGURE 9. Four conics meeting triply in 8 points with no other points of intersection, where A is $5x^2+9y^2 = 81$, B is $4x^2-5y^2 = 16$, C is $54x^2+61xy+24y^2 = 216$, and D is $54x^2-61xy+24y^2 = 216$.

curves, A and B, of degree d with no common components. Then take C to be the union of r elements of the pencil determined by A and B, so $\deg(C) = rd$ and C typically has d^2 points of multiplicity r, and no other singular points. This generalizes the trivial example of r concurrent lines, and so is not all that interesting. Such examples typically give $H(C) = 0$, but the components of C can themselves be singular (and the singular sets of the components can overlap) so one can also get $H(C) < 0$ this way, but it looks either hard or impossible for $H(C)$ to be more negative than the minimum $H(C_i)$ among the components C_i of C.

A somewhat more interesting example is given in [8]. Here the curve C is the union of four conics (see Figure 9), but in the background there again is a pencil. Note that each conic goes through 6 of the 8 singular points of C. If one blows up the singular points and the origin (as shown in Figure 9), then $|-K_X|$ on the resulting surface X is an elliptic fibration with four reducible singular fibers. Each reducible fiber consists of the proper transform of one of the conics together with the proper transform of the line through the two singular points of C not on that conic. In this case we get $H(C) = -1$.

A very interesting family of examples exists where each curve is a smooth plane cubic and the singular points all have multiplicity at least 4 was given in [46], and is the basis for Fact 3.5.4; see [4] for an exposition. Again, pencils of curves are in the background; the cubics are taken from four pencils of cubics with overlapping base points. For each curve C in this family we have $H(C) > -4$ but the H-values achieve -4 in the limit.

One can get even more examples by taking finite maps $\mathbb{P}^2 \to \mathbb{P}^2$, and pulling back known examples. But this again does not seem to lead to examples with H-constants more negative than what are already known.

3.6. **Waldschmidt constants and the ideal containment problem.** Let $Z = m_1p_1 + \cdots + m_rp_s \subset \mathbb{P}^N$ be a fat point subscheme. Recall that the ideal of Z is $I = I(Z) = \cap_i I(p_i)^{m_i} \subset K[\mathbb{P}^N]$. The mth symbolic power of I is denoted $I^{(m)}$. In the situation here, we can take it to be defined as $I^{(m)} = I(mZ) = \cap_i I(p_i)^{mm_i}$.

3.6.1. *The resurgence.* We know from results of [19, 34] that $I^{(Nr)} \subseteq I^r$. This raises the question of of for which m and r do we have $I^{(m)} \subseteq I^r$. One can also ask what is the smallest real $c \geq 0$

such that $m > rc$ implies $I^{(m)} \subseteq I^r$. We know $c \leq N$ by [19, 34], but could there be $c < N$ that works for all I?

Examples of [7] show for every N and $c < N$, that there is an ideal $I = I(Z)$ depending on c and positive integers m and r such that $m > cr$ but $I^{(m)} \not\subseteq I^r$.

But we can still ask what is the best c for a given ideal. This leads to the definition of the resurgence $\rho(I)$ [7]:

$$\rho(I(Z)) = \sup\left\{\frac{m}{r} : I(mZ) \not\subseteq I(Z)^r\right\}.$$

Note that $\rho(I(Z))$ is the least c such that $m/r > c$ guarantees that $I^{(m)} \subseteq I^r$.

Then [7] proves the following bounds on $\rho(I(Z))$, in terms of $\widehat{\alpha}(Z)$, $\alpha(I(Z))$ and the Castelnuovo-Mumford regularity $\mathrm{reg}(I(Z))$ of $I(Z)$. (For a fat point subscheme $Z = m_1p_1 + \cdots + m_rp_r \subset \mathbb{P}^N$, $\mathrm{reg}(I(Z)) - 1$ is the least $t \geq 0$ such that $\dim I(Z)_t = \binom{t+N}{N} - \sum_i \binom{m_i+N-1}{N}$.)

Theorem 3.6.1.

$$\frac{\alpha(I(Z))}{\widehat{\alpha}(Z)} \leq \rho(I(Z)) \leq \frac{\mathrm{reg}(I(Z))}{\widehat{\alpha}(Z)}$$

Proof. The upper bound is somewhat more involved; for this we refer to [7]. The proof for the lower bound works by showing for any m and r with $m/r < \alpha(I(Z))/\widehat{\alpha}(Z)$ that there are infinitely many s such that $I(smZ) \not\subseteq I(Z)^{sr}$.

