

Recollections from last week

- Definition of $a \mid b$, emphasizing the difference from the fraction a/b
- Basic properties of divisibility

Resumed from Section 5, page 6 of the worksheet ‘Divisibility – Part 1’.

More properties of divisibility

Last time we saw that, for integers a, b and c :

$$a \mid b \quad \text{or} \quad a \mid c \implies a \mid bc.$$

In this session, students observed that the converse does *not* hold:

$$a \mid bc \not\implies (a \mid b \quad \text{or} \quad a \mid c).$$

In fact, we noticed the following general result: given any **composite** number a , one can always find natural numbers b and c such that $a \mid bc$, but $a \nmid b$ and $a \nmid c$.

Complementarily, we made the following observation: given any **prime** number p and integers b, c , we have

$$p \mid bc \implies p \mid b \quad \text{or} \quad p \mid c.$$

Claim: If a, b, c are integers such that $a \mid bc$ and $\gcd(a, c) = 1$, then $a \mid b$.

One of the students remarked that in fact the following generalization should hold.

Conjecture:

$$a \mid bc \implies \frac{a}{\gcd(a, c)} \mid b.$$

An illustration of Bézout’s identity

One standard proof of the claim above uses the following fact.

Bézout’s identity

If a and c are relatively prime integers, then there exist integers r, s such that

$$ar + cs = 1.$$

In order to motivate Bézout’s identity, we tackled a real-life incarnation of it.

Question: Given a weighing balance, what quantities can be measured using only a kg and c kg weights? One may assume that there are as many such weights as needed, and that weights can be placed on either pan.

By playing around with examples, the students observed that whenever a and c share no common factor other than 1, it is possible to measure 1 kg, and therefore any integer weight. Some students also observed in examples that, more generally, the weights that can be measured are exactly the multiples of $\gcd(a, c)$.