NPTEL MOOC,JAN-FEB 2015
Week 8, Module 1

DESIGN AND ANALYSIS OF ALGORITHMS

Linear Programming

MADHAVAN MUKUND, CHENNAI MATHEMATICAL INSTITUTE
http://www.cmi.ac.in/~madhavan

Optimization problems

- Many computational problems are optimization tasks …
  - *Shortest* paths, *Minimum cost* spanning tree, *Longest* common subsequence
- … subject to constraints
  - Path follows edges in a graph, tree is a subset of given graph, subsequence has same letters

Linear programming

- Optimization problem where constraints and quantity to be optimized are linear functions
  - Constraints: ax + by + …. ≤ K, ax + by + …. ≥ K
  - Quantity (objective function): ax + by + …

Example: Maximize profits

Grandiose Sweets sells cashew barfis and dry fruit halwa.

- Each box of barfis earns a profit of Rs 100, while each box of halwa earns a profit of Rs 600
- Daily demand for barfis is at most 200 boxes, for halwa is at most 300 boxes
- Staff can produce 400 boxes a day, altogether
- What is the most profitable mix of barfis and halwa to produce?

Linear programming model

* b : number of boxes of barfis produced in a day
* h : number of boxes of halwa produced in a day
* Profit is $100b + 600h$
* Demand constraints: $b \leq 200$, $h \leq 300$
* Production constraint: $b + h \leq 400$
* Implicit constraints: $b \geq 0$, $h \geq 0$

