NPTEL MOOC,JAN-FEB 2015
Week 7, Module 3

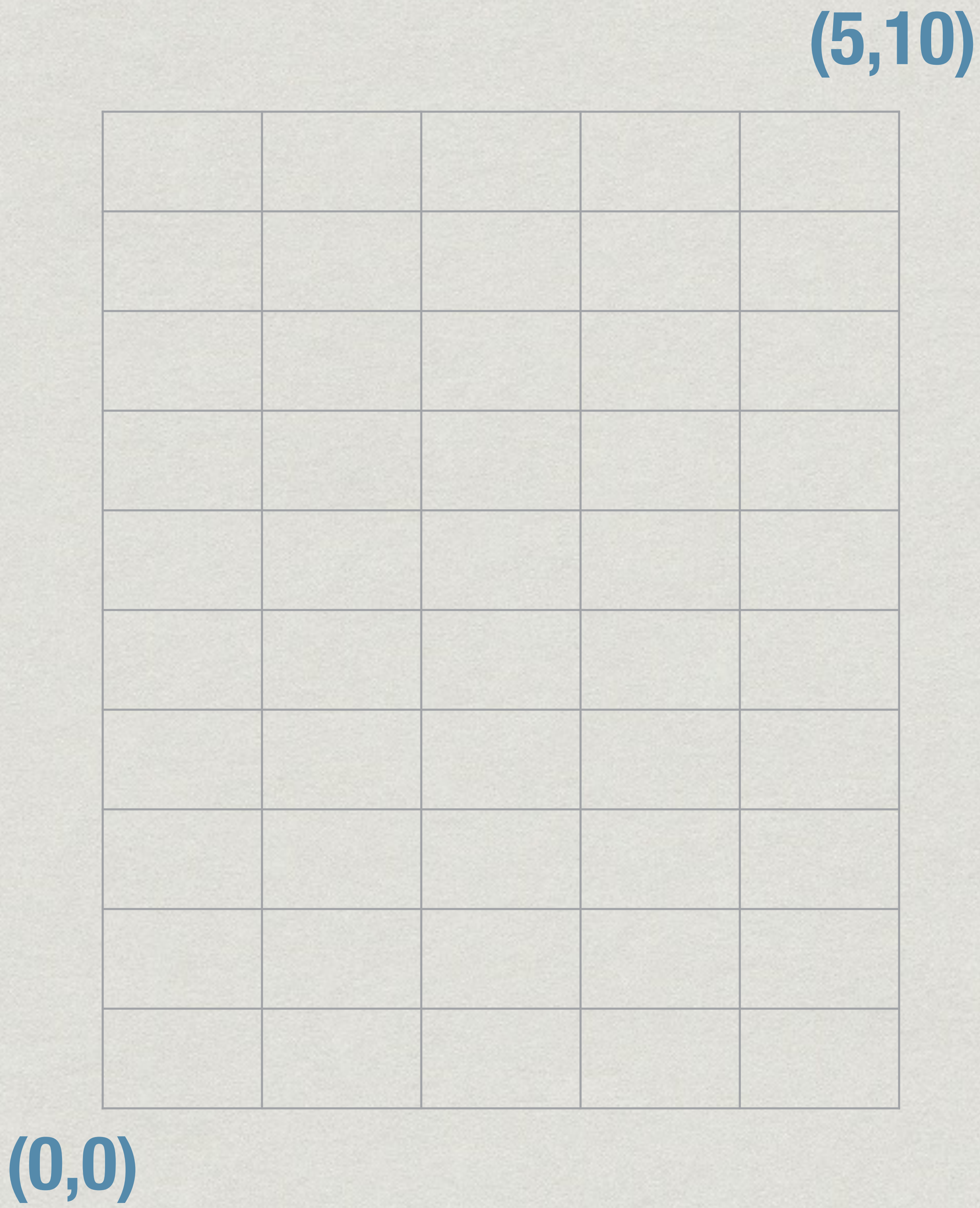
DESIGN AND ANALYSIS OF ALGORITHMS

Grid Paths

MADHAVAN MUKUND, CHENNAI MATHEMATICAL INSTITUTE
http://www.cmi.ac.in/~madhavan

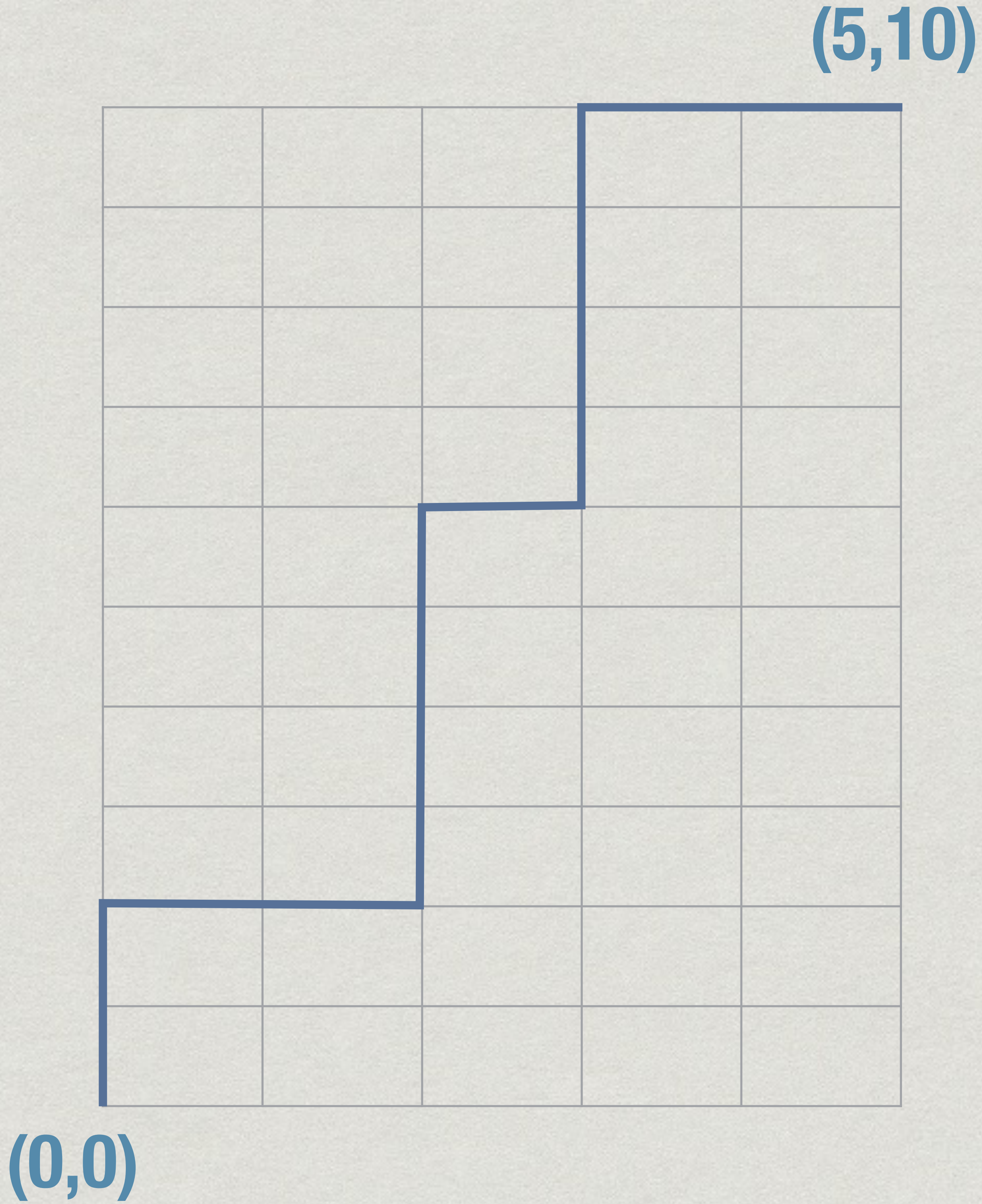
Grid Paths

- Roads arranged in a rectangular grid
- Can only go up or right
- How many different routes from (0,0) to (m,n)?



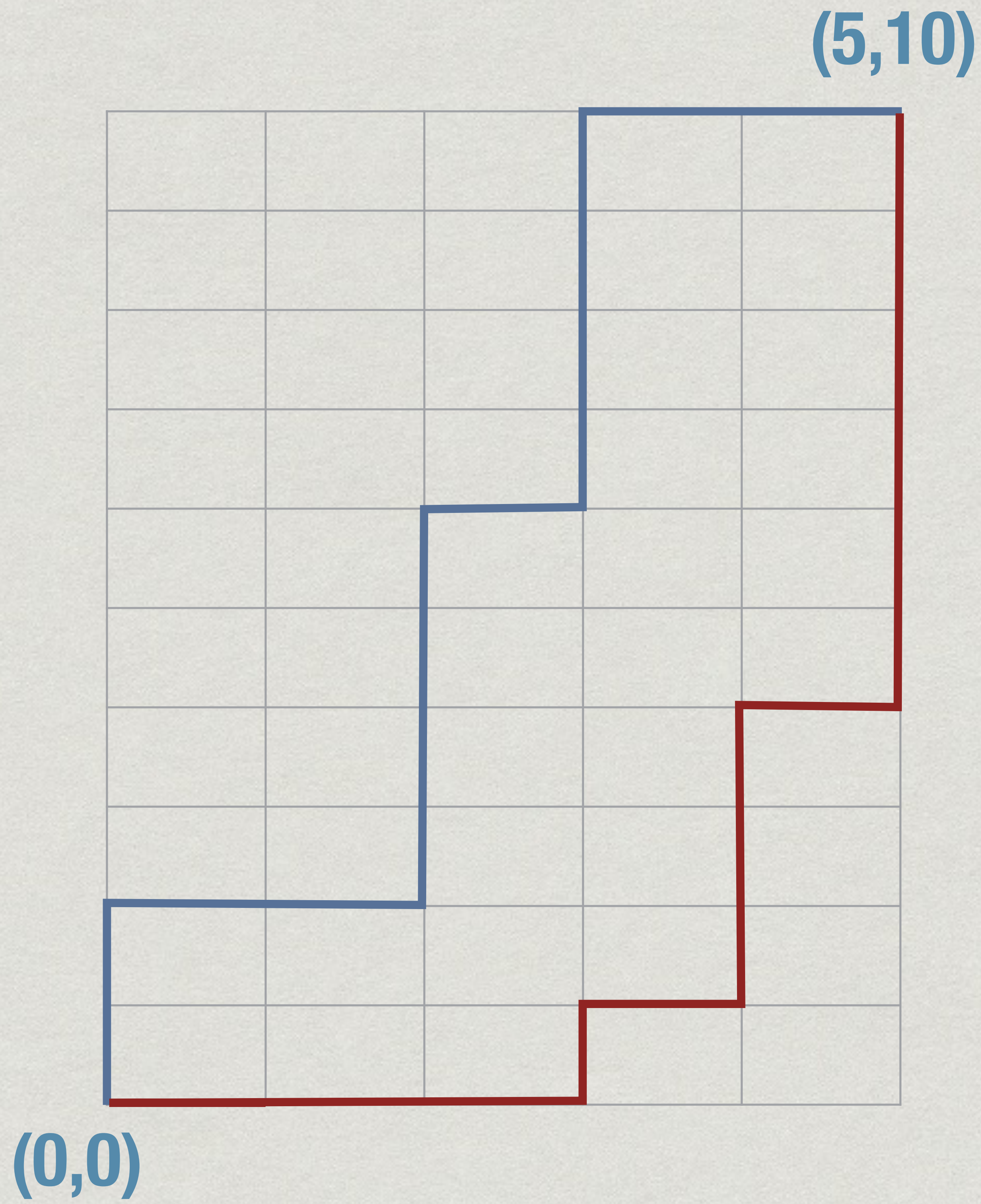
Grid Paths

* Roads arranged in a rectangular grid
* Can only go up or right
* How many different routes from (0,0) to (m,n)?



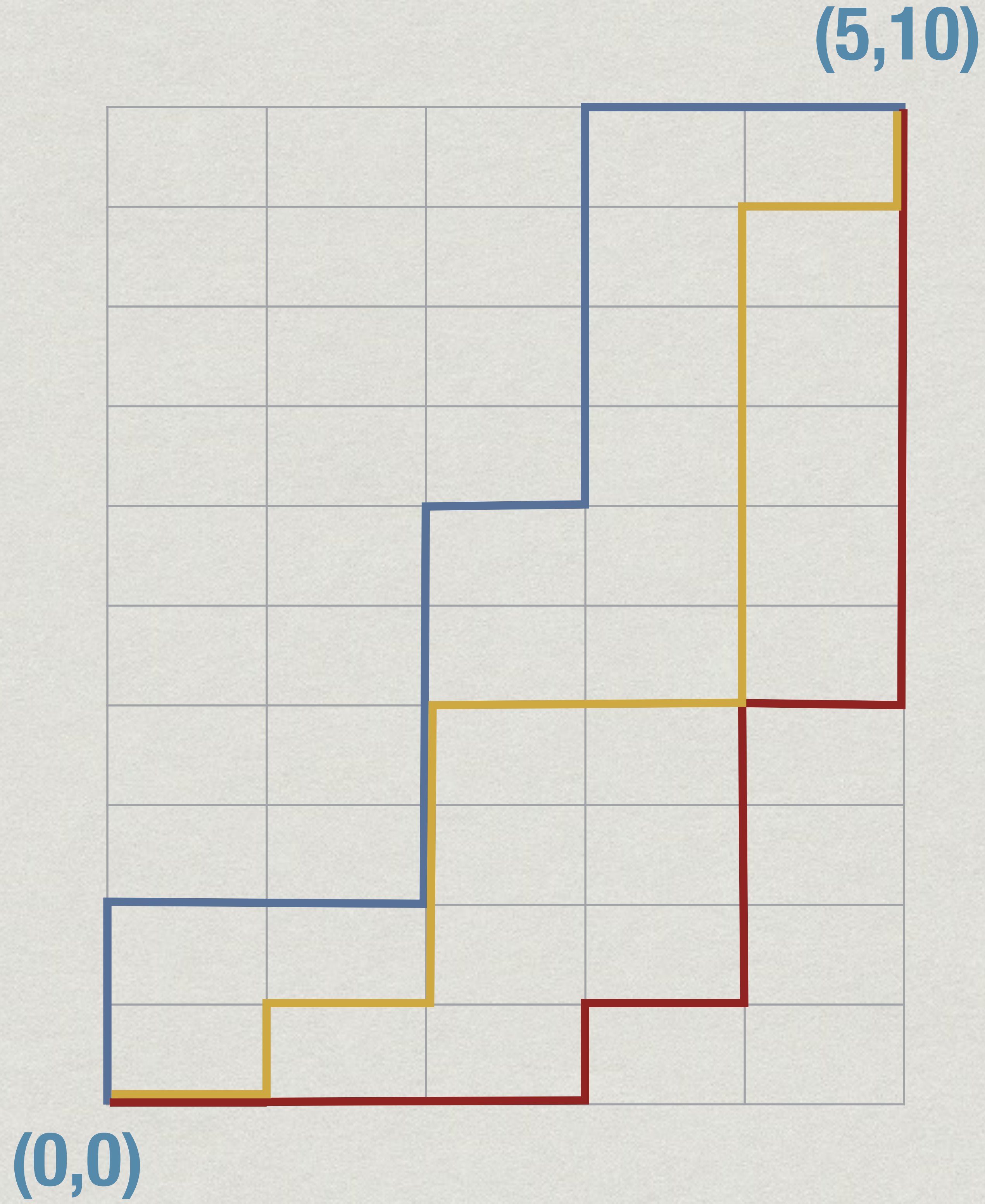
Grid Paths

- Roads arranged in a rectangular grid
- Can only go up or right
- How many different routes from (0,0) to (m,n)?



