NPTEL MOOC,JAN-FEB 2015
Week 8, Module 6

DESIGN AND ANALYSIS OF ALGORITHMS

Intractability: Checking algorithms

MADHAVAN MUKUND, CHENNAI MATHEMATICAL INSTITUTE
http://www.cmi.ac.in/~madhavan

Efficient algorithms

- Shortest path, minimum cost spanning tree, maximum flow, … have polynomial time algorithms
- Search space for solutions is exponential
  - All possible paths, all possible spanning trees, all possible subsets of edges, …
  - Brute force: scan exponential possibilities and choose the best

Efficient algorithms …

- Do all problems admit such efficient solutions?
- Unfortunately not
- For a large class of “natural” problems, no shortcut is known to exist

Generating vs checking

- A teacher assigns homework:
  - Factorize a large number that is the product of two primes
- Student: Given N, find p,q such that pq = N
  - Generate a solution
- Teacher: Given a student's solution p,q, verify that pq = N
  - Check a solution

Checking algorithms

- Checking algorithm C for problem P
- Takes in an input instance I for P and a solution "certificate" S for I
- C outputs yes if S represents a valid solution for I, no otherwise
- For factorization, I is N, S is {p,q} and C involves verifying that pq = N

Boolean satisfiability

- Boolean variables x,y,z,…
- !x — negation of x, x || y — x or y, x & y — x and y
- Clause — formula C of the form

  (x || !y || z || …. || w)

  - Disjunction of literals (variables, negated variables)
- Formula — conjunction of clauses

  C & D & … & E

Boolean satisfiability

- Assign suitable values {True,False} to x,y,z,… so that the formula evaluates to true

(x || y || z) & (x || !y) & (y || !z) & (!x || !y || !z)

- x = True, y = True, z = False makes this true

(x || y || z) & (x || !y) & (y || !z) & (z || !x) & (!x || !y || !z)

- Now there is no satisfying assignment

Boolean satisfiability

- Generating a solution
  - Try each possible assignment to x,y,z,…
  - N variables — 2^N possible assignments
  - Is there a better algorithm? Not known
- Checking a solution
  - Given formula F and valuation V(x) for each x, substitute into formula and evaluate

Boolean satisfiability

- Input format is important
- Suppose a clause is a conjunction of literals …

(x & !y & z & …. & w)

- … and a formula is a disjunction of clauses

C || D || … || E

- Each clause forces a unique valuation
- Try each clause in sequence

Travelling salesman

- A network of cities with distances between each pair
  - A complete graph G = (V,E) with edge weights
- Find the shortest tour that visits each city exactly once
  - Simple cycle x,y,z,…,x visiting all vertices, of minimum cost

Travelling salesman

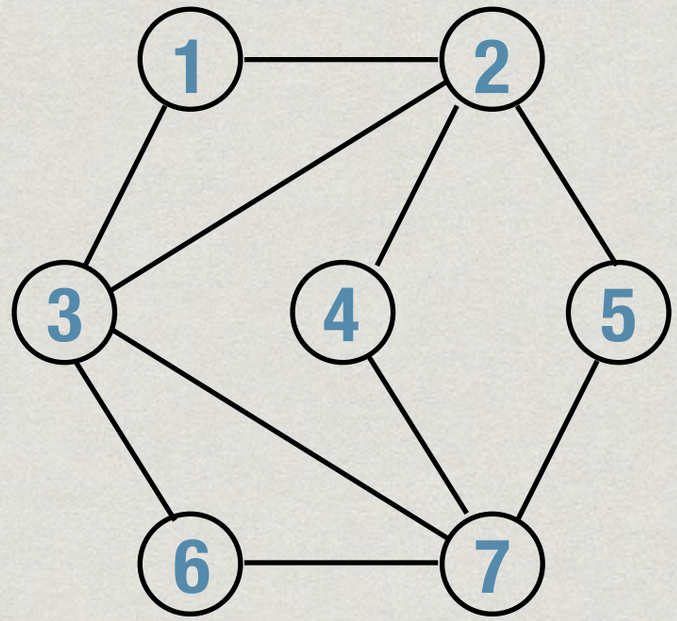
- Designing a checking algorithm
- Checking algorithm must give a yes/no answer
- Given a graph G and a proposed solution S we can
  - Verify that S is a cycle
  - Compute its cost
  - How to check that S is the least cost cycle?