Linear program

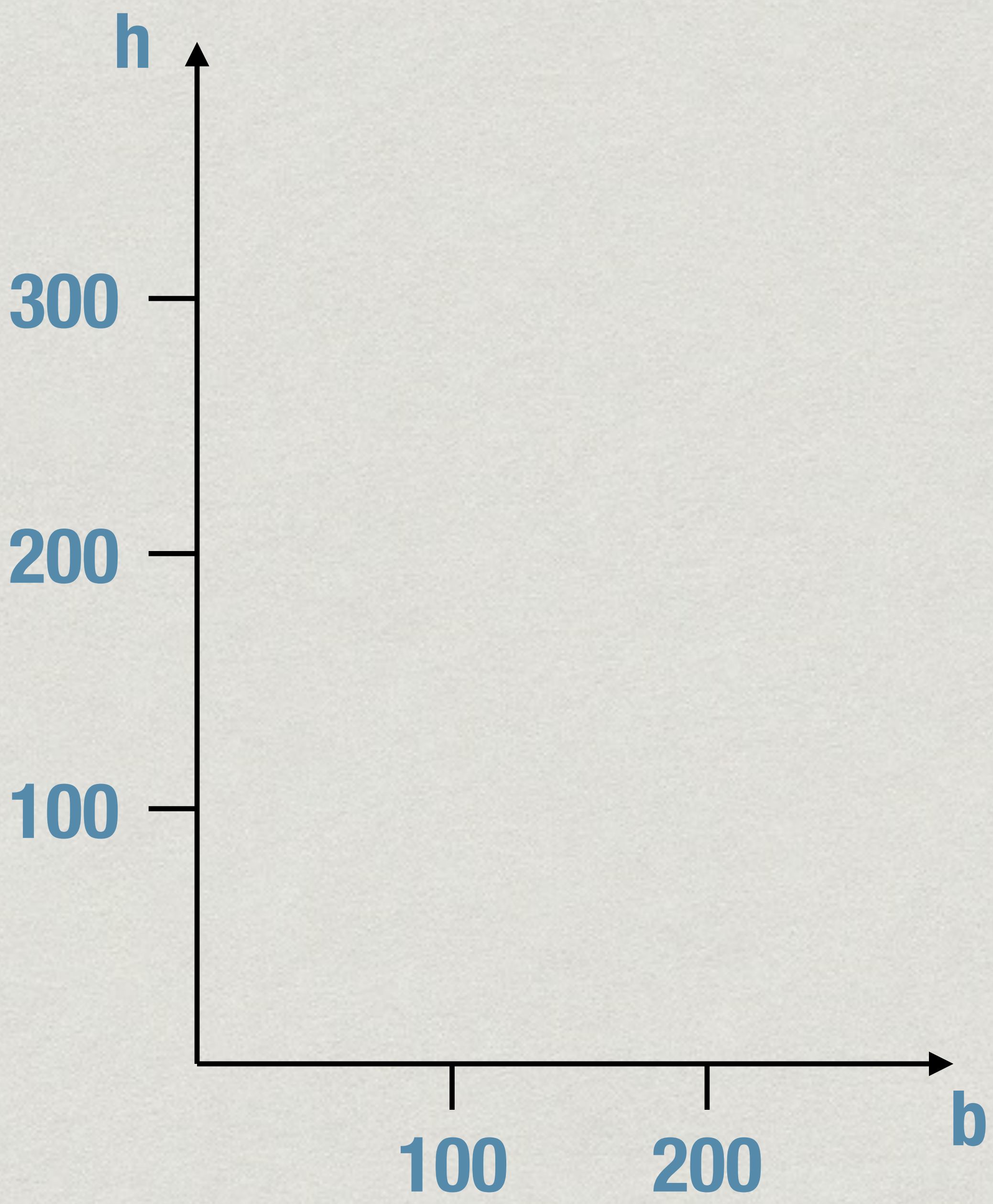
Objective function

- Maximize $100b + 600h$

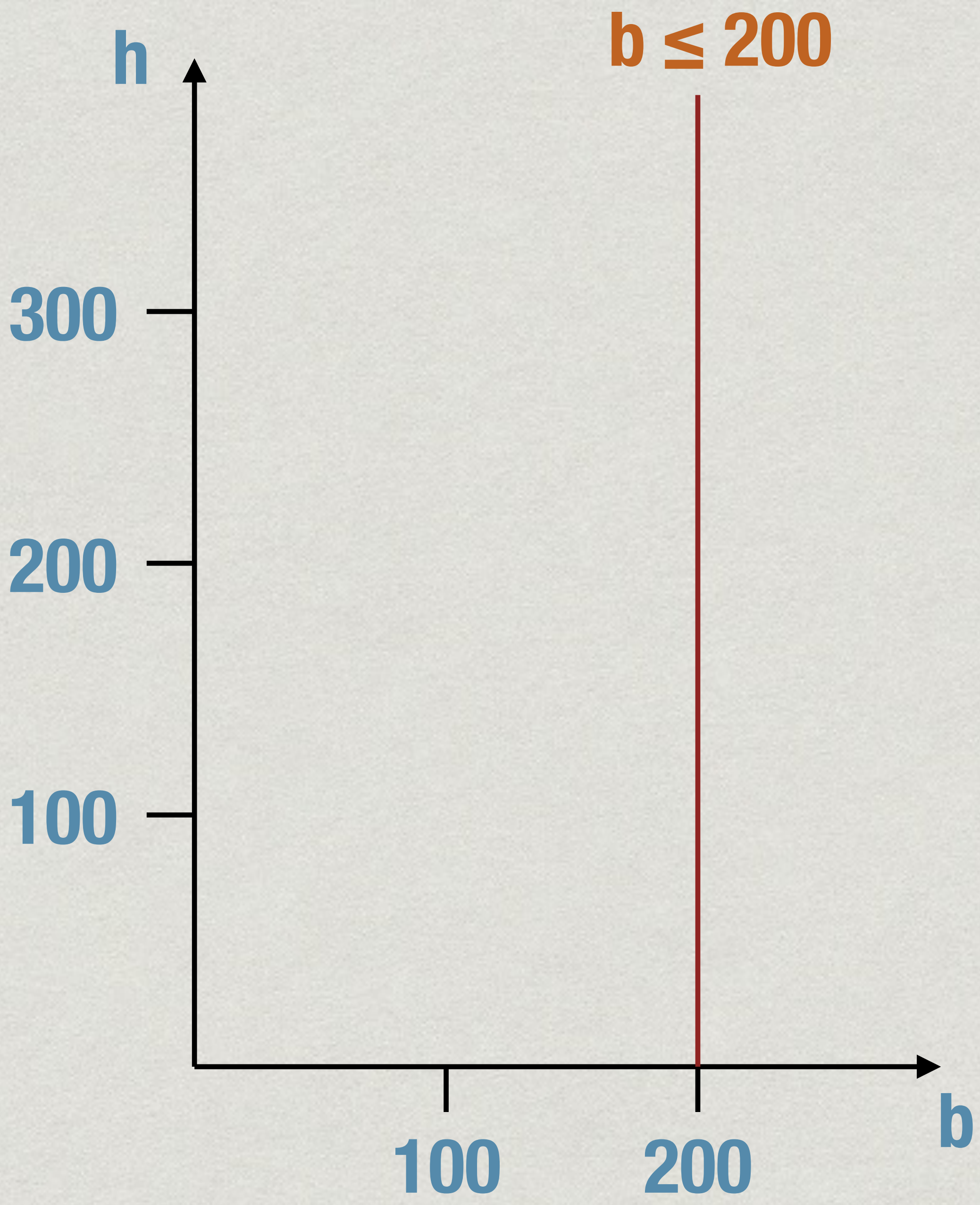