Grid Paths

- Roads arranged in a rectangular grid
- Can only go up or right
- How many different routes from (0,0) to (m,n)?

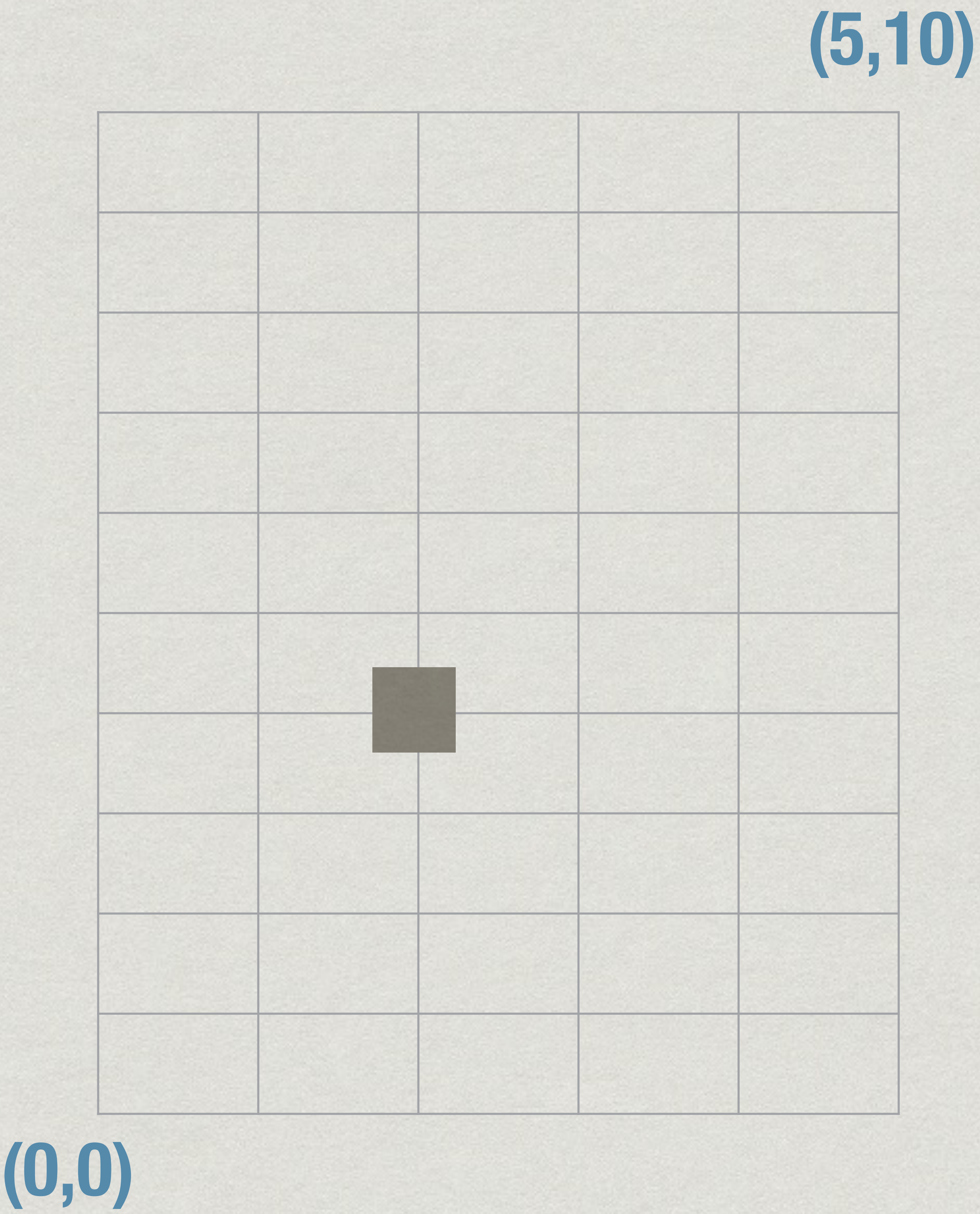


Combinatorial solution

- Every path from (0,0) to (5,10) has 15 segments
  - In general m+n segments from (0,0) to (m,n)
- Of these exactly 5 are right moves, 10 are up moves
- Fix the positions of the 5 right moves among the overall 15 positions
  - 15 choose 5 = (15!)/(10!)(5!) = 3003
  - Same as 15 choose 10: fix the 10 up moves

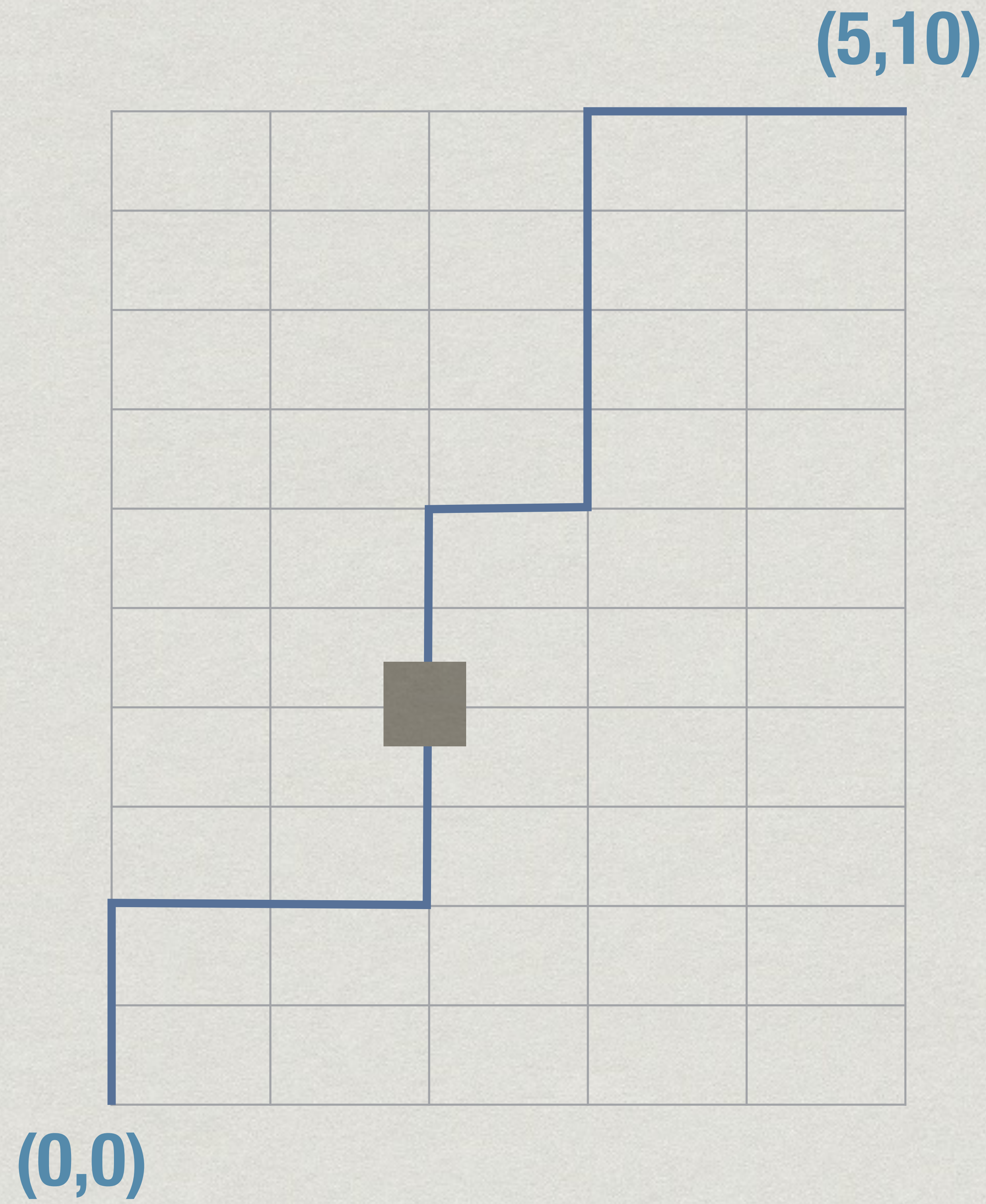
Holes

- What if an intersection is blocked?
  - (2,4), for example
- Paths through (2,4) need to be discarded
  - Two of our earlier examples are invalid paths



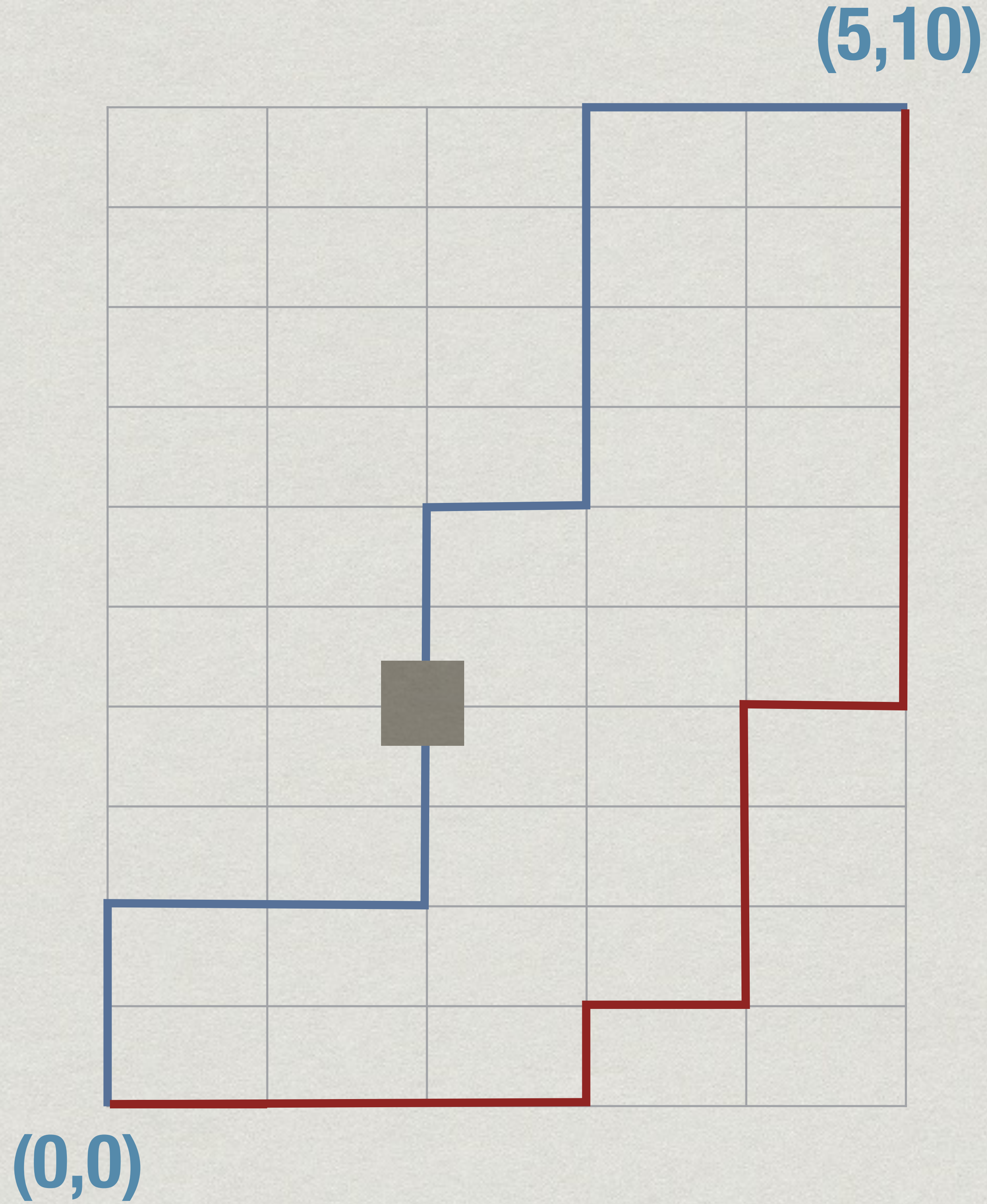
Holes

- What if an intersection is blocked?
  - (2,4), for example
- Paths through (2,4) need to be discarded
  - Two of our earlier examples are invalid paths



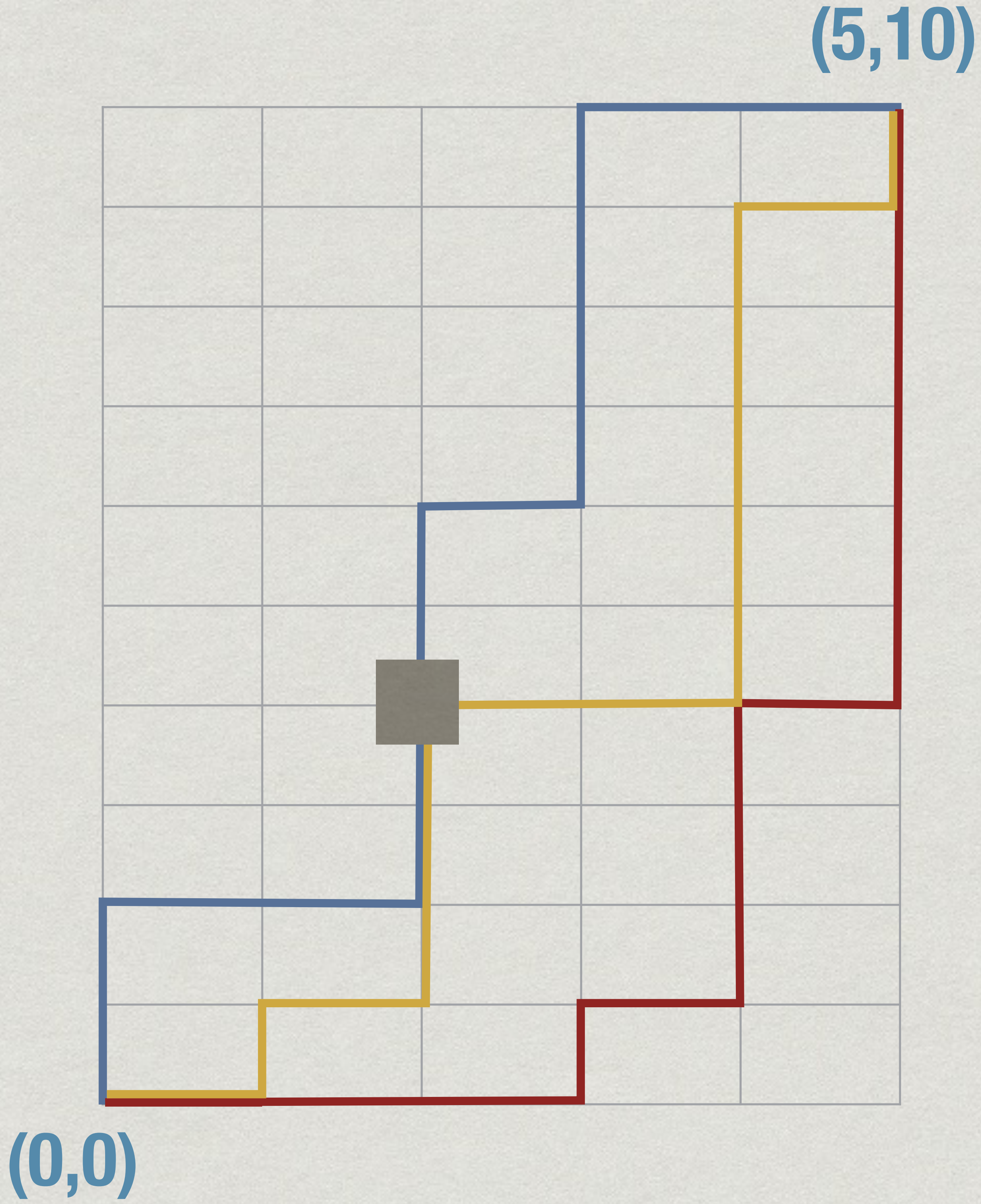
Holes

- What if an intersection is blocked?
  - (2,4), for example
- Paths through (2,4) need to be discarded
  - Two of our earlier examples are invalid paths



