Synopsis for August 16, 2026

1 Dividing up a region efficiently

The aim of this section was to introduce the optimization problem without mentioning the honeycomb theorem. The important distinction to establish is between: *the efficiency of one cell* vs. *the efficiency of an arrangement of many cells.*

- We started by asking *what shapes could you use for the individual plots?*

  We explored some responses: squares, rectangles, triangles, hexagons and circles. The important question is whether the proposed shapes can actually be used to fill a region without gaps or overlaps.

- We began with the simpler question:

  For a fixed area, which of several shapes has the smallest perimeter?

  This restricted question allowed students to discover why the hexagon looks promising.

- We compared the possible side lengths for squares, rectangles, triangles and hexagons required for area 1 square unit. Using these to calculate the corresponding perimeters, we concluded:

  $$P_{hexagon} < P_{\square} < P_{\triangle}.$$

  So the regular hexagon wins among these three shapes.

- **Point to note:** The circle gives us a counterexample to the naive claim that the hexagon is the most efficient possible shape. This is because for a circle of area 1, $\pi r^2 = 1$, so $r = \frac{1}{\sqrt{\pi}}$. Its circumference is therefore $C = 2\pi r = 2\sqrt{\pi} \approx 3.545$.

  This is smaller than the hexagon's perimeter, $P_{hexagon} \approx 3.72$.

  Thus the circle is more efficient than the hexagon when considered as a single isolated cell. However, we need that the cells must fit together without gaps or overlaps. We see that circles cannot tile the plane: if congruent circles are placed next to one another, there are unavoidable gaps between them.

  Thus, *The best shape for one cell need not be the best shape for a collection of cells.*

- The next question we explore is: which shapes fit together? For regular polygons, the question can be reduced to the angles at a vertex.

  Finally, at a vertex, the interior angles must add to $360°$.

  Thus $3(60°) = 180°$ is not the right way to think about the triangle if the question is whether the polygons can fill the whole neighbourhood of a point. Instead, $6(60°) = 360°$, $4(90°) = 360°$, and $3(120°) = 360°$.

  For the pentagon, $360°/108°$ is not an integer, and similarly for the heptagon.

  In the neighbourhood of a vertex, the three possibilities are:

  six triangles, four squares, three hexagons.

- We conclude that:

  A regular hexagon encloses more area for a given perimeter than a square or an equilateral triangle.

- We have now seen:

  - the circle is more efficient than the hexagon for one isolated region;
  - circles cannot tile the plane;
  - the regular hexagon is the most efficient of the three regular tiling cells considered so far.

  We conclude that we have a plausible conjecture, though not a proof.

2 Origami geometry

We learned how to create a hexagon from a square piece of paper. Then we ask the question: why do the folds used in the process actually give us the angles we desire?