Constraints

- $b \leq 200$
- $h \leq 300$
- $b + h \leq 400$
- $b, h \geq 0$

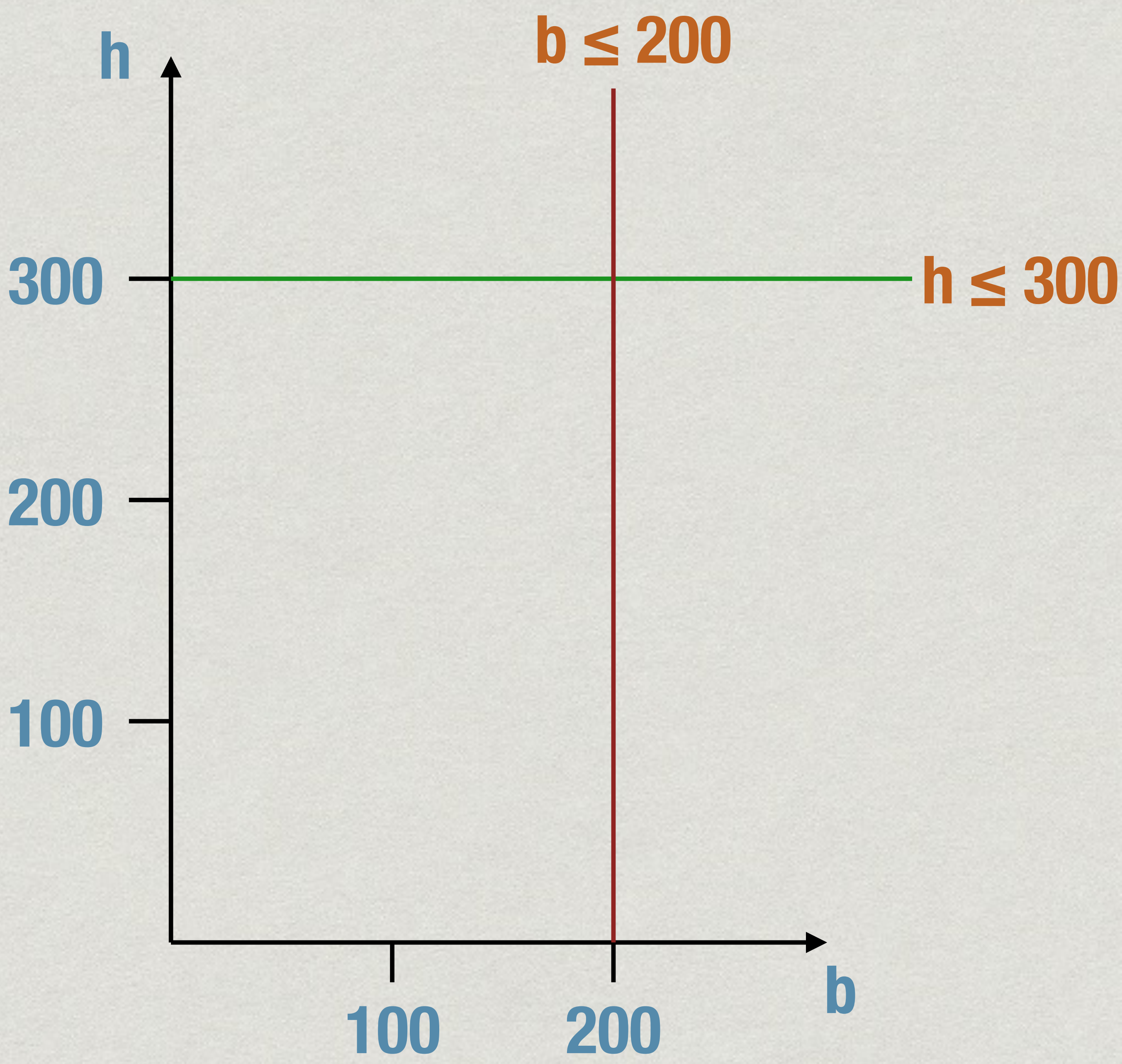