Holes

- What if an intersection is blocked?
  - (2,4), for example
- Paths through (2,4) need to be discarded
  - Two of our earlier examples are invalid paths

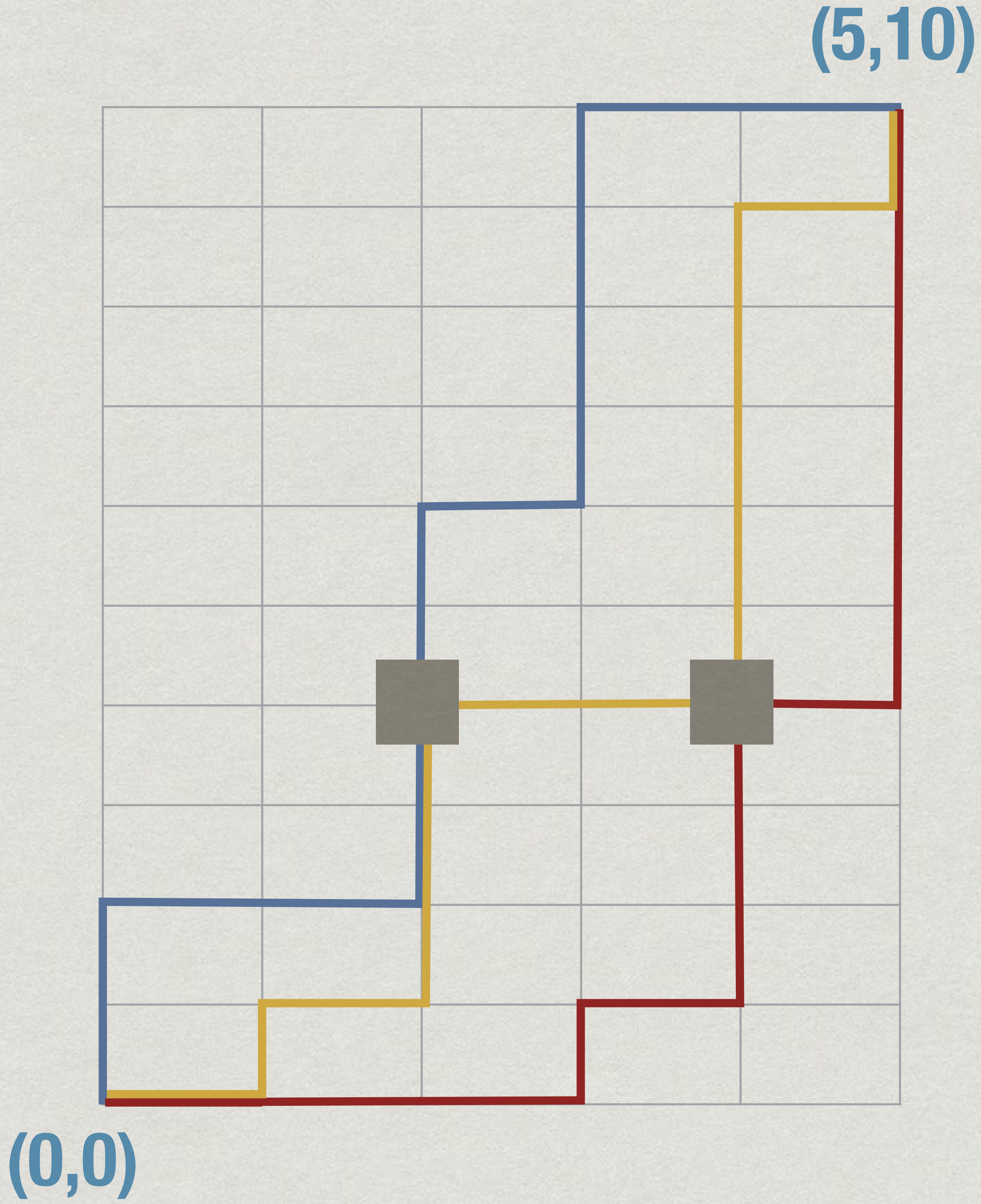


Combinatorial solution

- Every path through (2,4) goes from (0,0) to (2,4) and then from (2,4) to (5,10)
  - Count these separately:
    - (4+2) choose 2 = 15
    - (6+3) choose 3 = 84
  - Multiply to get all paths through (2,4): 1260
  - Subtract from 15 choose 5 = 3003 to get valid paths that avoid (2,4): 1743

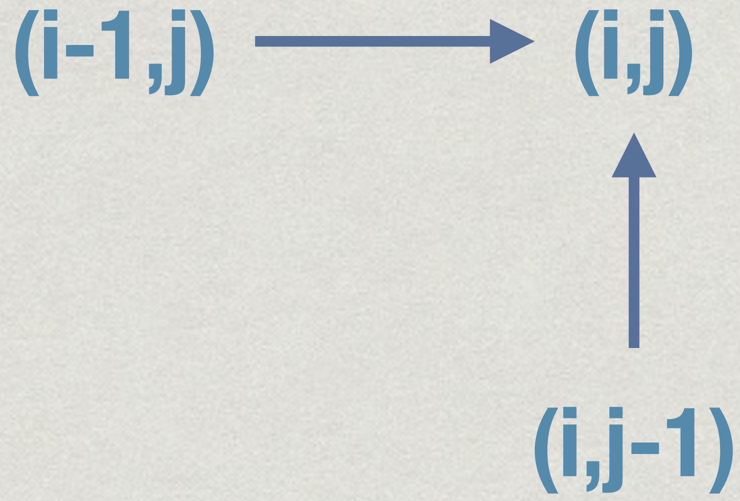
Holes

- What if two intersections are blocked?
- Subtract paths through (2,4), (4,4)
  - Some paths are counted twice!
- Add back paths through both holes
- Inclusion-exclusion: messy



Inductive formulation

- How can a path reach (i,j)
  - Move up from (i,j-1)
  - Move right from (i-1,j)
- Every path to these neighbours extends in a unique way to (i,j)



Inductive formulation

- Paths(i,j) : Number of paths from (0,0) to (i,j)
- Paths(i,j) = Paths(i-1,j) + Paths(i,j-1)
- Boundary cases
  - Paths(i,0) = Paths(i-1,0) # Bottom row
  - Paths(0,j) = Paths(0,j-1) # Left column
  - Paths(0,0) = 1 # Base case

Dealing with holes

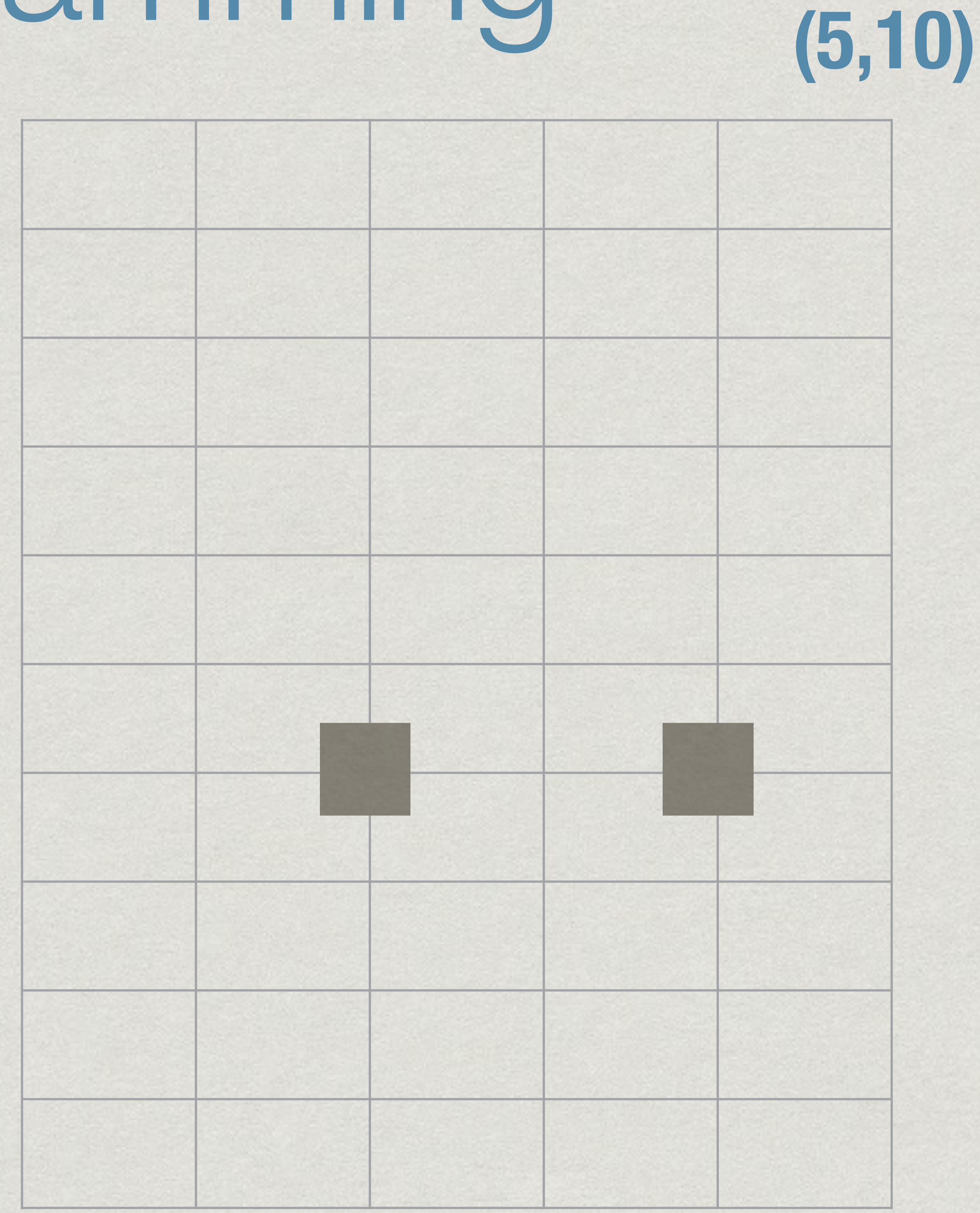
- Paths(i,j) = 0, if there is a hole at (i,j)
- Paths(i,j) = Paths(i-1,j) + Paths(i,j-1), otherwise
- Boundary cases
  - Paths(i,0) = Paths(i-1,0) # Bottom row
  - Paths(0,j) = Paths(0,j-1) # Left column
  - Paths(0,0) = 1 # Base case

Computing Paths(i,j)

- Naive recursion will recompute multiple times
  - Paths(5,10) requires Paths(4,10) and Paths(5,9)
  - Both Paths(4,10) and Paths(5,9) require Paths(4,9)
- Use memoization …
- … or compute the subproblems directly in a suitable way

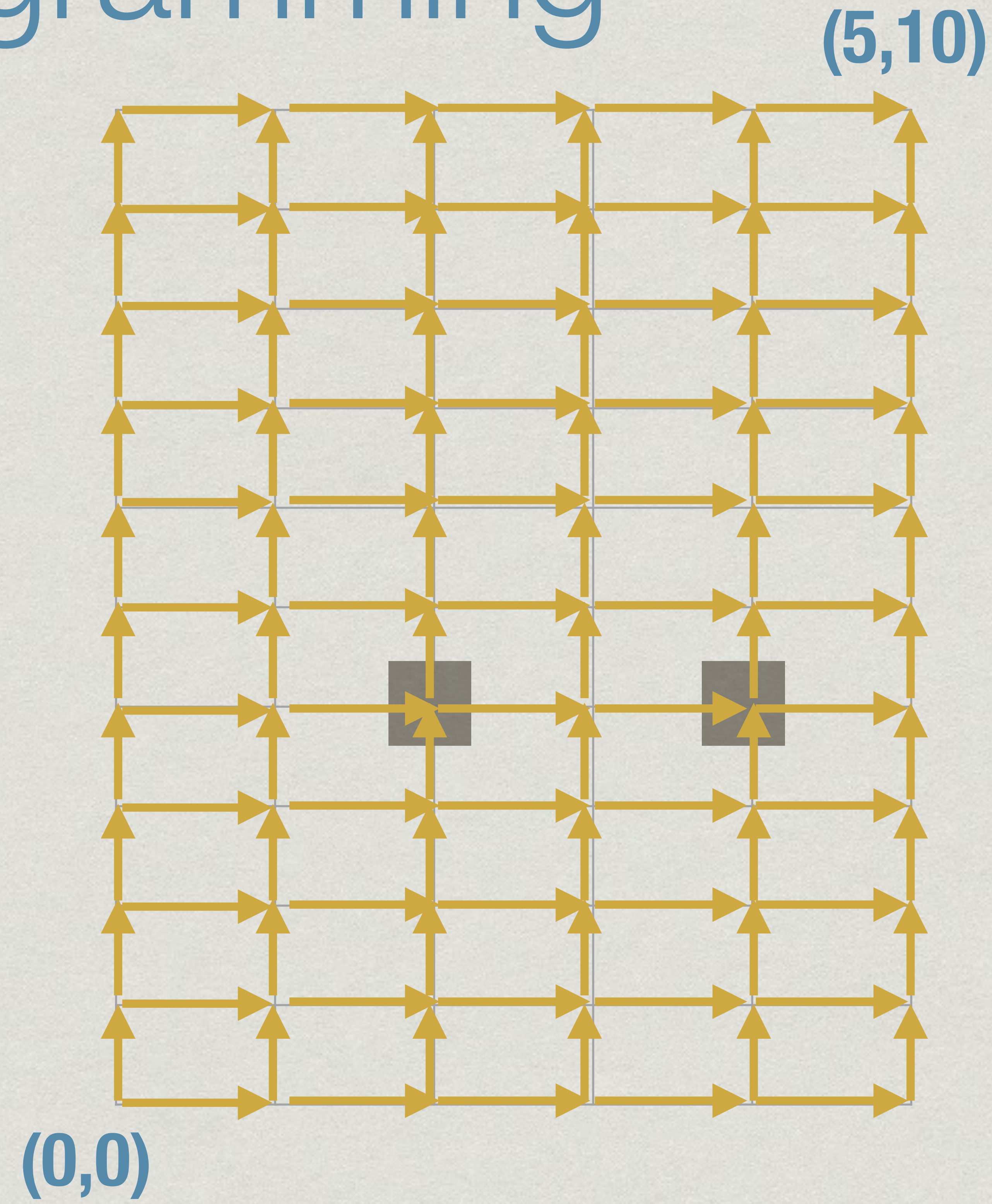
Dynamic programming

- Identify DAG structure
- Paths(0,0) has no dependencies
- Start at (0,0)