So suppose that $m/r < \alpha(I(Z))/\widehat{\alpha}(Z)$. Thus $m\widehat{\alpha}(Z) < r\alpha(I(Z)) = \alpha(I(Z)^r)$, but $\widehat{\alpha}(Z) = \inf\{\alpha(I(sZ))/s\} = \lim_{s\to\infty} \alpha(I(sZ))/s$, so for all $s \gg 0$ we have $m\widehat{\alpha}(Z) \leq m\alpha(I(sZ))/s < \alpha(I(Z)^r)$. But $\alpha(I(smZ))/s \leq m\alpha(I(sZ))/s$ so $\alpha(I(smZ)) < s\alpha(I(Z)^r) = \alpha(I(Z)^{sr})$, hence $I(smZ) \not\subseteq I(Z)^{sr}$, so $m/r = sm/(sr) \leq \rho(I(Z))$. Since this holds for all m and r with $m/r < \alpha(I(Z))/\widehat{\alpha}(Z)$, we see that $\alpha(I(Z))/\widehat{\alpha}(Z) \leq \rho(I(Z))$. □

3.6.2. *A question of Huneke.* According to Theorem 2.2.5, given the ideal $I = I(Z)$ of a fat point subscheme $Z \subset \mathbb{P}^2$, we have $I^{(4)} \subseteq I^2$. In wondering whether there were a more general universal containment, Huneke asked if it were true that $I^{(3)} \subset I^2$.

After many experiments and partial results, I conjectured [2] for the ideal $I = I(Z)$ of any fat point subscheme $Z \subset \mathbb{P}^N$ in any characteristic that $I^{(rN-N+1)} \subseteq I^r$. In particular for $r = N = 2$ this would imply $I^{(3)} \subset I^2$.

The first counterexample [15] used the singular points Z of the plane curve $(x^n - y^n)(x^n - z^n)(y^n - z^n) = 0$ for $n = 3$, whose components comprise the Fermat line arrangement (see Figure 8), but in fact every $n \geq 3$ gives a counterexample [31, 47]. Many additional counterexamples are known [6, 12, 16, 1], but over the complex numbers no counterexamples to $I^{(rN-N+1)} \subseteq I^r$ are currently known for any $N > 2$ when $r > 2$. In positive characteristics, counterexamples for larger r and N are much easier to find [31].

It is in fact true for all currently known nontrivial complex line arrangements in $\mathbb{P}^2$ with $t_2 = 0$ (see Remark 3.5.7), that $I(3Z) \not\subseteq I(Z)^2$, where Z is the set of singular points of the curve given by the union of the lines [3]. Moreover, all known examples of $I(3Z) \not\subseteq I(Z)^2$ currently come from line arrangements (where Z is a subset of the singular points of the curve given by the union of the lines, or Z comes from such using flat extensions [1]). It would be interesting to characterize which line arrangements give Z for which $I(3Z) \not\subseteq I(Z)^2$.

3.7. **Negative curves and an SHGH type problem.** By Exercise 3.3.2(d), if X is the blow up of a finite set of sufficiently general points of $\mathbb{P}^2$, then the SHGH Conjecture (Conjecture 3.3.1) implies that the only reduced irreducible curves C with $C^2 < 0$ have $C^2 = -1$. One way to look at this is to consider the birational morphism $\pi : X \to \mathbb{P}^2$ given by blowing up the points and ask if there are prime divisors C on X whose class is not in the image of $\pi^* : \mathrm{Cl}(Y) \to \mathrm{Cl}(\mathbb{P}^2)$ but have $C^2 < -1$. And SHGH says that there aren't any.

In this form, the question can be made relative. Let Y be any smooth rational projective surface and let $\pi : X \to Y$ be the birational morphism obtained by blowing up general points of Y. Then [10] raises the question: Are there ever prime divisors C on X whose class is not in the image of $\pi^* : \mathrm{Cl}(Y) \to \mathrm{Cl}(Y)$ but have $C^2 < -1$? Surprisingly the answer is yes [10].