In pictures …



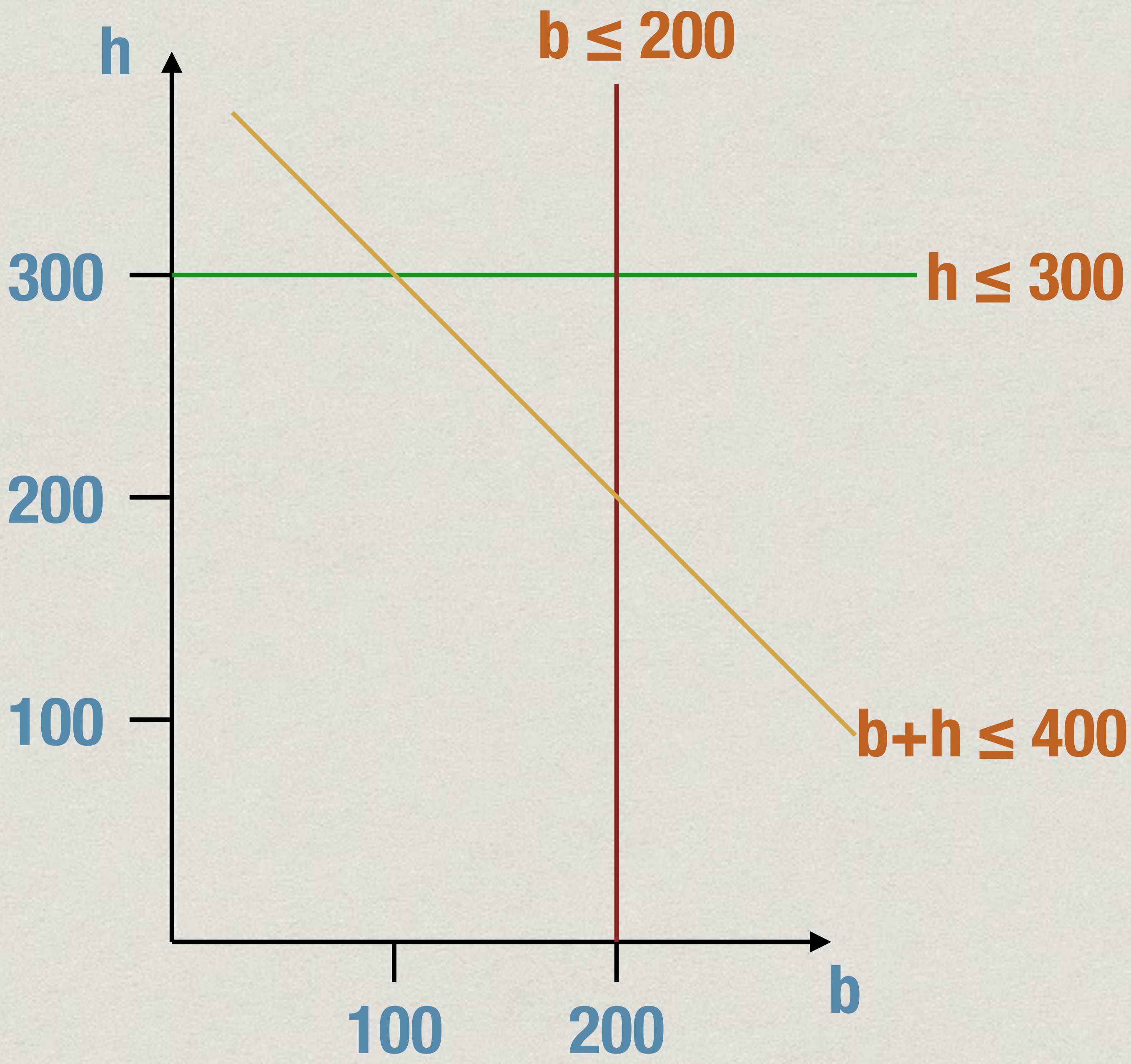
In pictures …



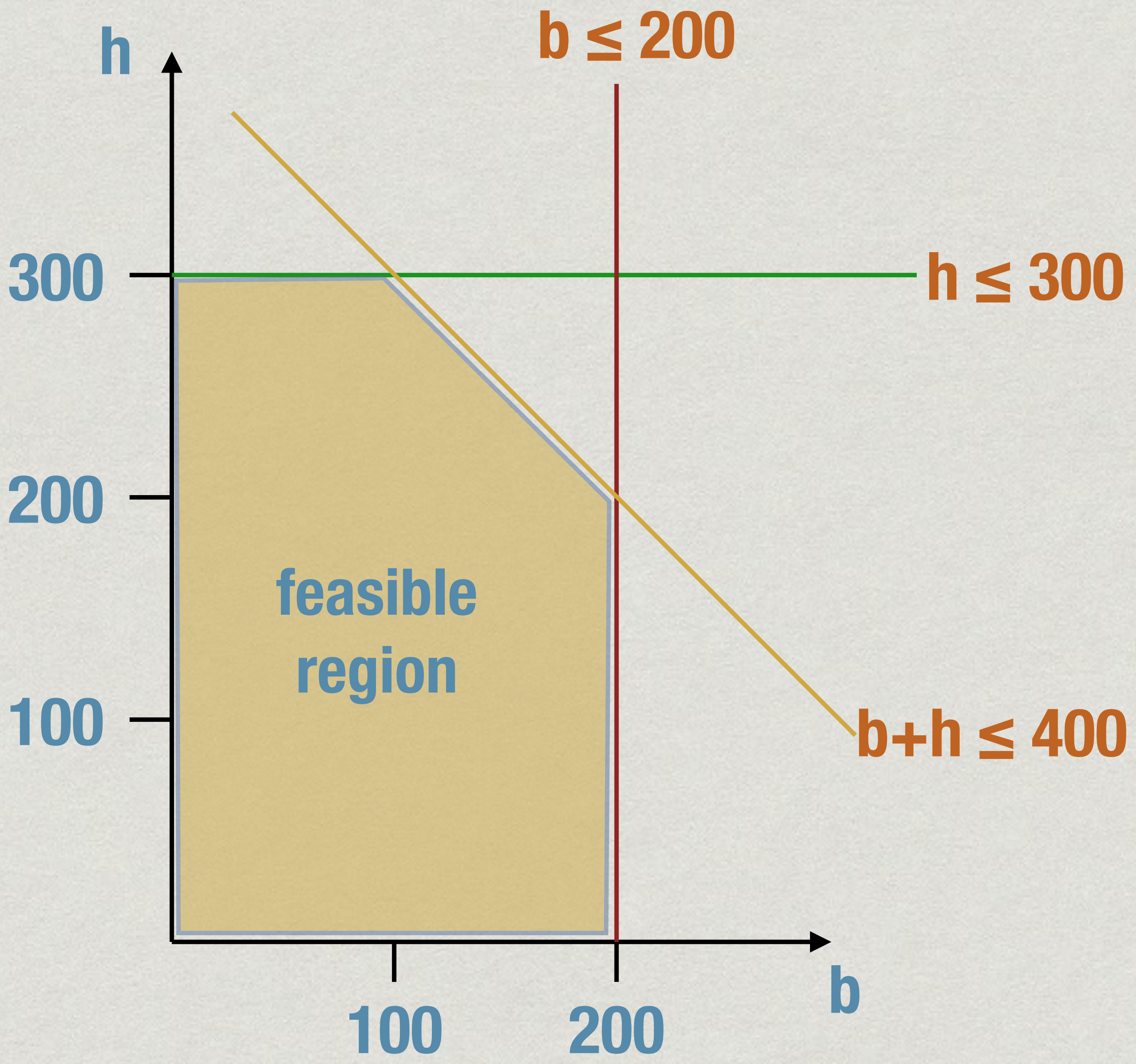
In pictures …