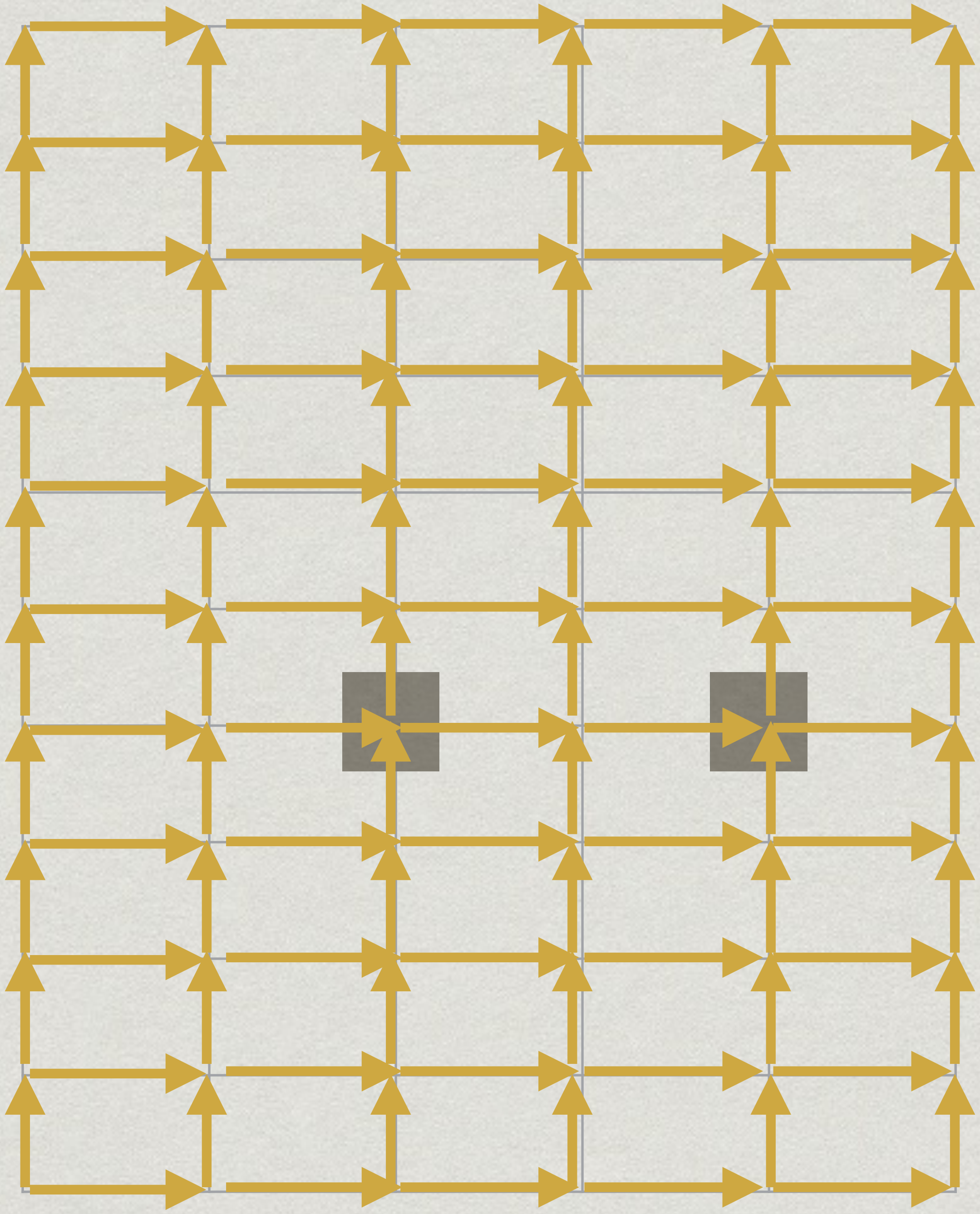
Dynamic programming

- Identify DAG structure
- Paths(0,0) has no dependencies
- Start at (0,0)