Example 3.7.1. One example is attributed to Serre [32, Exercise III.10.7]. Start with $\mathbb{P}^2$ over a field of characteristic 2 and let Y be the blow up of the 7 points whose coordinates involves just 0 and 1 (i.e., Y is the blow up of the 7 points $p_1, \dots, p_7$ of the Fano plane $\mathbb{P}^2$). Now let $X \to Y$ be the blow up of an eighth general point p_8. Then $C = 3\ell - e_1 - \cdots - e_7 - 2e_8$ is reduced and irreducible and has $C^2 = -2$ [10].

Another example comes from [14]. Let Y be the blow up of the 9 points $p_1, \dots, p_9 \in \mathbb{P}^2$ as shown in Figure 3 (see Example 2.4.5. Let $X \to Y$ be the blow up of a general 10 point p_{10}. Then $C = 4\ell - e_1 - \cdots - e_9 - 3e_{10}$ is reduced and irreducible and has $C^2 = -2$ [10].

Additional examples are given by the known line configurations with $t_2 = 0$. Let $p_1, \dots, p_r$ be the points dual to the lines of either the Fermat, Klein or Wiman arrangements of lines (see Remark 3.5.7), but in the case of the Fermat arrangement, take $n \geq 5$. Let Y be the blow up of the points p_i. Let X be the blow up of a general point p_{r+1}. Then $C = d\ell - e_1 - \cdots - e_r - me_{r+1}$ is reduced and irreducible for $d = m+1$ and has $C^2 < -1$ [10], where $r = 3n$ and $m = n+1$ for the Fermat arrangement, $r = 21$ and $m = 9$ for the Klein, and $r = 45$ and $m = 19$ for the Wiman. Thus we get $C^2 = 3 - n$ for the Fermat arrangement, -2 for the Klein and -6 for the Wiman.

Let's say that a fat point subscheme $Z = m_1p_1 + \cdots + m_sp_s \subset \mathbb{P}^2$ fails to impose independent conditions on forms of degree d if $\dim I(Z)_d > \max(0, \dim R_d - \sum_i \binom{m_i+1}{2})$. One way to think of the SHGH Conjecture is that it characterizes exactly when Z fails to impose independent conditions on forms of degree d, in case the points p_i are general.

A more general problem is to have a fat point subscheme $Y = n_1q_1 + \cdots + n_rq_r \subset \mathbb{P}^2$ where the points q_i are distinct but not necessarily general, in addition to $Z = m_1p_1 + \cdots + m_sp_s$ where the p_i are general, and ask to classify those m_i, Y and d such that Z fails to impose independent conditions on $I(Y)_d$, meaning $\dim I(Z+Y)_d > \max(0, \dim I(Y)_d - \sum_i \binom{m_i+1}{2})$.

The paper [10] characterizes (but does not classify) such Z, Y and d in the special case that $s = 1$ and $d = m_1 + 1$, for reduced $Y = q_1 + \cdots + q_r$.

References

[1] S. Akesseh. *Ideal Containments Under Flat Extensions*, Preprint, arXiv:1512.08053.

[2] T. Bauer, S. Di Rocco, B. Harbourne, M. Kapustka, A. Knutsen, W. Syzdek, T. Szemberg. *A primer on Seshadri constants*, pp. 33–70 in Contemporary Mathematics vol. 496, 2009, arXiv:0810.0728.

[3] T. Bauer, S. Di Rocco, B. Harbourne, J. Huizenga, A. Lundman, P. Pokora, T. Szemberg. *Bounded Negativity and Arrangements of Lines*, International Math. Res. Notices 19 (2015) 9456–9471.

[4] Th. Bauer, B. Harbourne, J. Roé, T. Szemberg. *The Halphen cubics of order two*, Collect. Math. (2016), arXiv:1603.04480.

[5] C. Bocci and B. Franci. *Waldschmidt constants for Stanley-Reisner ideals of a class of Simplicial Complexes*, Preprint, arXiv:1504.04201.

[6] C. Bocci, S. Cooper and B. Harbourne. *Containment results for ideals of various configurations of points in $\mathbb{P}^N$*, J. Pure Appl. Alg. 218 (2014), 65–75, arXiv:1109.1884.

[7] C. Bocci and B. Harbourne. *Comparing Powers and Symbolic Powers of Ideals*, J. Algebraic Geometry, 19 (2010), 399–417, arXiv:0706.3707.