Travelling salesman

- Transform the problem
- Is there a tour with cost at most K?
- Now, given a solution S, we can check it
- For the original problem, cost is at most the sum of all the edge weights in the graph
- Find optimum K — test different values using binary search

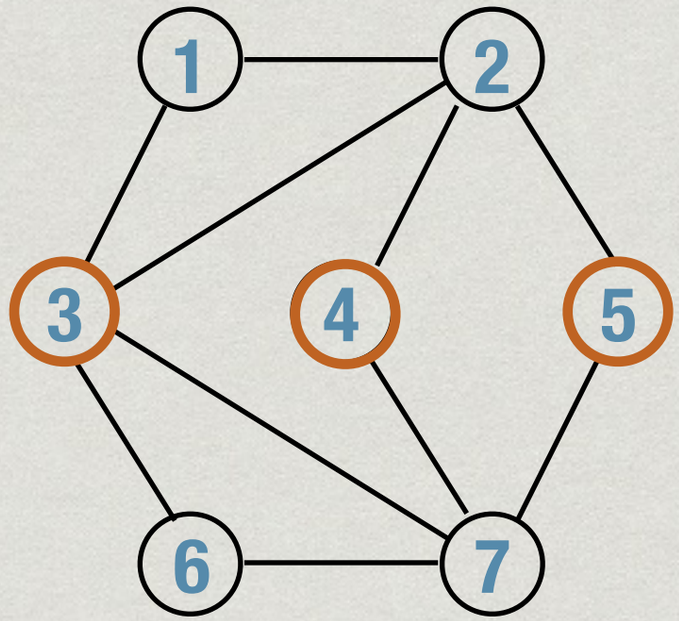
Independent set

- u,v are independent if there is no edge (u,v)
- U is an independent set if each pair {u,v} in U is independent
  - Constitute a neutral committee where none of the members know each other
- Find the largest independent set in a given graph
- Checking version: Is there an independent set of size K?



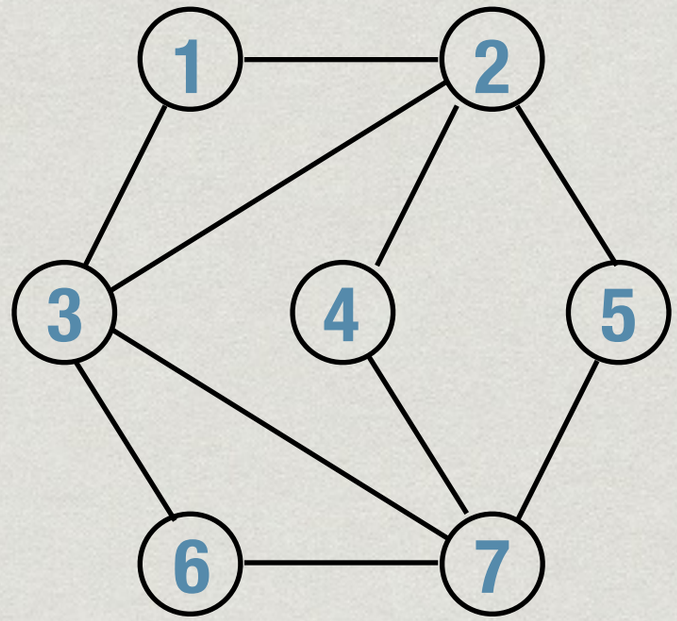
Independent set

- u,v are independent if there is no edge (u,v)
- U is an independent set if each pair {u,v} in U is independent
  - Constitute a neutral committee where none of the members know each other
- Find the largest independent set in a given graph
- Checking version: Is there an independent set of size K?