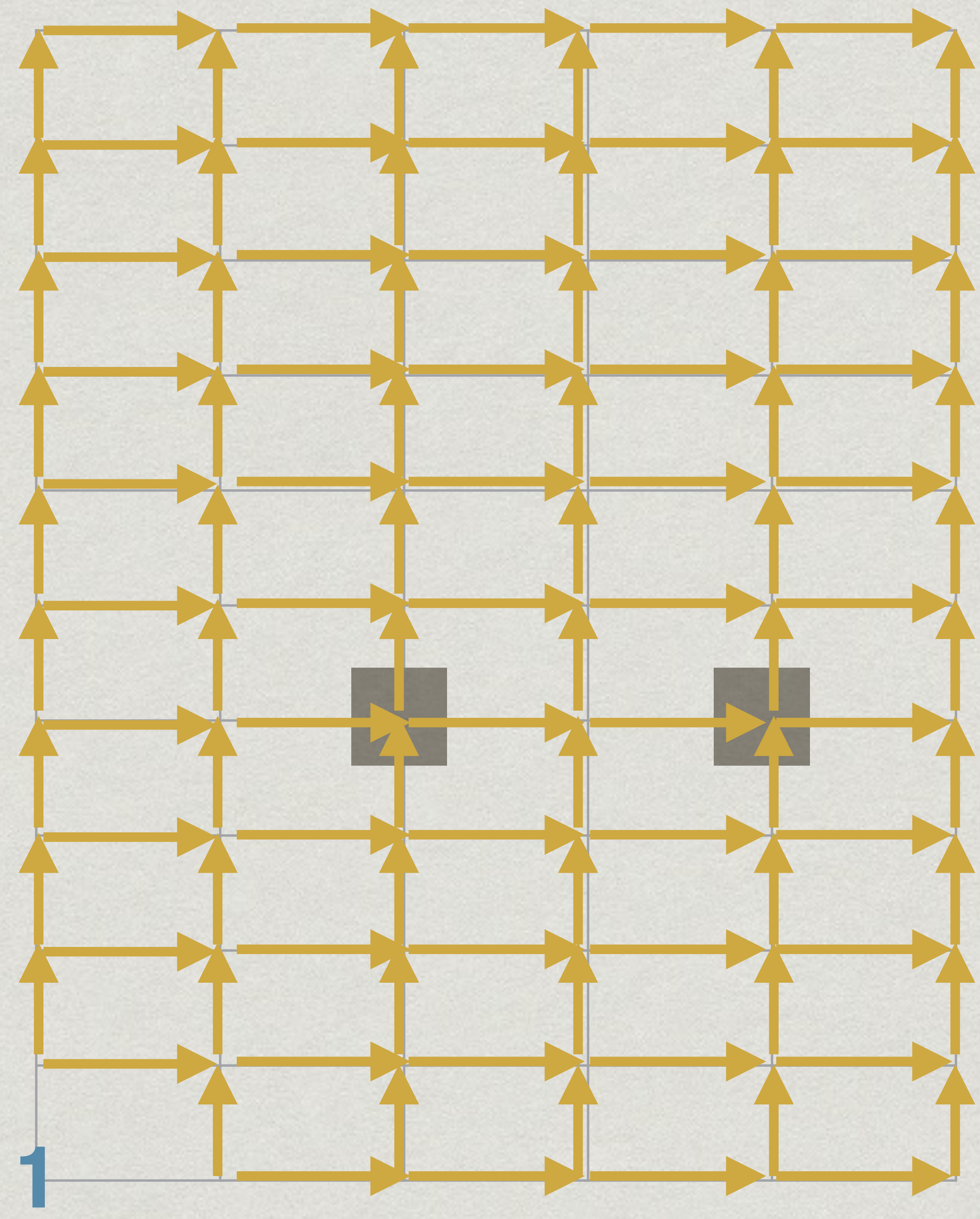
Dynamic programming

- Start at (0,0)
- Fill row by row



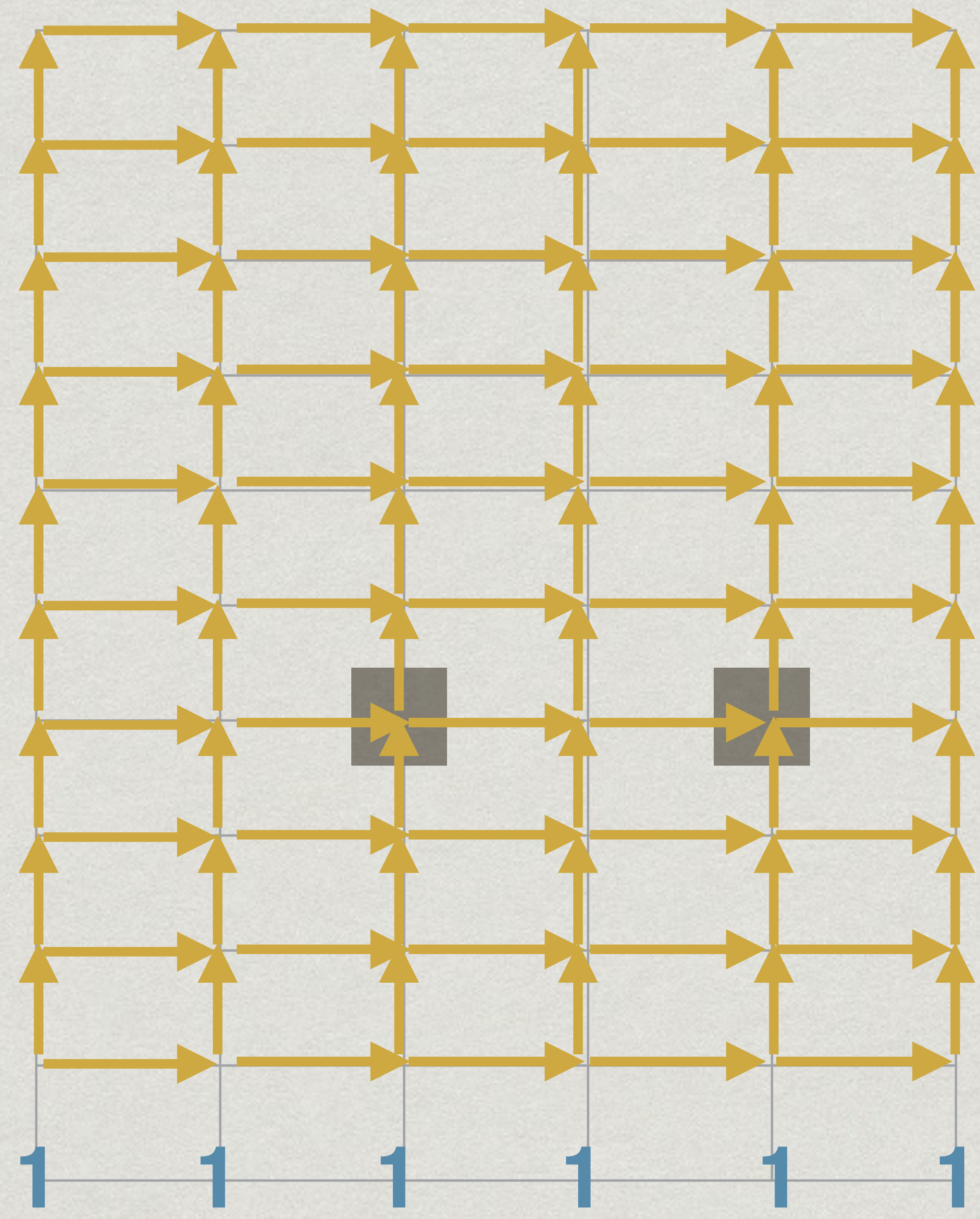
Dynamic programming

- Start at (0,0)
- Fill row by row



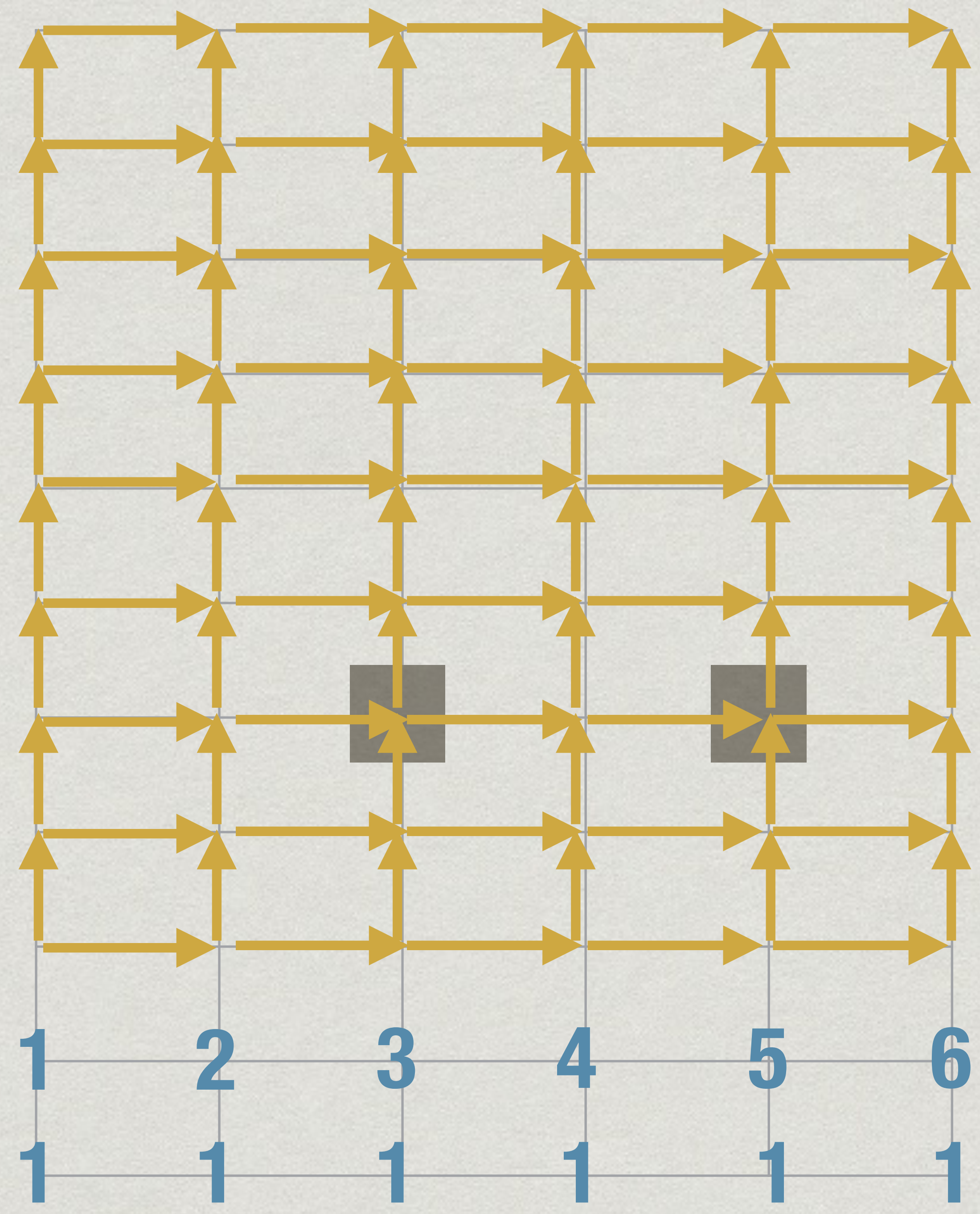
Dynamic programming

- Start at (0,0)
- Fill row by row



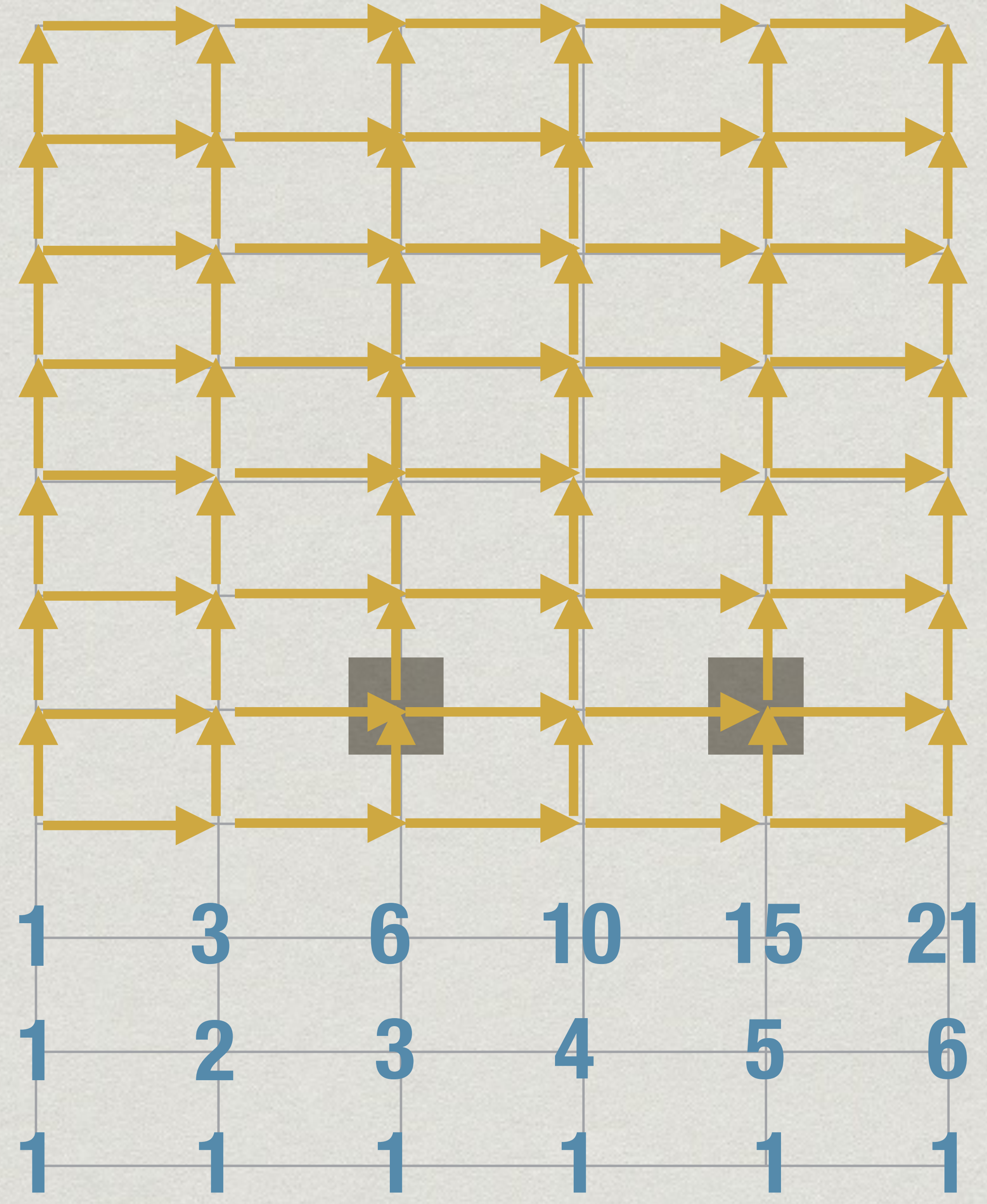
Dynamic programming

- Start at (0,0)
- Fill row by row



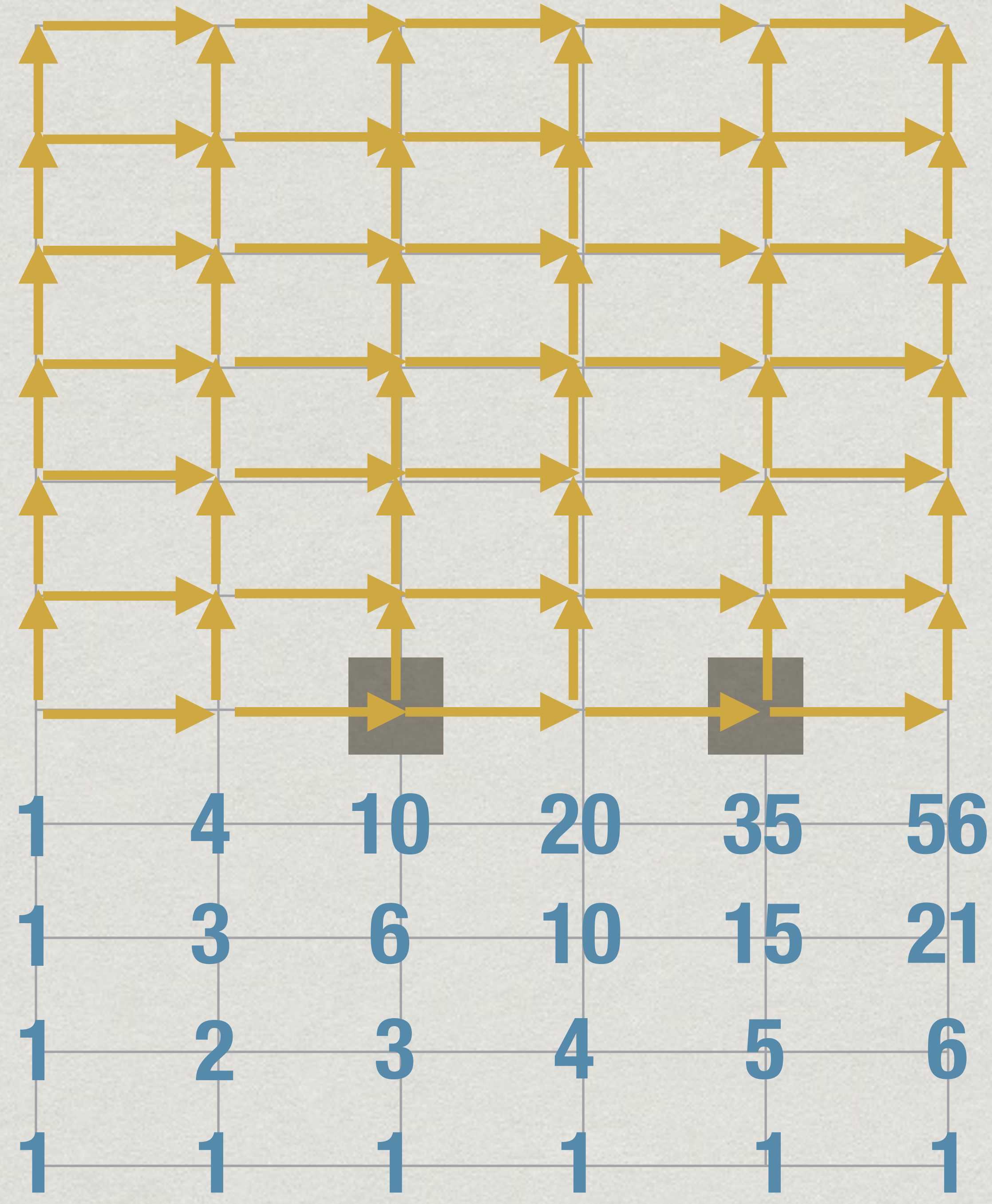
Dynamic programming

- Start at (0,0)
- Fill row by row