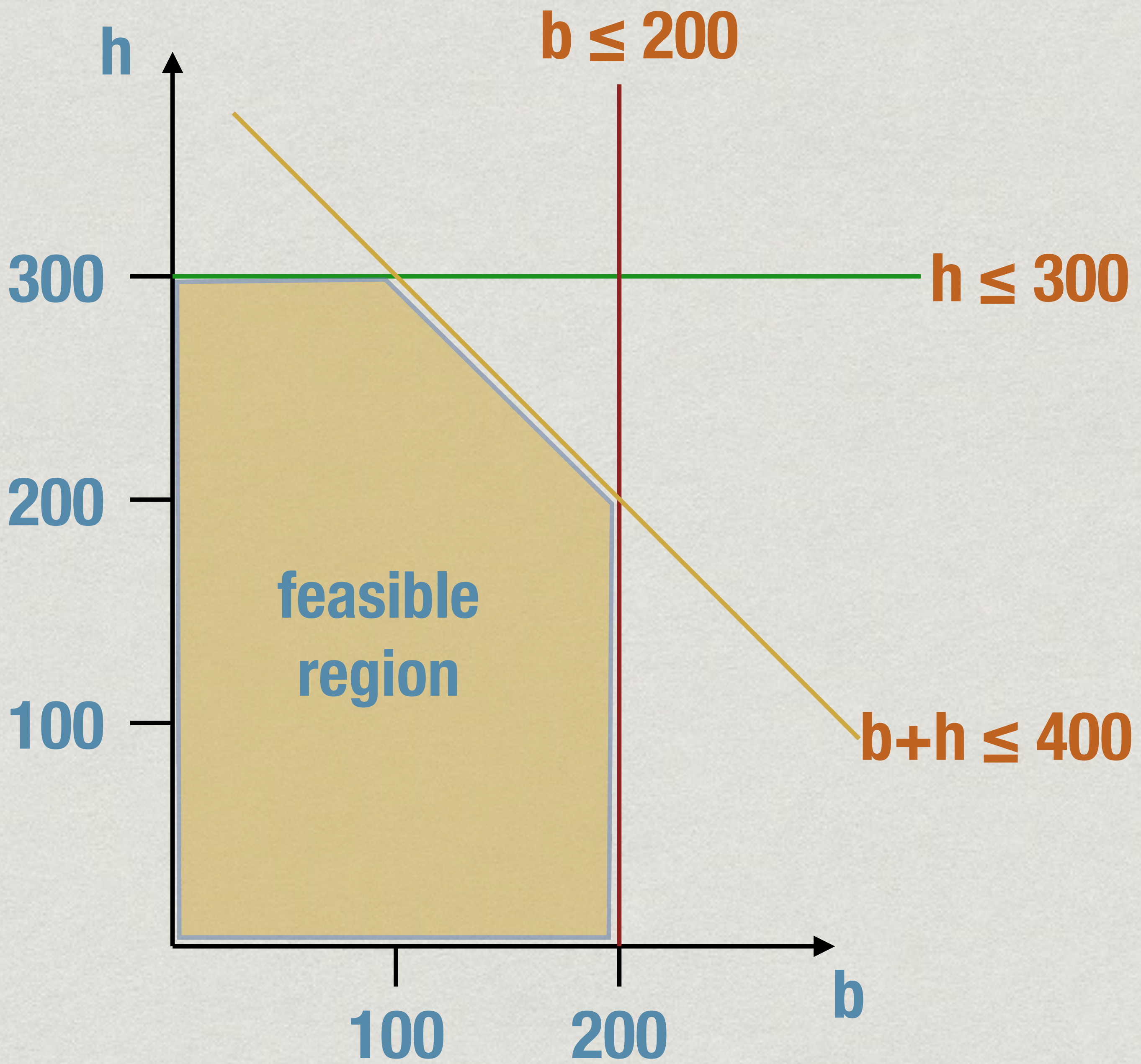
In pictures …



In pictures …

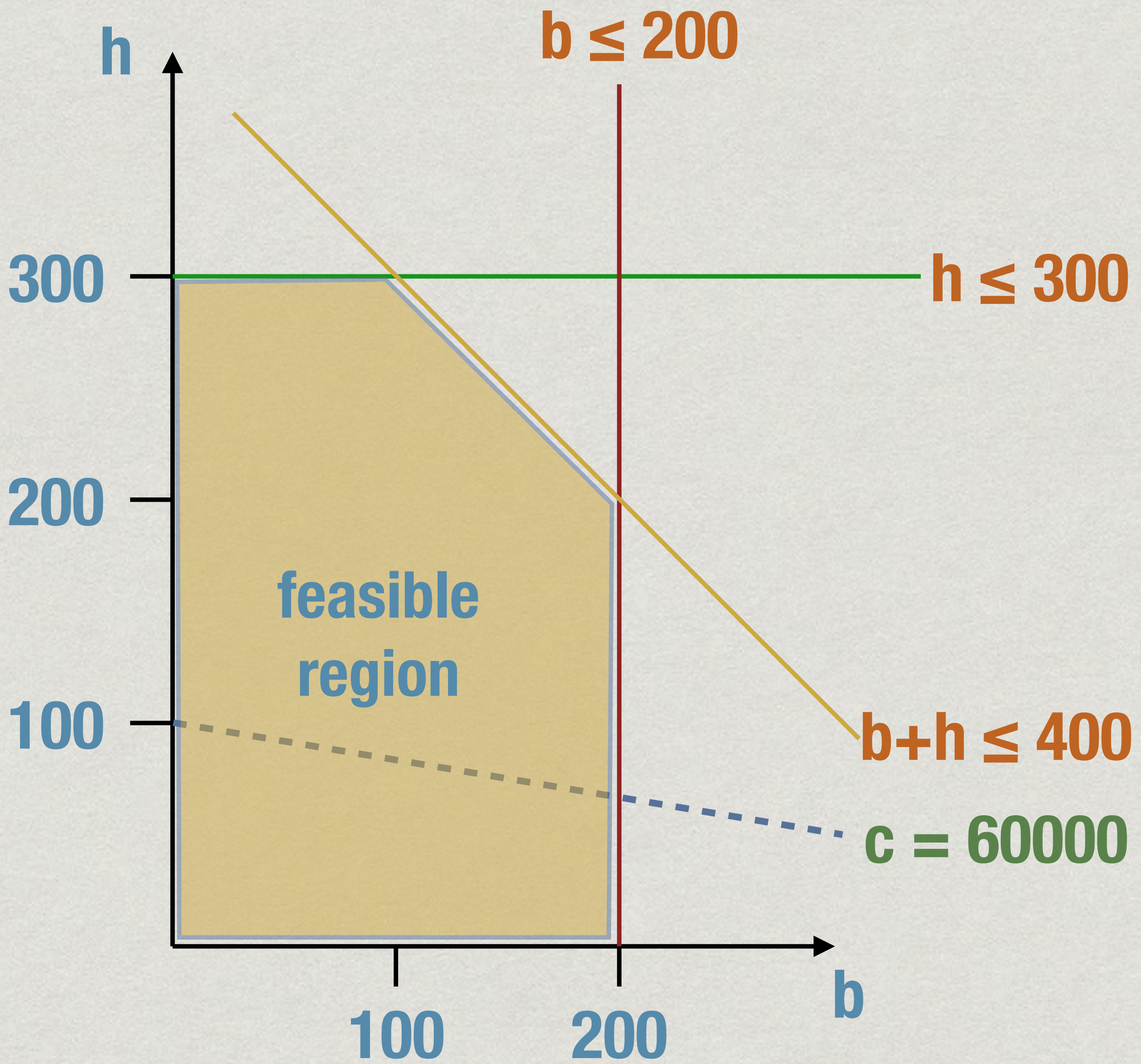


In pictures ...



