NPTEL MOOC,JAN-FEB 2015
Week 7, Module 6

DESIGN AND ANALYSIS OF ALGORITHMS

Matrix Multiplication

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Multiplying matrices

- To multiply matrices A and B, need compatible dimensions
  - A of dimension m x n, B of dimension n x p
  - AB has dimension mp
- Each entry in AB take O(n) steps to compute
  - AB[i,j] is A[i,1]B[1,j] + A[i,2]B[2,j] +…+ A[i,n]B[n,j]
- Overall, computing AB is O(mnp)

Multiplying matrices

- Matrix multiplication is associative
  - ABC = (AB)C = A(BC)
  - Bracketing does not change the answer …
  - … but can affect the complexity of computing it!

Multiplying matrices

- Suppose dimensions are A[1,100], B[100,1], C[1,100]
  - Computing A(BC)
    - BC is [100,100], 100 x 1 x 100 = 10000 steps
    - A(BC) is [1,100],1 x 100 x 100 = 10000 steps
  - Computing (AB)C
    - AB is [1,1], 1 x 100 x 1 = 100 steps
    - (AB)C is [1,100], 1 x 1 x 100 = 100 steps
- A(BC) takes 20000 steps, (AB)C takes 200 steps!

Multiplying matrices

- Given matrices $M_1, M_2, \ldots, M_n$ of dimensions $[r_1,c_1]$, $[r_2,c_2]$, …, $[r_n,c_n]$
  - Dimensions match, so $M_1 \times M_2 \times \ldots \times M_n$ can be computed
  - $c_i = r_{i+1}$ for $1 \leq i < n$
- Find an optimal order to compute the product
  - That is, bracket the expression optimally

Inductive structure

- Product to be computed: M_1 x M_2 x …x M_n
- Final step would have combined two subproducts
  - (M_1 x M_2 x …x M_k) x (M_{k+1} x M_{k+2} x …x M_n), for some $1 \leq k < n$
  - First factor has dimension (r_1,c_k), second (r_{k+1},c_n)
  - Final multiplication step costs $O(r_1c_kc_n)$
  - Add cost of computing the two factors

Subproblems

* Final step is
  $(M_1 \times M_2 \times \ldots \times M_k) \times (M_{k+1} \times M_{k+2} \times \ldots \times M_n)$
* Subproblems are $(M_1 \times M_2 \times \ldots \times M_k)$ and $(M_{k+1} \times M_{k+2} \times \ldots \times M_n)$
* Total cost is $\mathrm{Cost}(M_1 \times M_2 \times \ldots \times M_k) + \mathrm{Cost}(M_{k+1} \times M_{k+2} \times \ldots \times M_n) + r_1 c_k c_n$
* Which k should we choose?
* No idea! Try them all and choose the minimum!

Inductive formulation

- $Cost(M_1 \times M_2 \times \ldots \times M_n) =$

  minimum value, for $1 \leq k < n$, of

  $Cost(M_1 \times M_2 \times \ldots \times M_k) +$

  $Cost(M_{k+1} \times M_{k+2} \times \ldots \times M_n) +$

  $r_1 c_k c_n$

- When we compute $Cost(M_1 \times M_2 \times \ldots \times M_k)$ we will get subproblems of the form $M_j \times M_{j+1} \times \ldots \times M_k$

In general …

- $\text{Cost}(M_i \times M_{i+1} \times \ldots \times M_j)$ =

  minimum value, for $i \leq k < j$, of

  $\text{Cost}(M_i \times M_{i+1} \times \ldots \times M_k)$ +

  $\text{Cost}(M_{k+1} \times M_{k+2} \times \ldots \times M_j)$ +

  $r_i c_k c_j$

- Write Cost(i,j) to denote $\text{Cost}(M_i \times M_{i+1} \times \ldots \times M_j)$