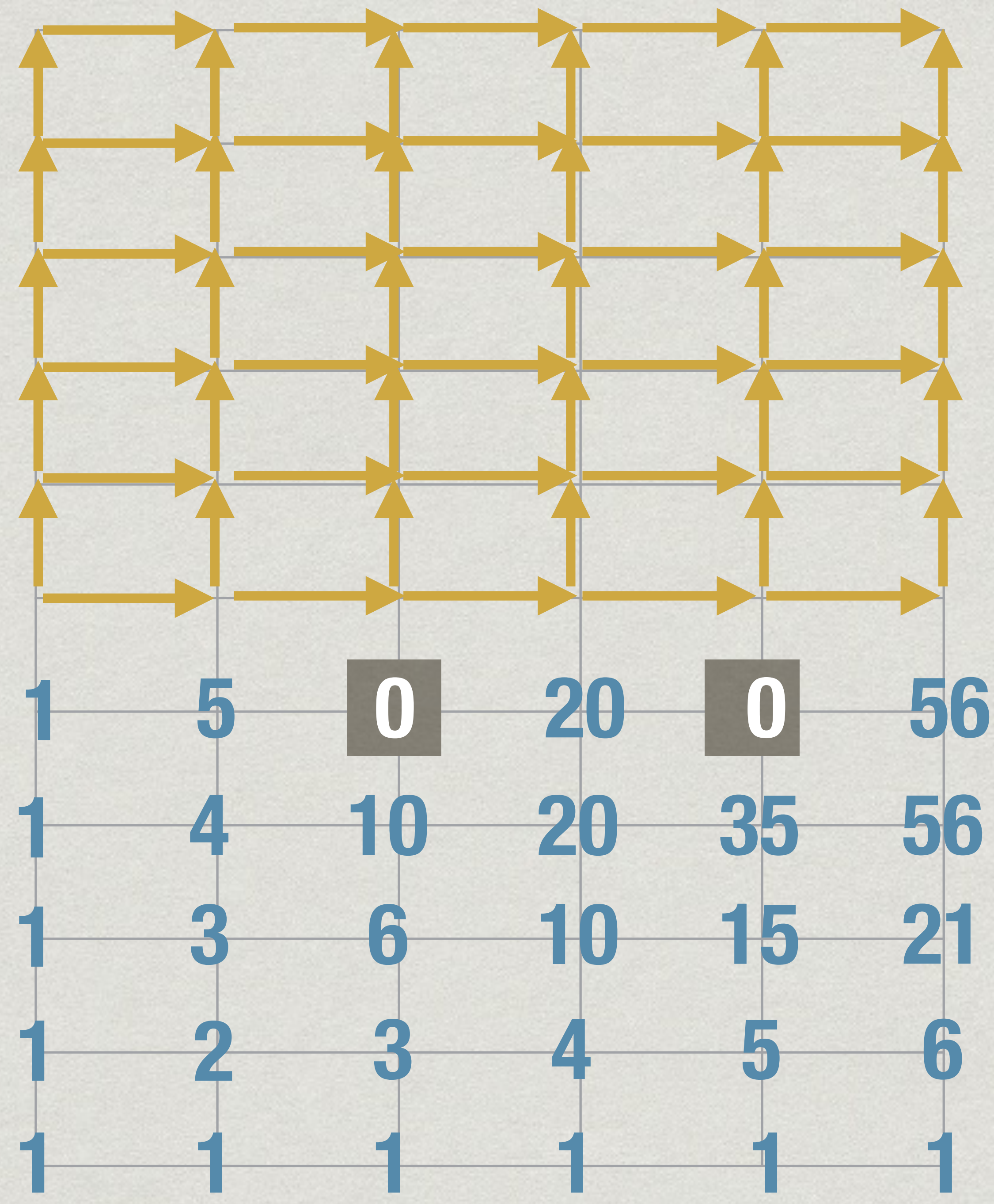
Dynamic programming

- Start at (0,0)
- Fill row by row



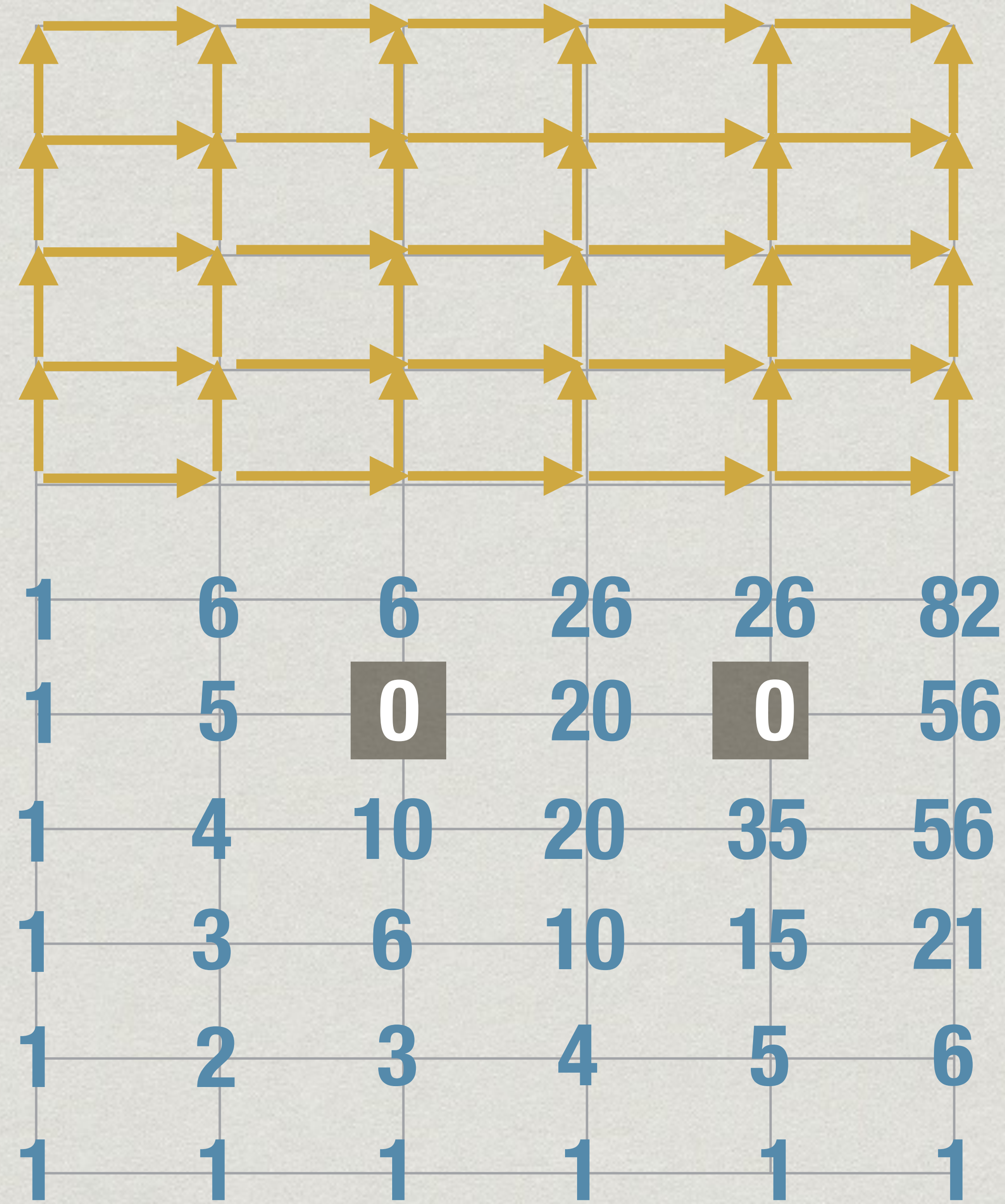
Dynamic programming

* Start at (0,0)
* Fill row by row



Dynamic programming

- Start at (0,0)
- Fill row by row

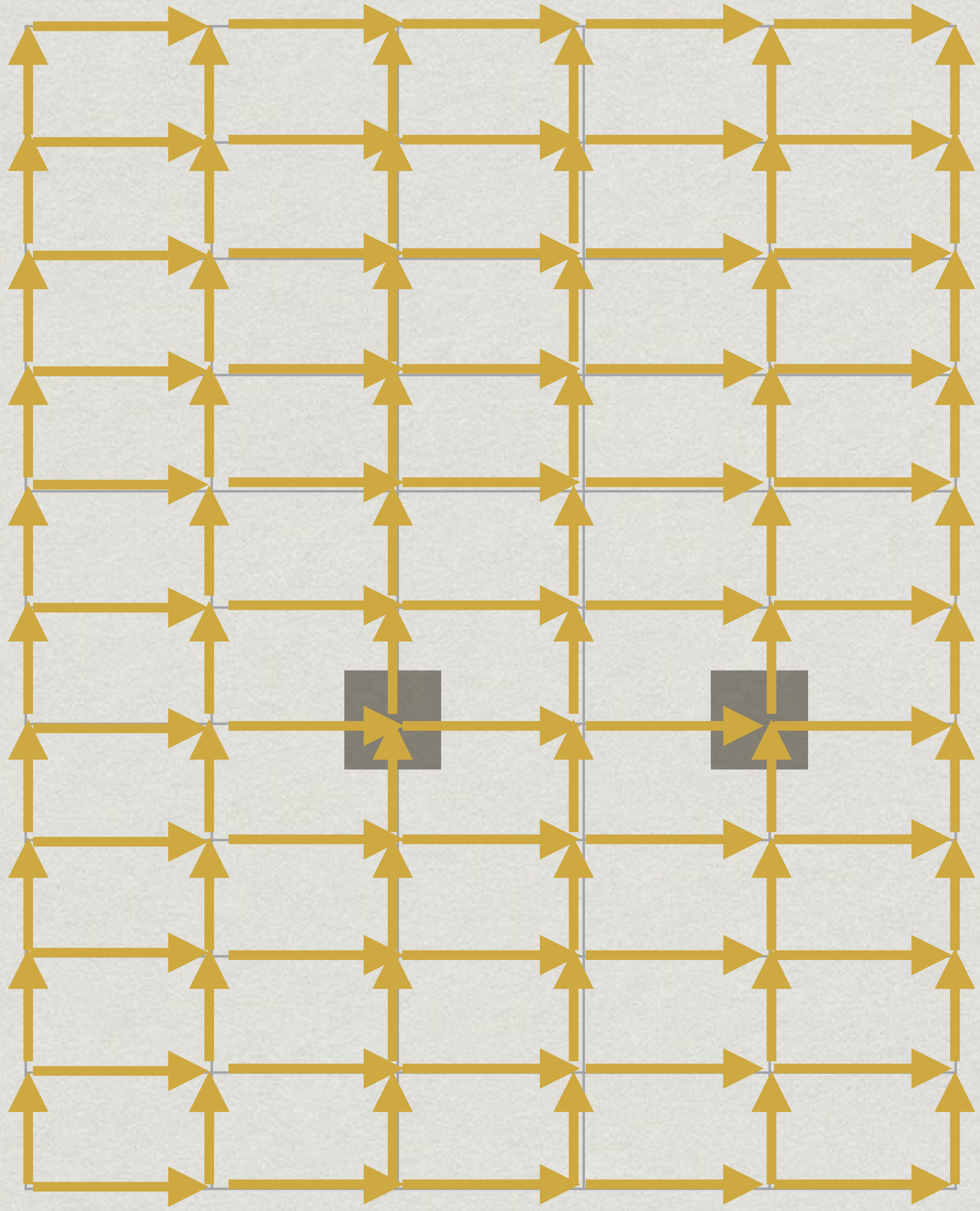


Dynamic programming

- Start at (0,0)
- Fill row by row

1	11	51	181	526	1363
1	10	40	130	345	837
1	9	30	90	215	492
1	8	21	60	125	272
1	7	13	39	65	147
1	6	6	26	26	82
1	5	0	20	0	56
1	4	10	20	35	56
1	3	6	10	15	21
1	2	3	4	5	6
1	1	1	1	1	1