[8] G. V. Chudnovsky. *Singular points on complex hypersurfaces and multidimensional Schwarz Lemma*, Séminaire de Théorie des Nombres, Paris 197980, Séminaire Delange-Pisot-Poitou, Progress in Math vol. 12, M-J Bertin, editor, Birkhäuser, Boston-Basel-Stutgart (1981).

[9] C. Ciliberto, R. Miranda. *Nagatas conjecture for a square number of points*, Ric. Mat., 55 (2006), pp. 71–78.

[10] D. Cook, B. Harbourne, J. Migliore, U. Nagel. *Line arrangements and configurations of points with an unusual geometric property*, Preprint, arXiv:1602.02300.

[11] S. Cooper and B. Harbourne. *Regina Lectures on Fat Points*, in: Connections Between Algebra, Combinatorics, and Geometry, S. Cooper and S. Sather-Wagstaff editors, vol. 76 of Springer proceedings in mathematics & statistics, 2014.

[12] A. Czapliński, A. Główka, G. Malara, M. Lampa - Baczyńska, P. Łuszcz - Świdecka, P. Pokora, J. Szpond. A counterexample to the containment $I^{(3)} \subset I^2$ over the reals. Advances in Geometry vol. 16:1 (2015) 77–82 (arXiv:1310.0904).

[13] J.-P. Demailly. *Singular hermitian metrics on positive line bundles*, In Complex Algebraic Varieties, Bayreuth 1990, volume 1507 of SLNM, pages 87–104, 1992.

[14] R. Di Gennaro, G. Ilardi and J. Vallès. *Singular hypersurfaces characterizing the Lefschetz properties*, J. London Math. Soc., (2) 89 (2014), no. 1, 194–212.

[15] M. Dumnicki, T. Szemberg, H. Tutaj-Gasińska. A counter-example to a question by Huneke and Harbourne. J. Algebra **393** (2013), 24–29.

[16] M. Dumnicki, B. Harbourne, U. Nagel, A. Seceleanu, T. Szemberg, H. Tutaj-Gasińska Resurgences for ideals of special point configurations in $\mathbb{P}^N$ coming from hyperplane arrangements. J. Algebra 443 (2015) 383–394.

[17] M. Dumnicki, Ł. Farnik, H. Tutaj-Gasińska. *Asymptotic Hilbert Polynomial and a bound for Waldschmidt constants*, Preprint, arXiv:1511.07633.

[18] M. Dumnicki, D. Harrer, J. Szpond. *On absolute linear Harbourne constants*, Preprint, arXiv:1507.04080.

[19] L. Ein, R. Lazarsfeld and K. Smith. *Uniform bounds and symbolic powers on smooth varieties*, Invent. Math. 144 (2001), 241-252.

[20] Hd. Esnault and E. Viehweg. *Sur une minoration du degré d'hypersurfaces s'annulant en certains points*, Math. Ann. 263 (1983), no. 1, 75–86.

[21] L. Evain. *Computing limit linear series with infinitesimal methods*, Ann. Inst. Fourier (Grenoble) 57 (2007) no. 6, 1947–1974. (arXiv:math/0407143)

[22] Ł. Farnik, J. Gwoździewicz, B. Hejmej, M. Lampa-Baczyńska, G. Malara, J. Szpond. *Initial sequences and Waldschmidt constants of planar point configurations*,

[23] H. Fekete. *Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten.* Math. Z. 17 (1923) 228–249.

[24] A. Gimigliano. *On linear systems of plane curves*, Thesis, Queen's University, Kingston, (1987).

[25] B. Harbourne. *Asymptotics of linear systems with connections to line arrangements*, to appear, Proceeding of 2016 miniPages mini-Semester, Banach Center Publications.

[26] B. Harbourne. *Blowings-up of* $\mathbf{P}^2$ *and their blowings-down*, Duke Math. J. 52, 129–148 (1985).

[27] B. Harbourne. *The geometry of rational surfaces and Hilbert functions of points in the plane*, Proceedings of the 1984 Vancouver Conference in Algebraic Geometry, CMS Conf. Proc., **6** Amer. Math. Soc., Providence, RI, (1986) 95–111.