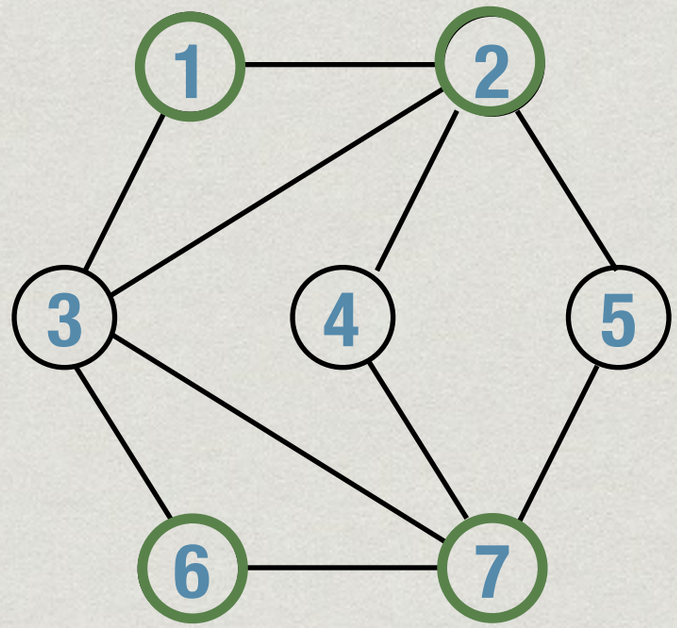
Vertex cover

- Node u covers every edge (u,v) incident on u
- U is a vertex cover if each edge in the graph is covered by some vertex in U
  - Position surveillance cameras at intersections to watch all roads
- Find the smallest vertex cover in a given graph
- Checking version: Is there an vertex cover of size K?



Vertex cover

- Node u covers every edge (u,v) incident on u
- U is a vertex cover if each edge in the graph is covered by some vertex in U
  - Position surveillance cameras at intersections to watch all roads
- Find the smallest vertex cover in a given graph
- Checking version: Is there an vertex cover of size K?

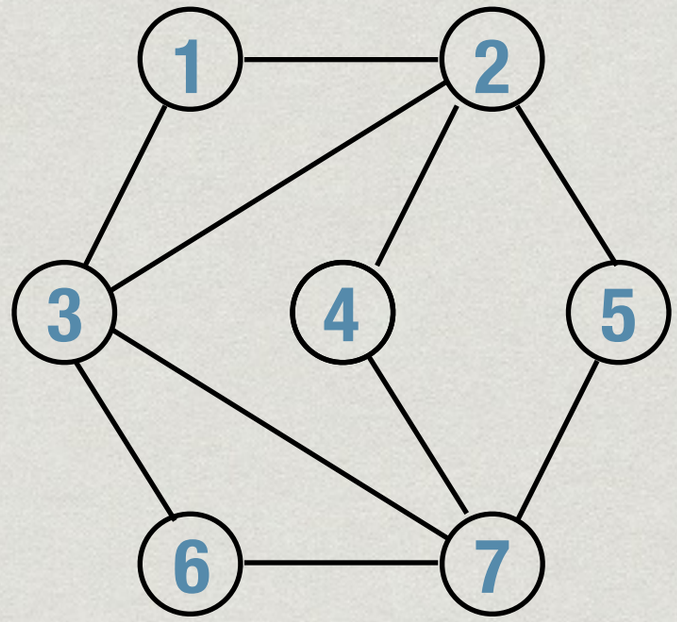


Connecting independent set and vertex cover

- U is an independent set of size K iff V-U is a vertex cover of size N-K
  - (⇒)