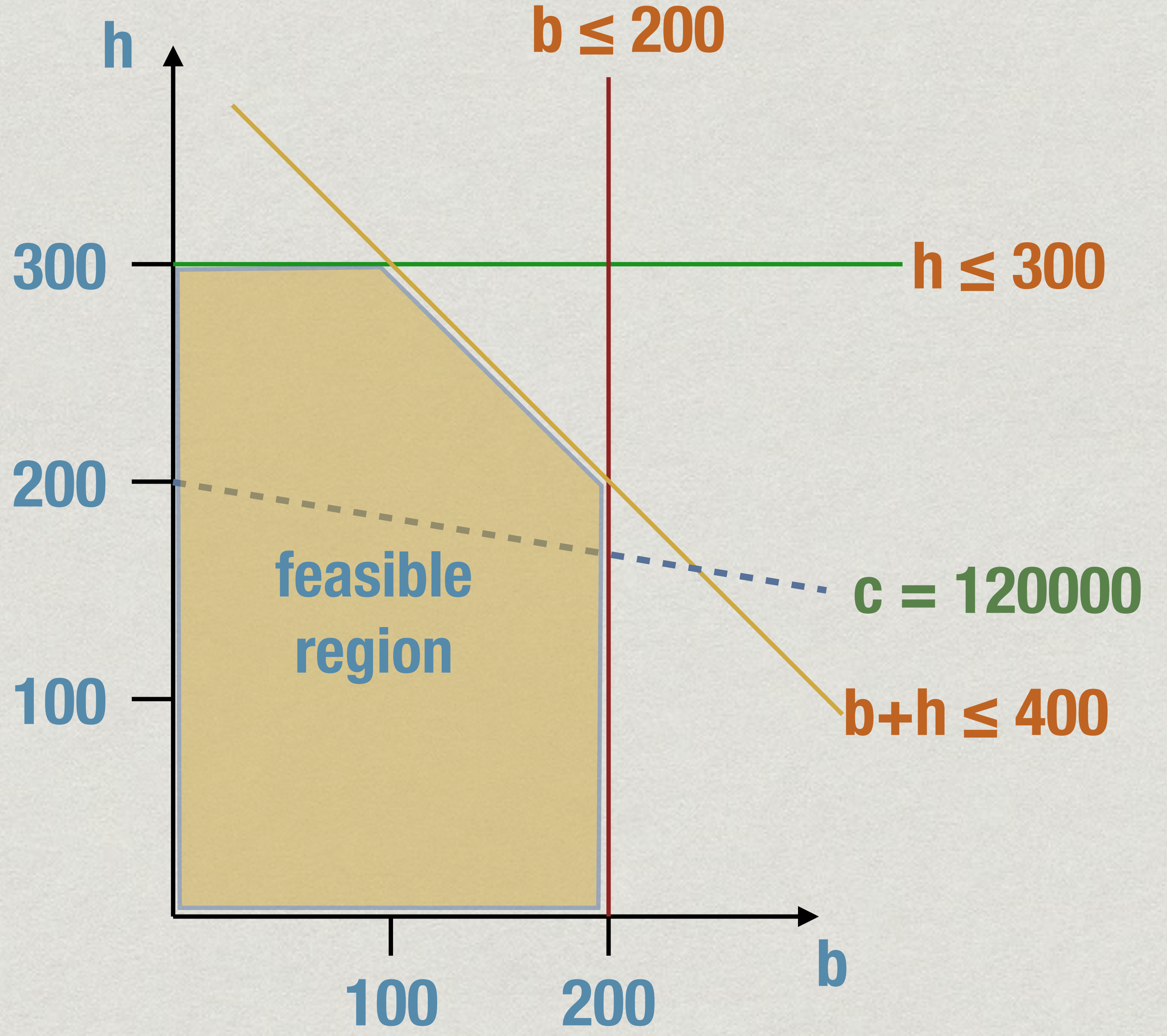
Objective: 100b+600h = c

In pictures …



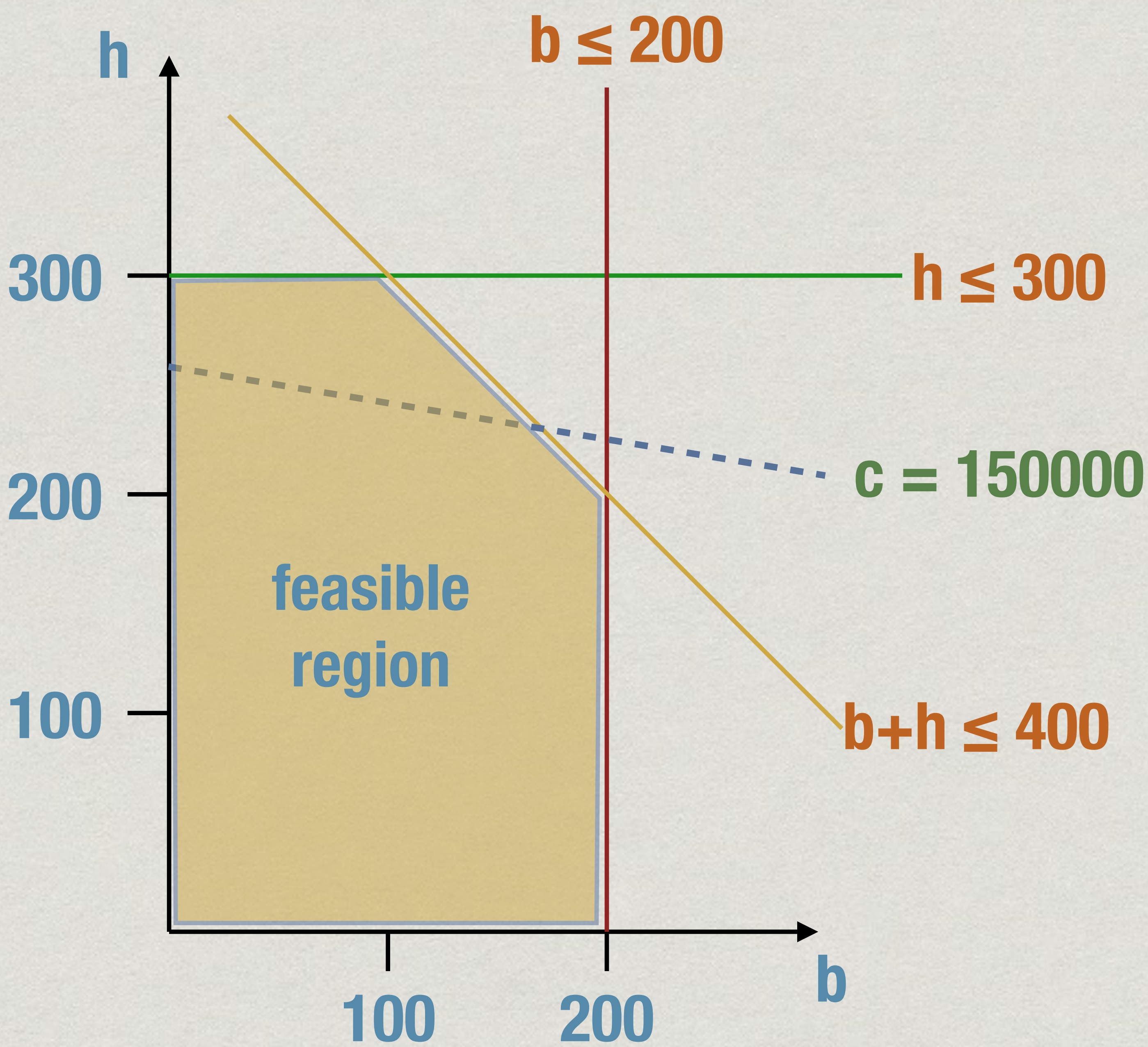
Objective: $100b+600h = c$

In pictures …