[28] B. Harbourne. *Complete linear systems on rational surfaces*, Trans. Amer. Math. Soc. 289, 213–226 (1985).

[29] B. Harbourne, C. Huneke. *Are symbolic powers highly evolved?*, J. Ramanujan Math. Soc. 28, No.3 (Special Issue-2013) 311–330, arXiv:1103.5809.

[30] B. Harbourne, J. Roé. *Extendible estimates of multipoint Seshadri constants*, Preprint, 2003, arXiv:math/0309064v1.

[31] B. Harbourne, A. Seceleanu. Containment Counterexamples for ideals of various configurations of points in $\mathbb{P}^N$. J. Pure Appl. Algebra (2014),

[32] R. Hartshorne. *Algebraic Geometry.* Springer, GTM #52, 1977, pp. xvi+496.

[33] A. Hirschowitz. *Une conjecture pour la cohomologie des diviseurs sur les surfaces rationelles génériques*, J. Reine Angew. Math., **397** (1989), 208–213.

[34] M. Hochster and C. Huneke. *Comparison of symbolic and ordinary powers of ideals*, Invent. Math. **147** (2002), no. 2, 349–369.

[35] F. Klein. Über die Transformation siebenter Ordnung der elliptischen Functionen. Math. Annalen **14** (1879), 428 – 471.

[36] R. Laface, P. Pokora. *Bounds on Harbourne indices*, Preprint, arXiv:1602.05418

[37] R. Lazarsfeld. Positivity in Algebraic Geometry I.-II. Ergebnisse der Mathematik und ihrer Grenzgebiete, Vols. 48-49., Springer Verlag, Berlin, 2004.

[38] E. Looijenga. *Rational surfaces with effective anticanonical divisor*, Ann. of Math. 114 (1981), 267–322.

[39] E. Melchior. Über Vielseite der Projektive Ebene. Deutsche Mathematik 5: 461–475, 1941.

[40] M. Nagata. *On the 14-th problem of Hilbert*, Amer. J. Math. 33 (1959), 766–772.

[41] M. Nagata. *On rational surfaces II*, Mem. Coli. Sci. Kyoto (A) 32 (1960), 271–293.

[42] P. Pokora. *Harbourne constants and arrangements of lines on smooth hypersurfaces in* $\mathbb{P}^3_{\mathbb{C}}$, Taiwanese Journal of Mathematics 20(1): 25-31 (2016), arXiv:1505.03822

[43] P. Pokora, H. Tutaj-Gasińska. *Harbourne constants and conic configurations on the projective plane*, preprint 2015, arXiv:1506.06097.

[44] P. Pokora, X. Roulleau, T. Szemberg. *Bounded negativity, Harbourne constants and transversal arrangements of curves*, Preprint, arXiv:1602.02379

[45] J. Roé. *Limit linear systems and applications*, Preprint (2006), arXiv:math/0602213.

[46] X. Roulleau. *Bounded negativity, Miyaoka-Sakai inequality and elliptic curve configurations*, preprint 2015 (arXiv:1411.6996).

[47] A. Seceleanu. A homological criterion for the containment between symbolic and ordinary powers for some ideals of points in $\mathbb{P}^2$. J. Pure Appl. Algebra 219:11 (2015) 4857–4871 (arXiv:1410.0312).

[48] B. Segre. *Alcune questioni su insiemi finiti di punti in geometria algebrica*, Atti Convegno Intern. di Geom. Alg. di Torino, (1961), 15–33.

[49] H. Skoda. *Estimations L^2 pour l'opérateur $\overline{\delta}$ et applications arithmétiques*, Journées sur les Fonctions Analytiques (Toulouse, 1976), Lecture Notes in Mathematics 578, Springer, 1977, 314–323.

[50] J. Szpond, *On linear Harbourne constants*, preprint 2015 (arXiv:1502.04334).

[51] M. Waldschmidt. *Propriétés arithmétiques de fonctions de plusieurs variables II*, Séminaire P. Lelong (Analyse), 1975–76, Lecture Notes Math. 578, Springer, 1977, 108–135.

[52] Z. Wan. *Introduction to Kac-Moody algebras*, World Scientific, 1991.

[53] A. Wiman. *Zur Theorie der endlichen Gruppen von birationalen Transformationen in der Ebene*, Math. Annalen **48** (1896), 195 – 240.