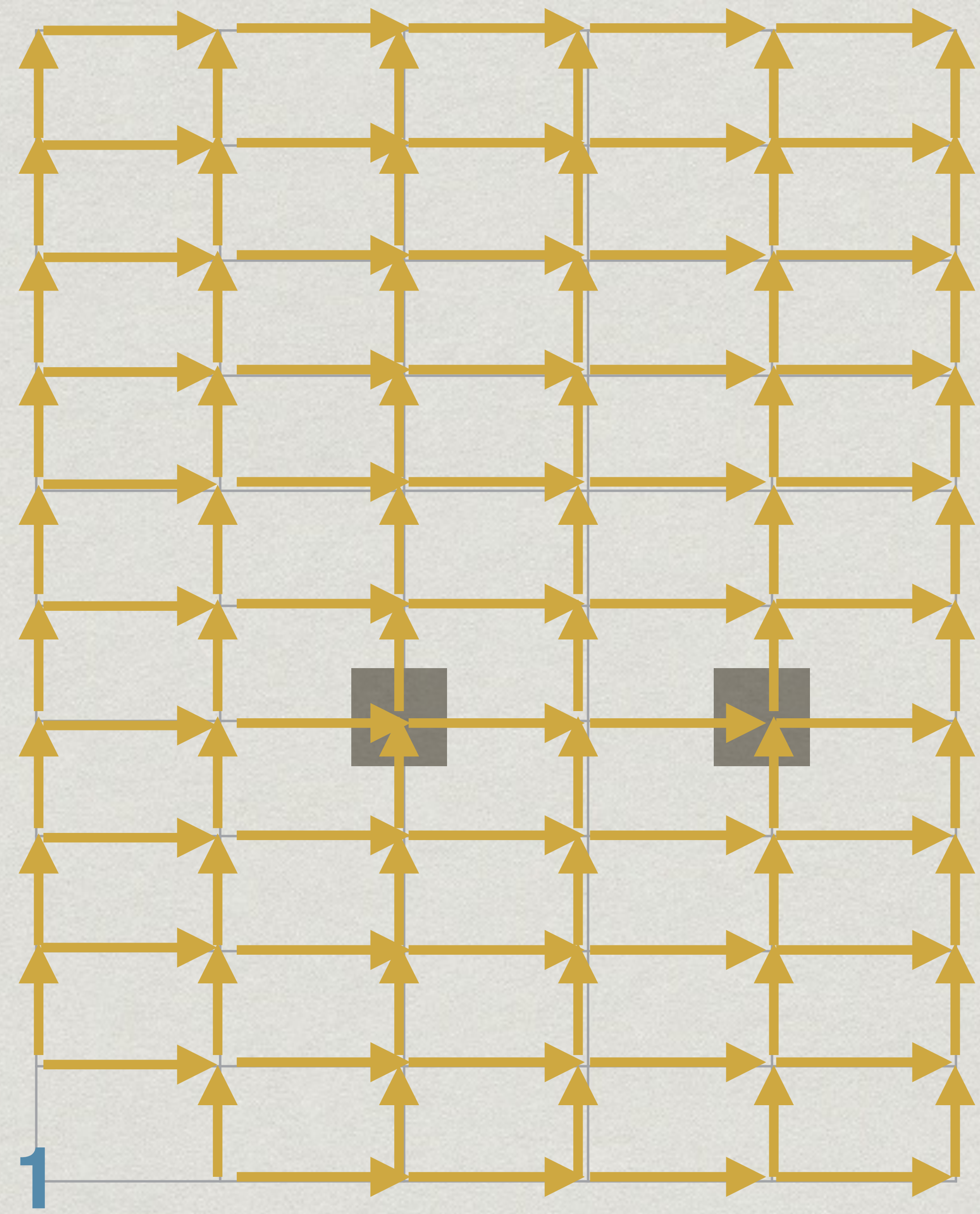
Dynamic programming



- Start at (0,0)
- Fill by column

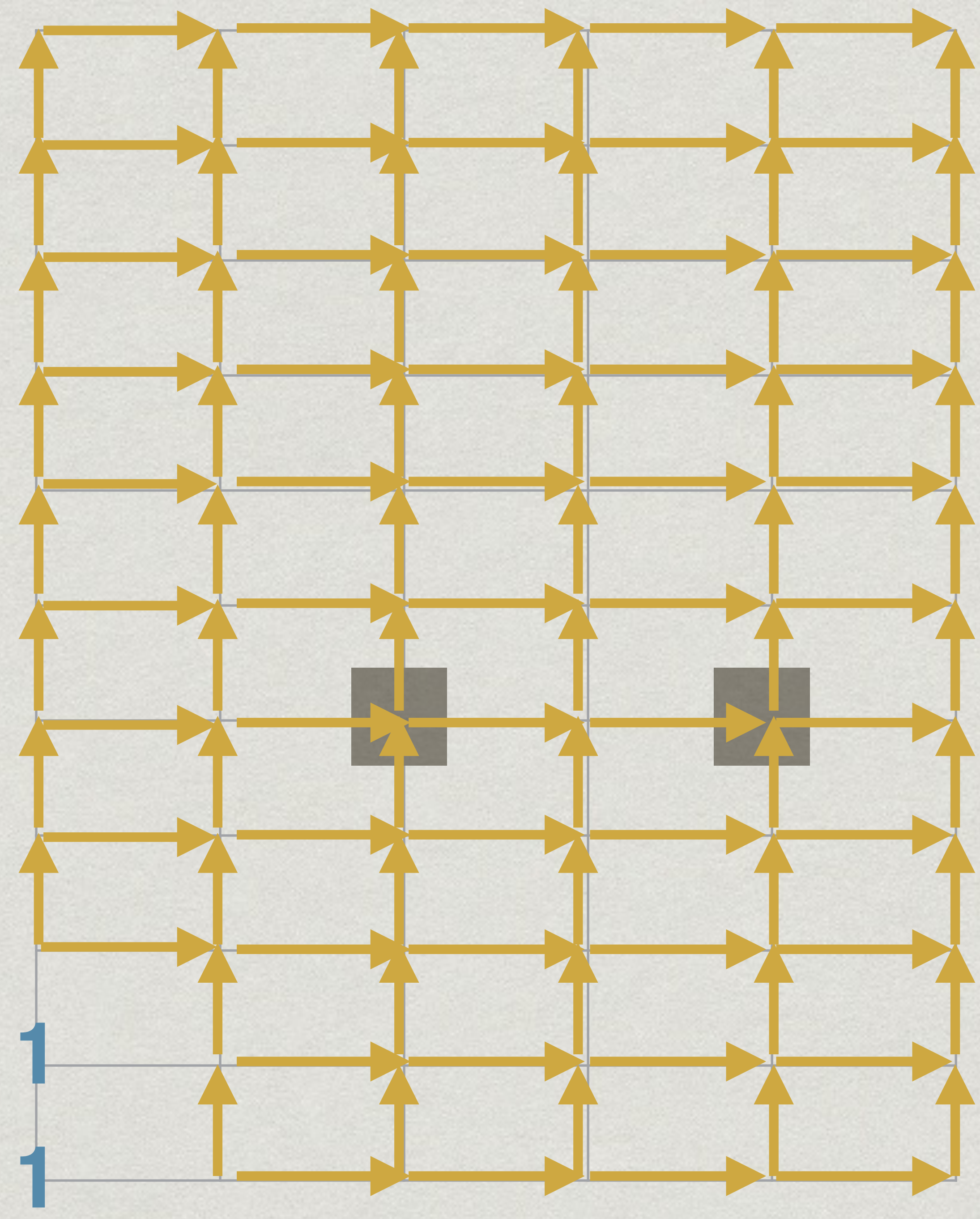
Dynamic programming

- Start at (0,0)
- Fill by column



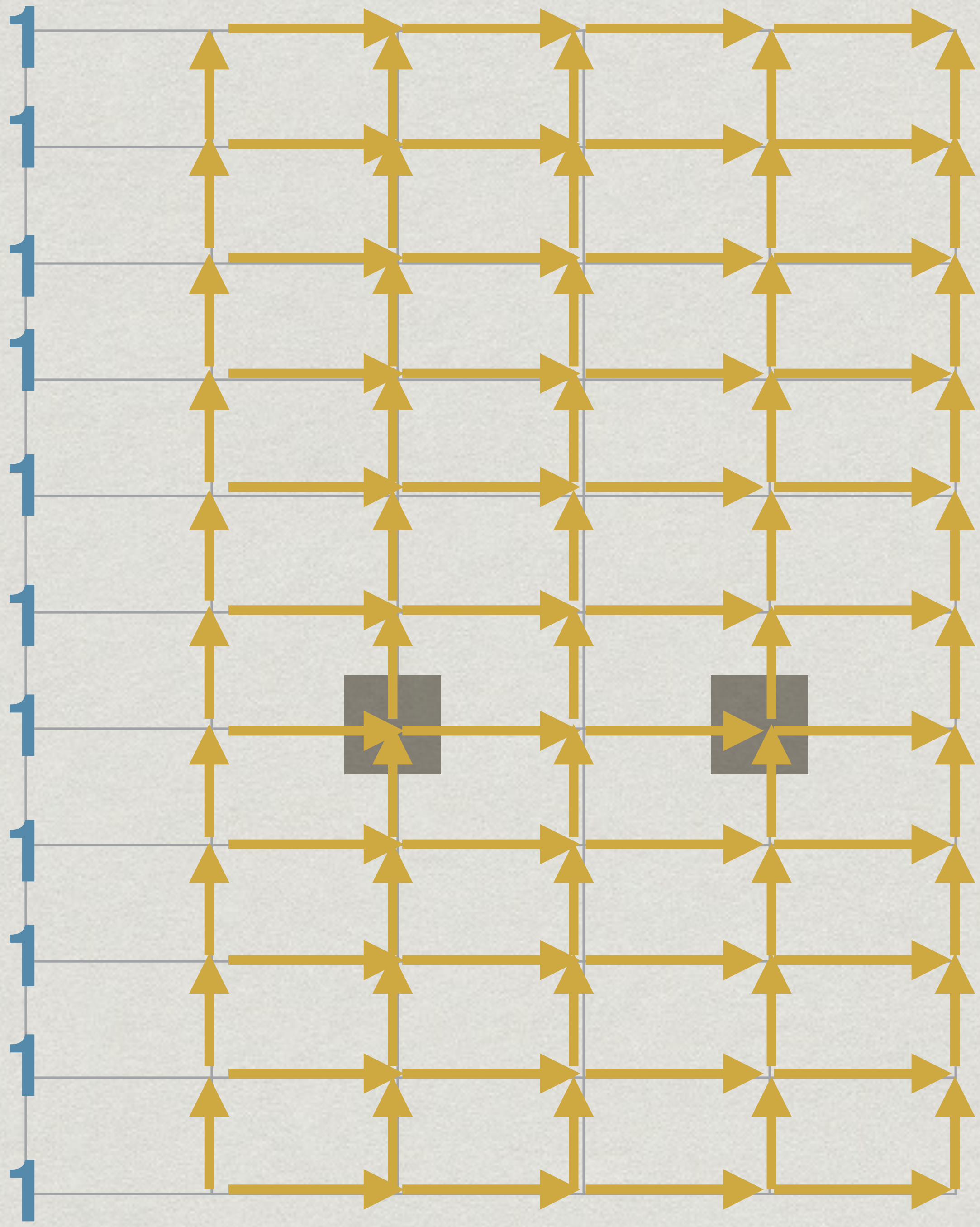
Dynamic programming

- Start at (0,0)
- Fill by column



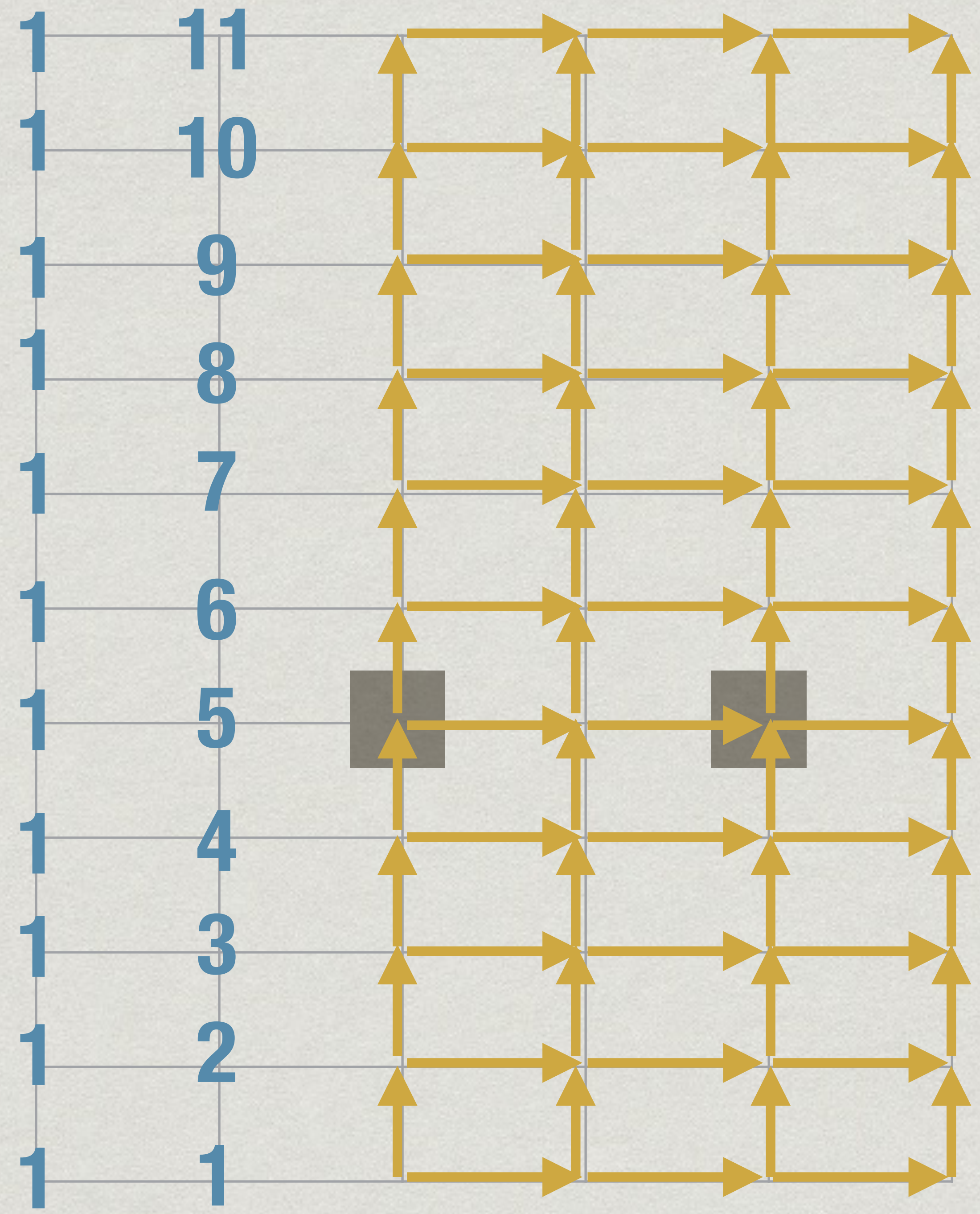
Dynamic programming

- Start at (0,0)
- Fill by column



Dynamic programming

- Start at (0,0)
- Fill by column



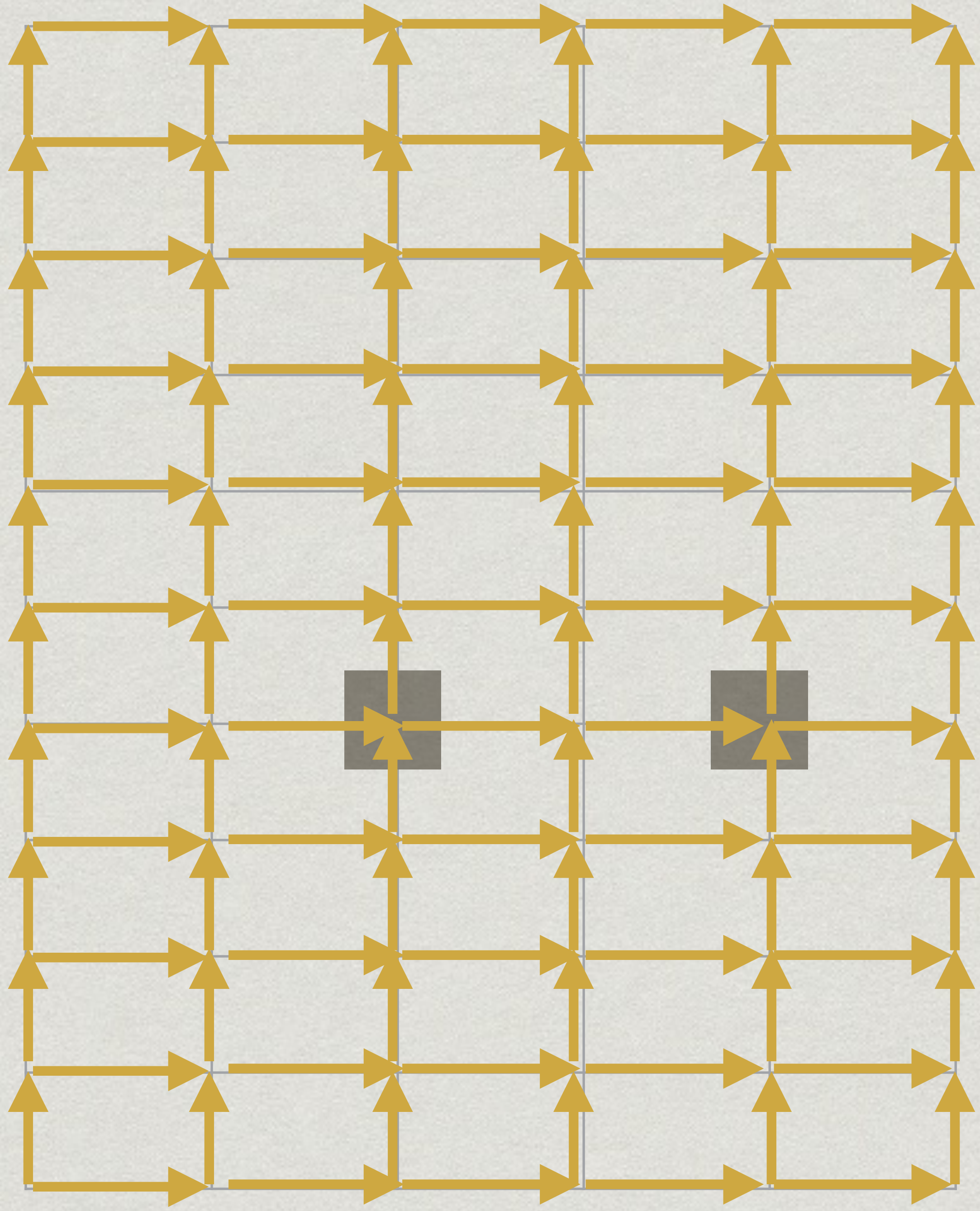
Dynamic programming

* Start at (0,0)
* Fill by column

1	11	51	181	526	1363
1	10	40	130	345	837
1	9	30	90	215	492
1	8	21	60	125	272
1	7	13	39	65	147
1	6	6	26	26	82
1	5	0	20	0	56
1	4	10	20	35	56
1	3	6	10	15	21
1	2	3	4	5	6
1	1	1	1	1	1