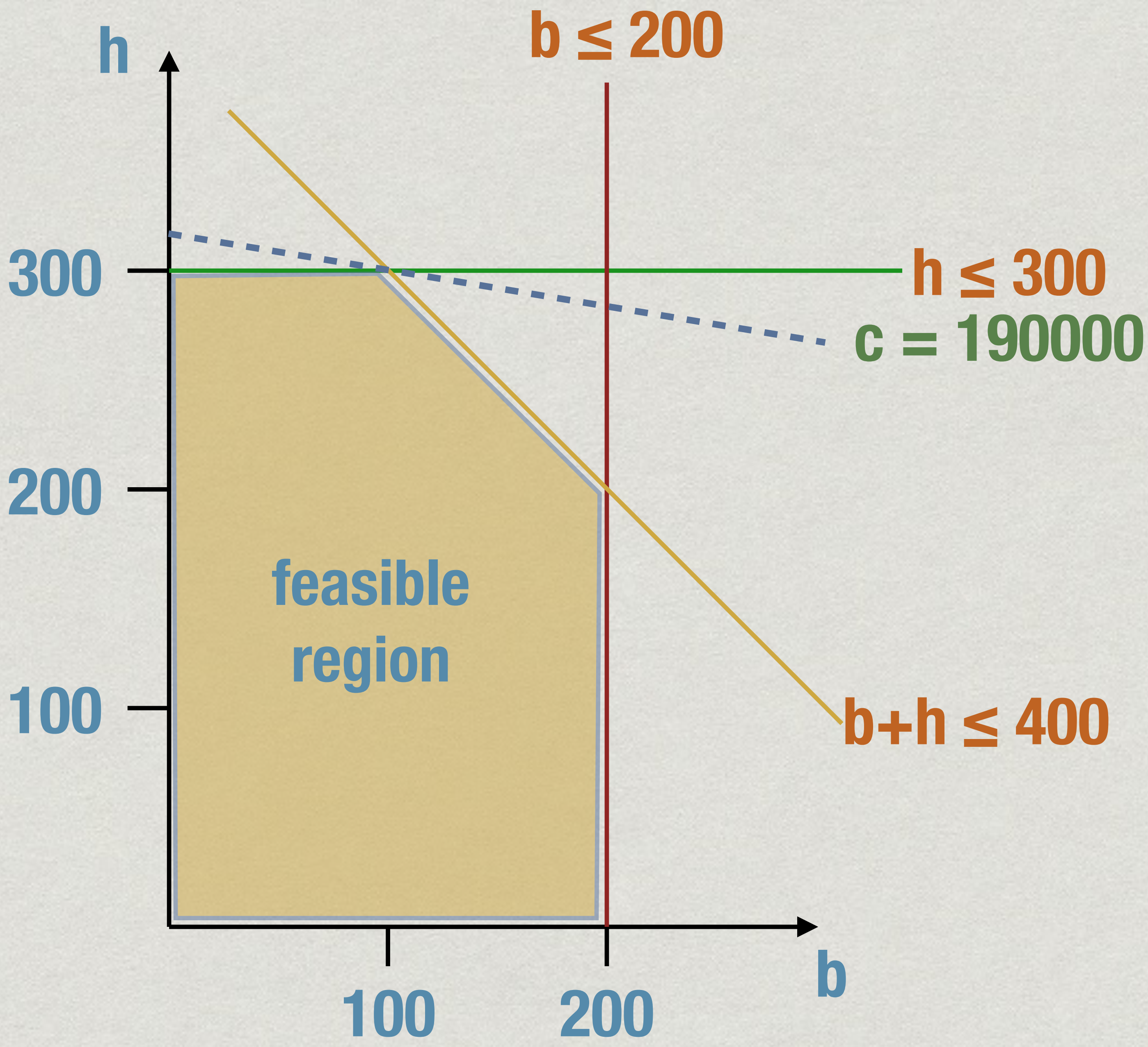
Objective: 100b+600h = c

In pictures ...



Objective: 100b+600h = c

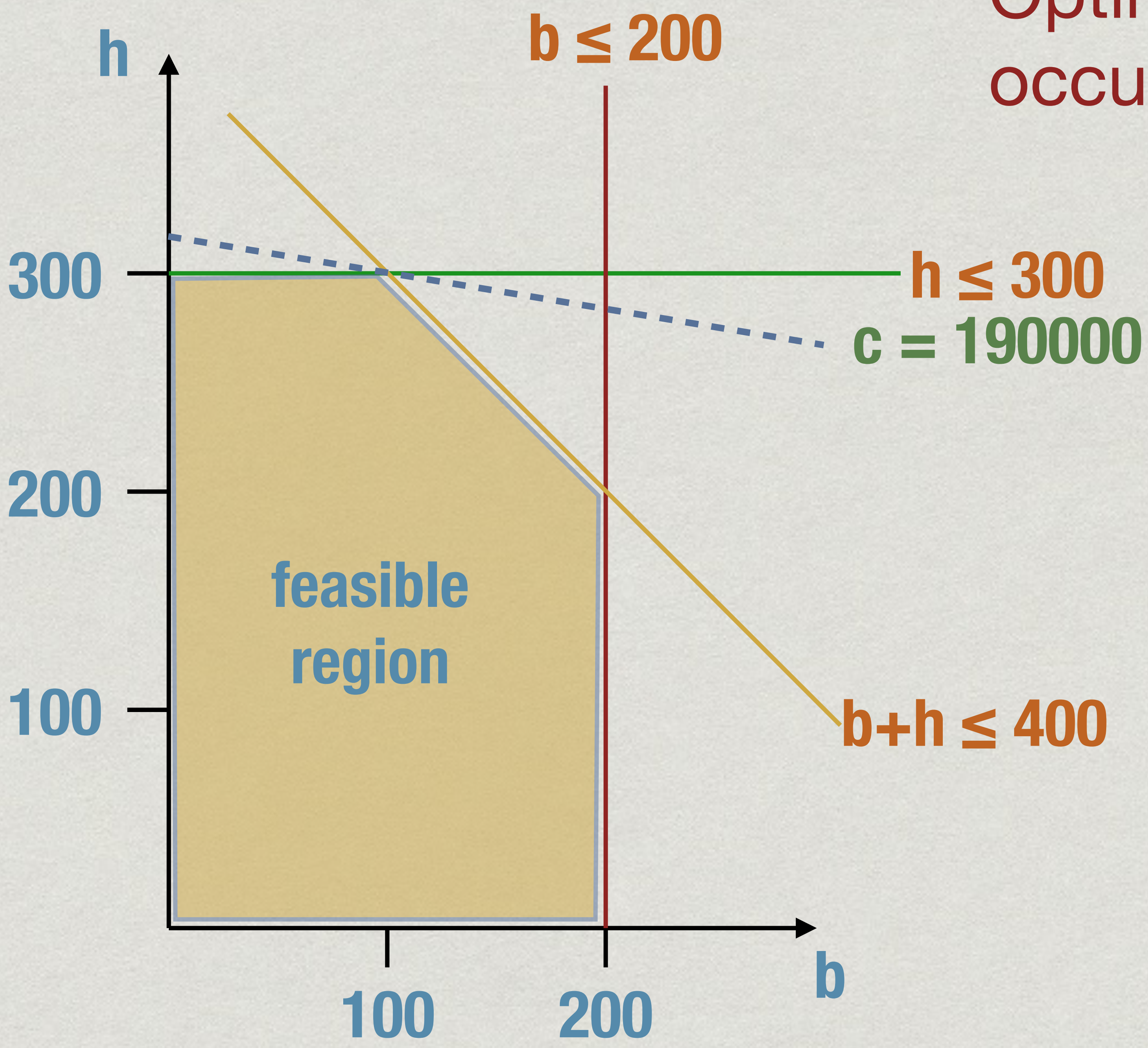
In pictures ...



