CMI Math Circle

# Synopsis for August 2 2026

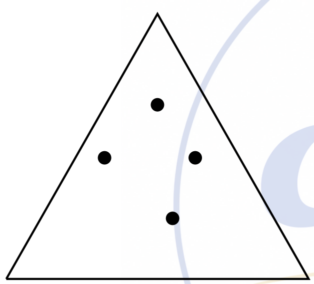
## Junior Batch

We started the session with a two-player game called the **No-Crossing Game**. This activity is inspired from criss-cross game by Sam Vandervelde.

- Students drew a triangle and placed a chosen number of dots inside it.
- Taking turns, players drew straight lines connecting any two dots.
- The rules were strict: a new line could **never cross** an already drawn line.
- The first player who could no longer make a legal move lost the game.

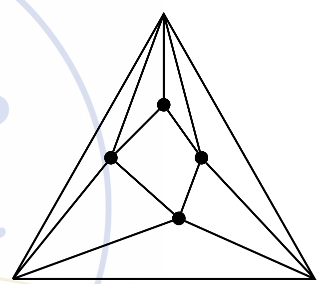
Students played several rounds, recording the number of dots, lines, and the ultimate winner. The goal of this activity was to find a hidden mathematical pattern that would allow them to predict the winner *before the game even started!*

**Start of the Game**



(4 dots chosen inside the boundary)

**End of the Game**



(No more lines can be drawn!)

To solve the mystery of the game, we had to learn some Graph Theory. We defined several important terms:

- **Vertices (V):** The dots or points.
- **Edges (E):** The line segments connecting the dots.
- **Regions (R):** The separate "pieces" the background is divided into (including the 1 big region on the *outside* of the graph).
- **Planar Graph:** A graph drawn on a flat surface where edges do not cross each other.
- **Connected Graph:** A graph where you can reach any vertex from any other vertex by traveling along the edges.

Students analysed various pre-drawn graphs, carefully counting $V$, $E$, and $R$ for each one. They calculated the value of $V - E + R$ and discovered a beautiful mathematical constant. No matter what the graph looked like, the answer was always the same!

**Euler's Formula**

For any connected, planar graph (drawn with no crossing lines):

$$V - E + R = 2$$

But mathematicians don't just guess from examples; they **prove** things. The students proved Euler's Formula step-by-step:

1. First, they looked at a simple graph with no enclosed regions (just a line of dots). For example, 5 vertices and 4 edges gives 1 outside region. $5 - 4 + 1 = 2$. It works!
2. Next, they imagined adding one new non-crossing edge. What happens? The number of vertices ($V$) stays the same. The number of edges ($E$) goes up by 1. The number of regions ($R$) goes up by 1 (because the new edge slices an existing region in half).
3. Because $E$ and $R$ both increase by exactly 1, they cancel each other out in the equation $V - E + R$. The total remains exactly 2 forever!

## Hacking the Game

With Euler's Formula proven, the students became math detectives to crack the No-Crossing Game. They realized that when the game is completely over, every single region inside the graph is a **triangle** (a 3-sided shape). If a region had 4 or more sides, a player could still draw a line through it, meaning the game wasn't over yet!

Using this fact, they built a equation:

- Because every region is a triangle, the total number of sides is $3 \times R$.
- However, every edge is shared like a fence between exactly 2 regions, so the total number of sides is also $2 \times E$.
- This gives us the equation: $\mathbf{3R = 2E}$, which means $\mathbf{R = \frac{2E}{3}}$.

The students then took this and plugged it directly into Euler's Formula ($V - E + R = 2$):

$$V - E + \frac{2E}{3} = 2 \qquad \text{therefore we have} \qquad V - \frac{E}{3} = 2 \qquad \text{and hence we have} \qquad \mathbf{E = 3V - 6}$$

**This was the secret to the game!** Because every turn adds exactly 1 edge, the total number of edges ($E$) is exactly the total number of moves made in the game.

By simply looking at the number of vertices ($V$) chosen at the very beginning, the students could calculate exactly how many moves the game would take using $E = 3V - 6$.

- If $E$ is an **ODD** number, Player 1 makes the last move and wins.
- If $E$ is an **EVEN** number, Player 2 makes the last move and wins.

Using pure algebra and graph theory, the students could now predict the winner of a 25-dot game without ever having to draw a single line!

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Although some of the students were able to completely hack the game and provide the correct mathematical justification, a few grasped the underlying reasoning but did not initially follow why a formal justification was necessary. However, almost all the students successfully understood and derived both Euler's formula and the $3R = 2E$ relationship!

## Take-Home Challenge

We wrapped up the session with the following take-home challenge: Does Euler's formula work for 3D shapes? Students were given shapes like a Cube, a Triangular Prism, and a Tetrahedron. By replacing "Regions" with "Faces" ($F$), they tested if $V - E + F = 2$ holds true for 3D solids (Spoiler: It does!).