    Every edge (u,v) has at most one end point in U, so at least one end point in V-U
  - (⇐)

    For any edge (u,v), at least one endpoint in V-U, so no edges (u,v) within U

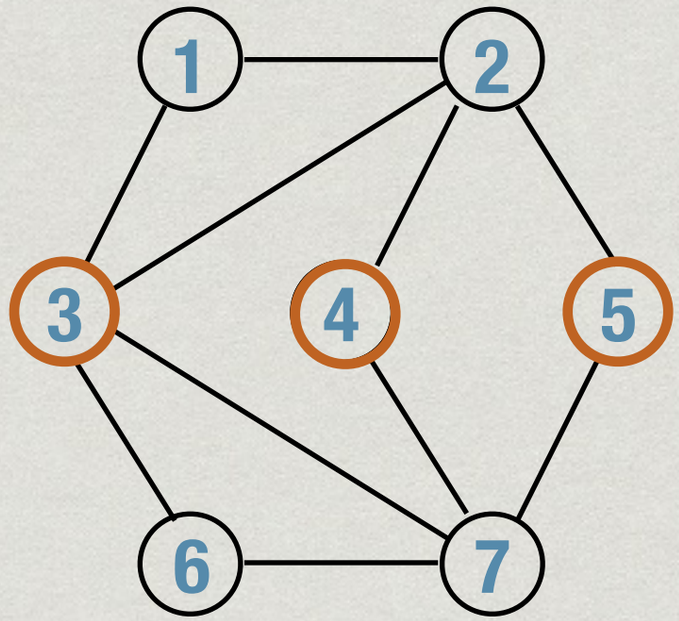


Connecting independent set and vertex cover

- U is an independent set of size K iff V-U is a vertex cover of size N-K
  - (⇒)

    Every edge (u,v) has at most one end point in U, so at least one end point in V-U
  - (⇐)

    For any edge (u,v), at least one endpoint in V-U, so no edges (u,v) within U

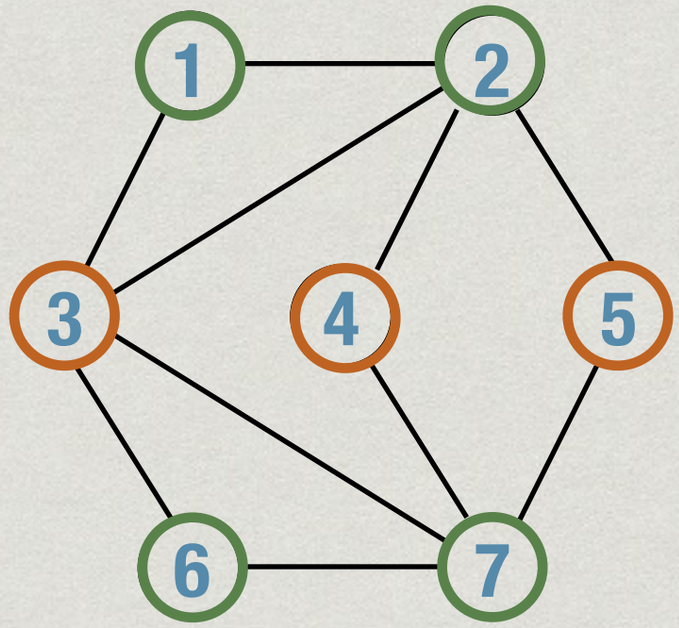


Connecting independent set and vertex cover

- U is an independent set of size K iff V-U is a vertex cover of size N-K
  - (⇒)

    Every edge (u,v) has at most one end point in U, so at least one end point in V-U
  - (⇐)

    For any edge (u,v), at least one endpoint in V-U, so no edges (u,v) within U



Reductions

- Independent set and vertex cover reduce to each other
- Recall: if A reduces to B and A is intractable, so is B
- Many pairs of checkable problems are inter-reducible
  - All “equally” hard