Objective: 100b+600h = c

In pictures ...

Optimal value always occurs at a vertex



Objective: $100b+600h = c$

Solving linear programs

Simplex Algorithm

- Start at any vertex, evaluate objective function
- If an adjacent vertex has a better value, move
- If current vertex is better than all neighbours, stop
- Can be exponential, but efficient in practice
- Theoretically efficient algorithms exist

Solving linear programs

Existence of solutions

- Feasible region is convex
- May be empty — constraints are unsatisfiable — no solutions
- May be unbounded — no upper/lower limit on objective function

Example, extended

Grandiose Sweets adds almond rasmalai

- Profit per box: barfis — Rs 100, halwa — Rs 600, rasmalai — Rs 1300
- Demand, in boxes: barfis — 200, halwa — 300, rasmalai — unlimited
- Production: 400 boxes a day, altogether
- Milk supply: 600 boxes of halwa, 200 of rasmalai or any combination (rasmalai needs 3 times as much milk)
- What is the most profitable mix to produce?

New linear program

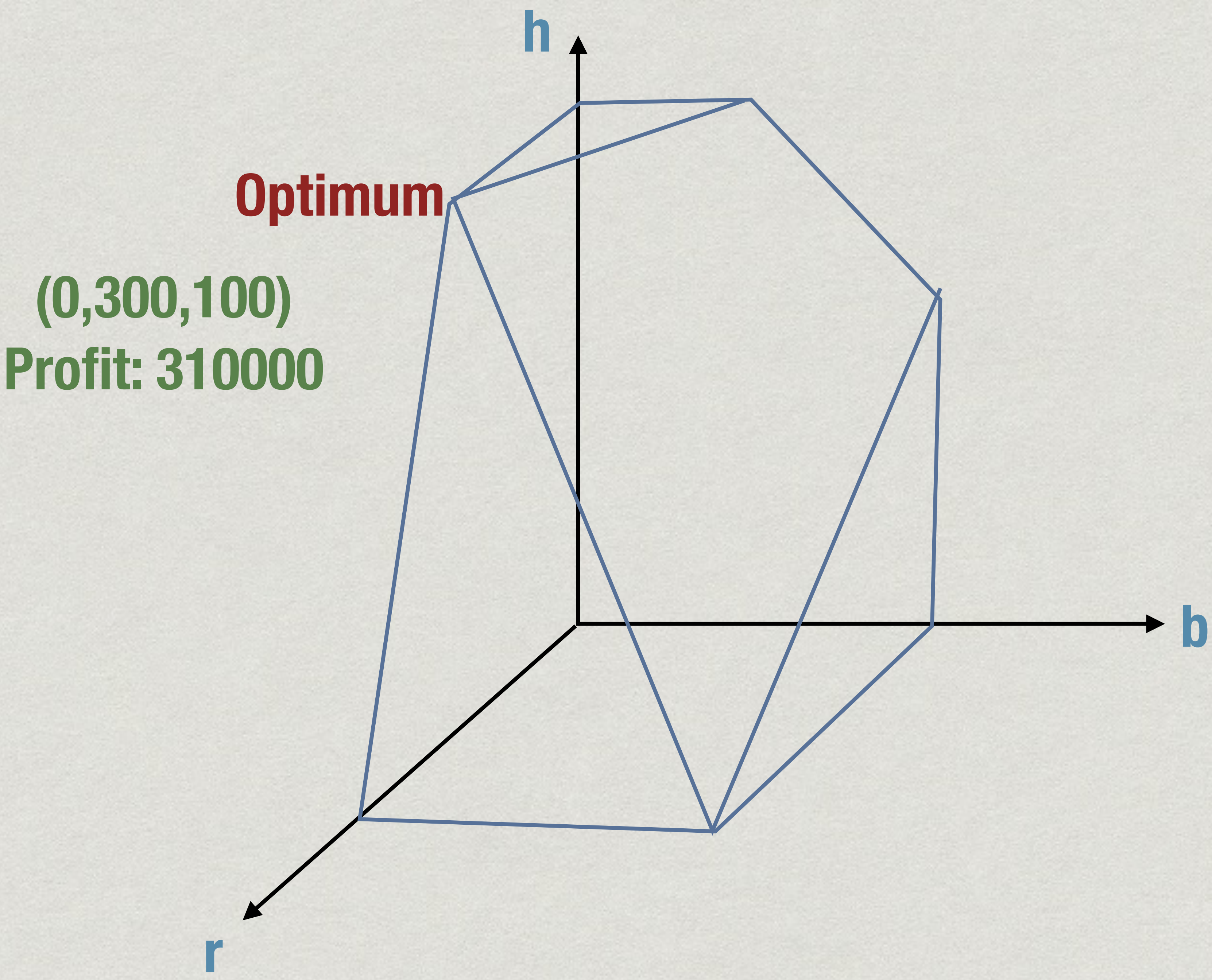
Objective function

- Maximize $100b + 600h + 1300r$

Constraints

- $b \leq 200$
- $h \leq 300$
- $b + h + r \leq 400$
- $h + 3r \leq 600$
- $b, h, r \geq 0$

Now a 3D picture



Why (0,300,100)?

Constraints

- $b \leq 200$
- $h \leq 300$ (A)
- $b + h + r \leq 400$ (B)
- $h + 3r \leq 600$ (C)
- 100x(A) + 100x(B) + 400x(C):
  $100b + 600h + 1300r \leq 310000$
- Profit is $100b + 600h + 1300r$,
  value at (0,300,100) is 310000, hence optimal

LP duality

- Can always construct a combination of constraints that tightly captures upper bound on objective function
- Dual LP problem
  - Minimize linear combination of constraints
  - Variables are the multipliers
  - Optimum solution solves both original (primal) and dual LP