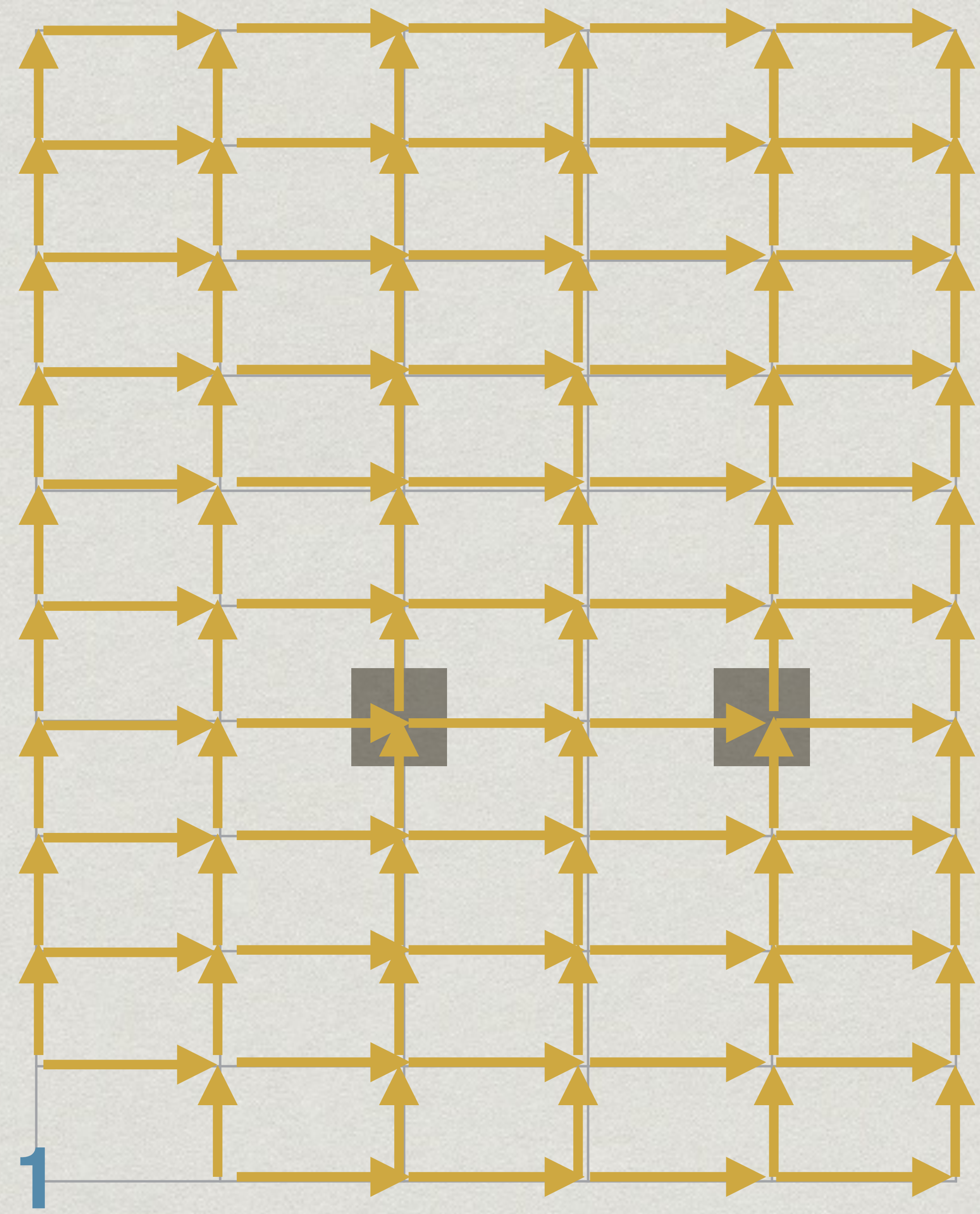
Dynamic programming

* Start at (0,0)
* Fill by diagonal



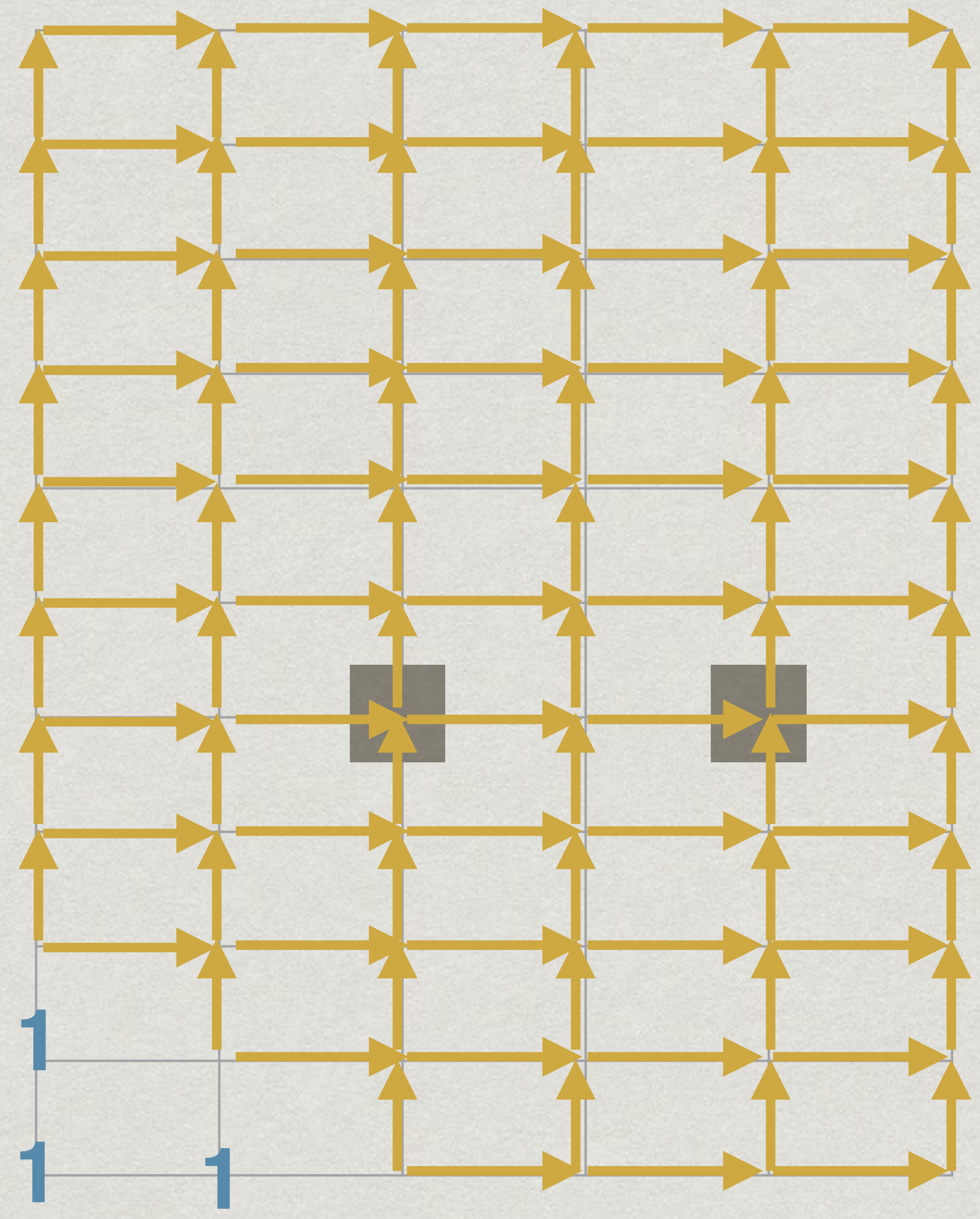
Dynamic programming

- Start at (0,0)
- Fill by diagonal



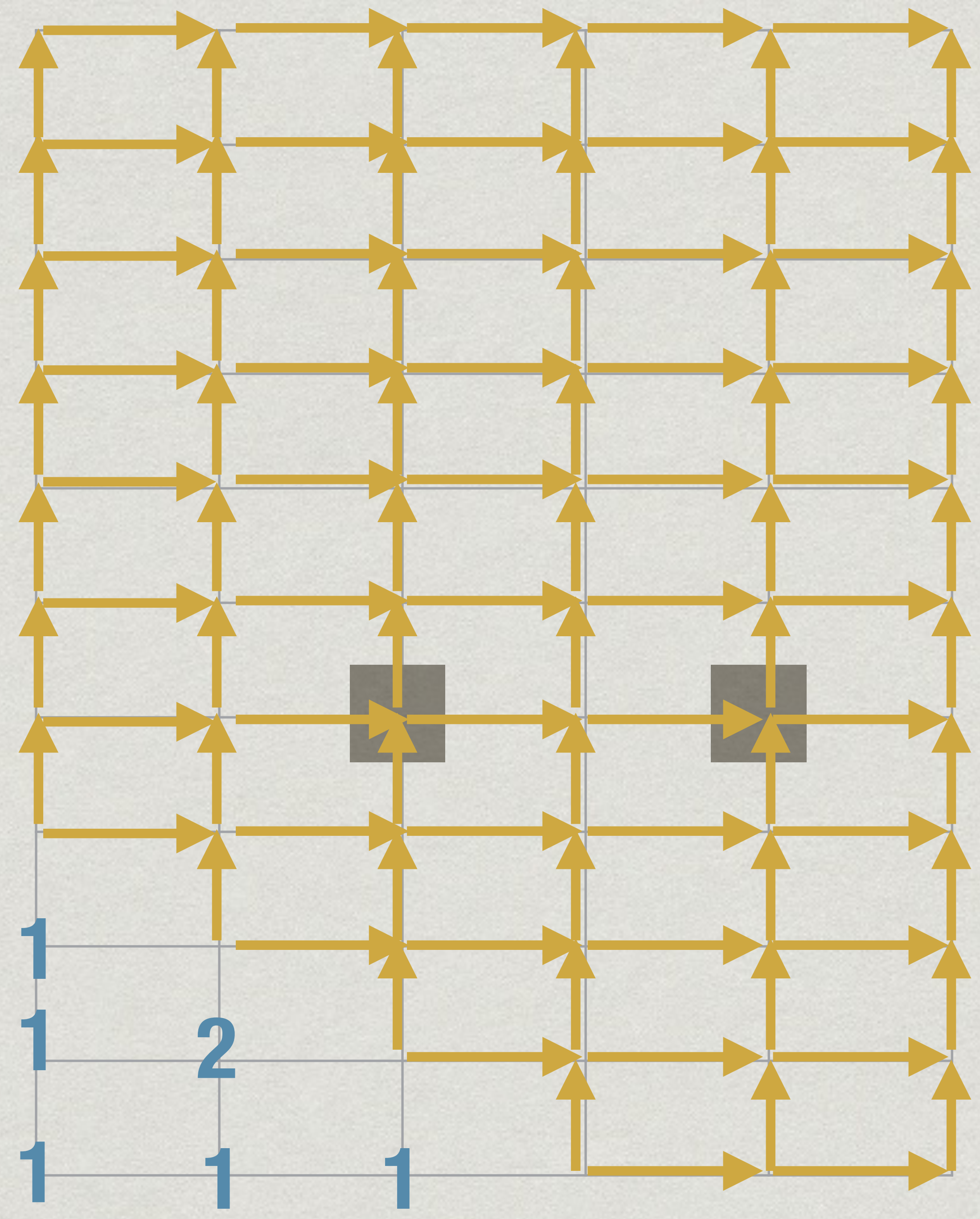
Dynamic programming

- Start at (0,0)
- Fill by diagonal



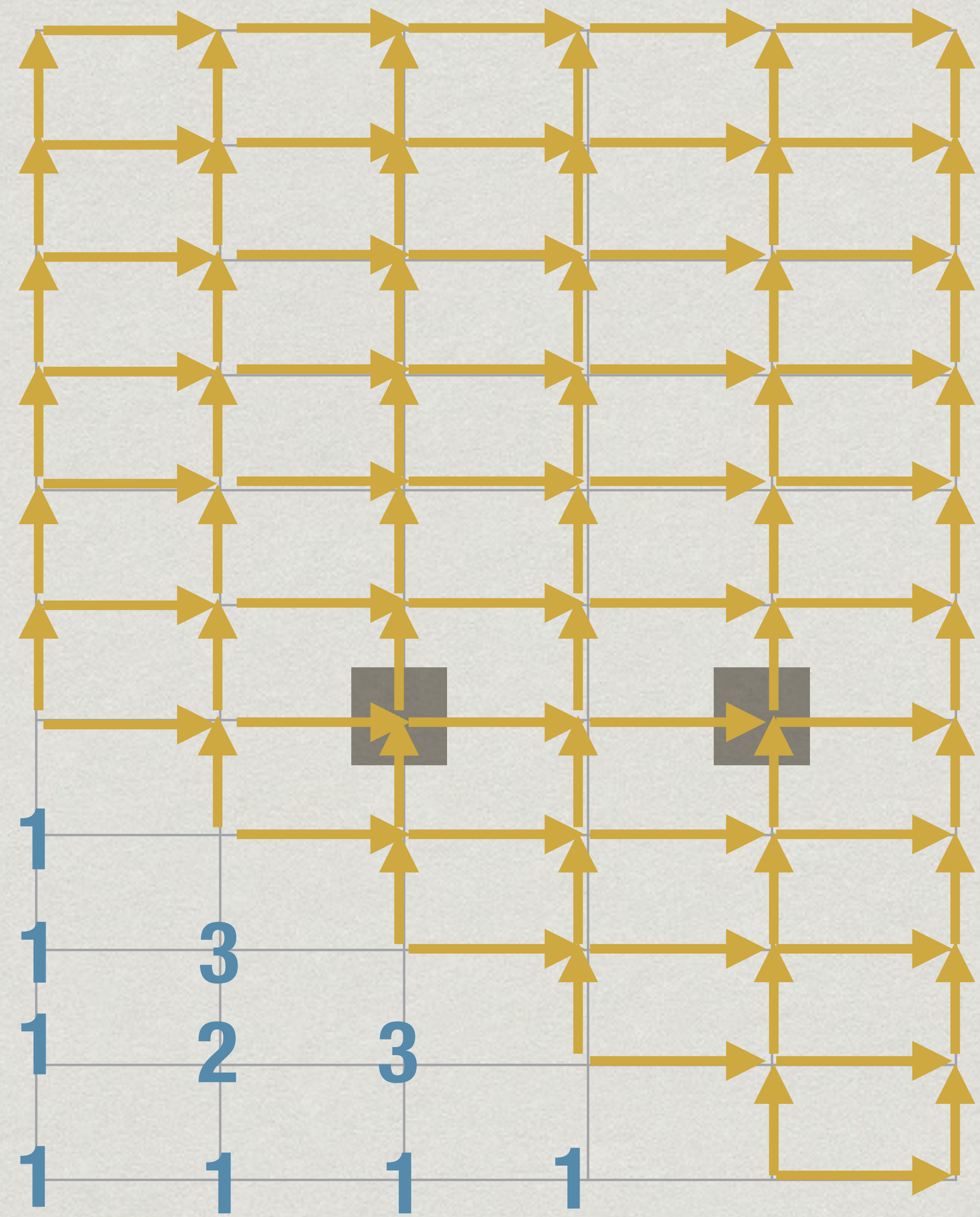
Dynamic programming

- Start at (0,0)
- Fill by diagonal



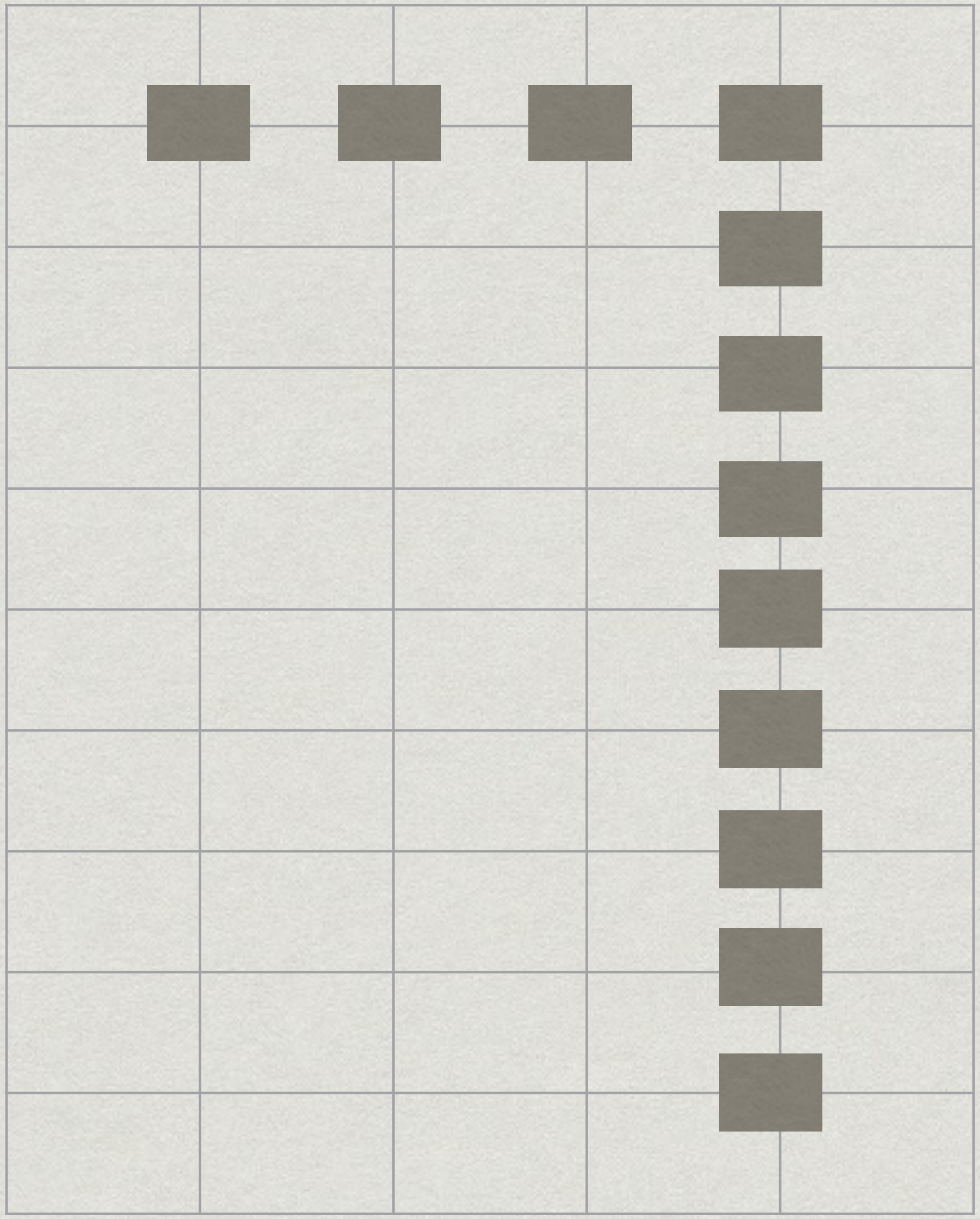
Dynamic programming

- Start at (0,0)
- Fill by diagonal



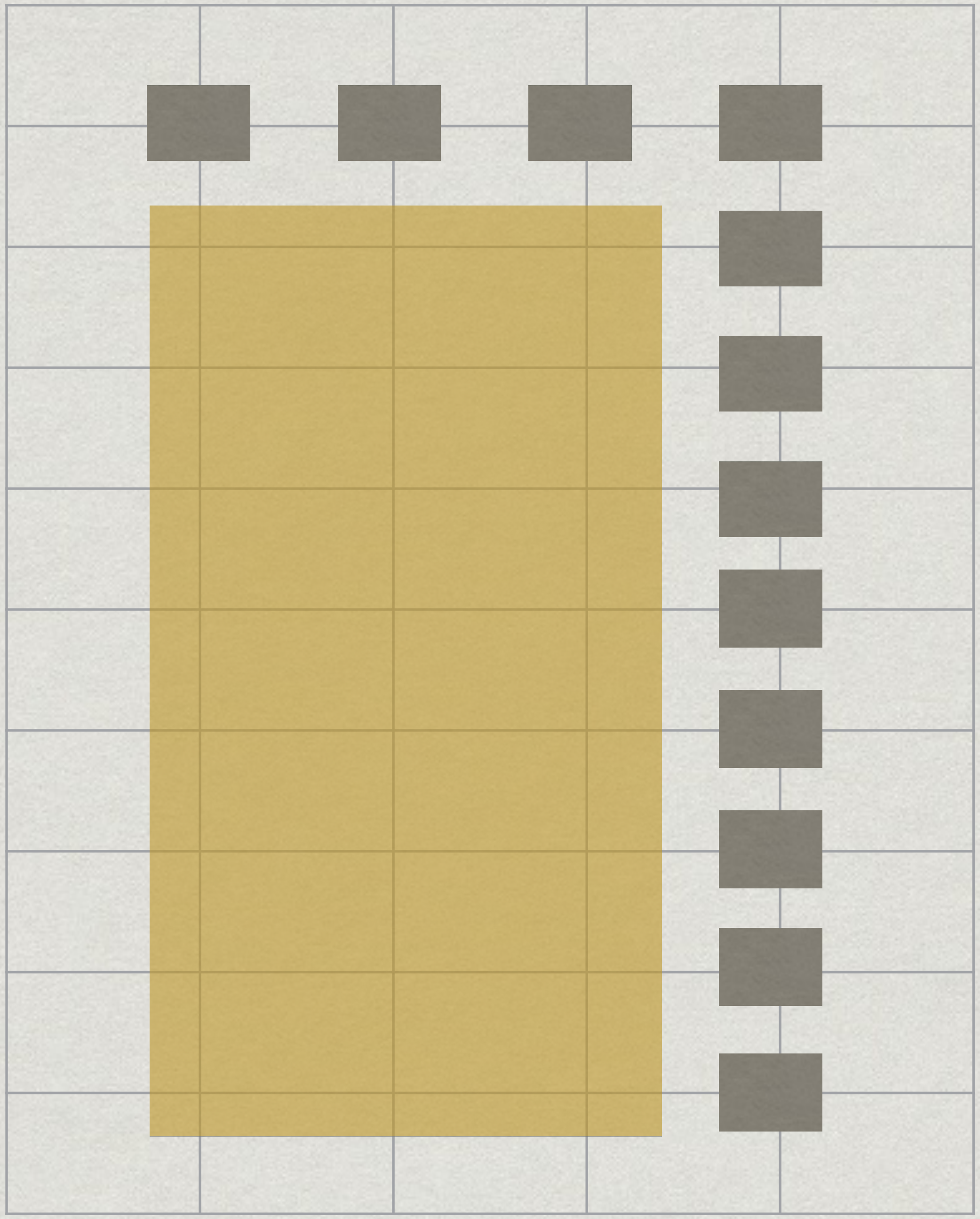
Memoization vs dynamic programming

- Holes just inside the border
- Memoization never explores the shaded region



Memoization vs dynamic programming

- Holes just inside the border
- Memoization never explores the shaded region



Memoization vs dynamic programming

- Memo table has O(m+n) entries
- Dynamic programming blindly fills all O(mn) entries
- Iteration vs recursion — "wasteful" dynamic programming is still better, in general