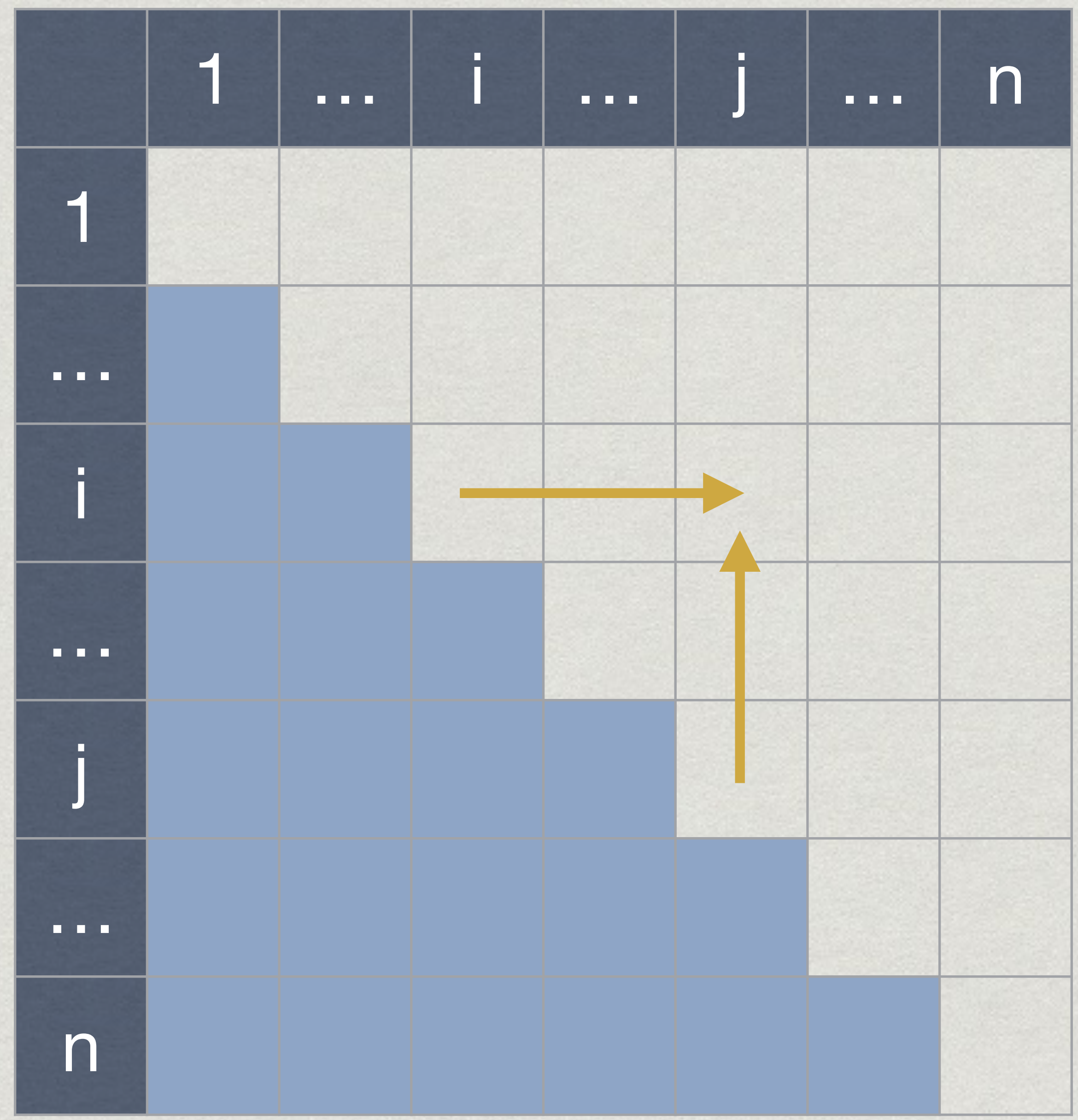
Final equation

- Cost(i,i) = 0 — No multiplication to be done
- Cost(i,j) = min over $i \leq k < j$

  [Cost(i,k) + Cost(k+1,j) + $r_i c_k c_j$]
- Note that we only require Cost(i,j) when $i \leq j$

Subproblem dependency

- Cost(i,j) depends on Cost(i,k), Cost(k+1,j) for all $i \leq k < j$
- Can have O(n) dependent values, unlike LCS, LCW, ED
- Start with main diagonal and fill matrix by columns, bottom to top, left to right



MMCost(M1,...,Mn), DP

```
function MMC(R,C)
# R[1..n],C[1..n] have row/column sizes

for r = 1,…,n
  MMC[r][r] = 0

for c = 2,3,…,n+1
  for r = c,c-1,…,1
    MMC[r][c] = infinity
    for k = r,r+1,…,c-1
      subprob = MMC[r][k] + MMC[k][c] +
                R[r]C[k]C[c]
      if (subprob < MMC[r][c])
        MMC[r][c] = subprob
```

Complexity

- As with LCS, ED, we to fill an $O(n^2)$ size table
- However, filling MMC[i][j] could require examining $O(n)$ intermediate values
- Hence, overall complexity is $O(n^3)$