NPTEL MOOC,JAN-FEB 2015
Week 4, Module 6

DESIGN AND ANALYSIS OF ALGORITHMS

Spanning trees: Prim's algorithm

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Spanning tree

- Weighted undirected graph, G = (V,E,w)
  - Assume G is connected
- Identify a spanning tree with minimum weight
  - Tree connecting all vertices in V
- **Strategy 1:**
  - Start with minimum cost edge
  - Keep extending the tree with smallest edge

Prim's algorithm

```
algorithm Prim_V1

Let e = (i,j) be minimum cost edge in E

TE = [e]       //List of edges in tree
TV = [i,j]     //List of vertices connected by tree

for i = 3 to n
   choose edge f = (u,v) of minimum cost
      such that u in TV and v not in TV
   TE.append(f)
   TV.append(v)

return(TE)
```

Correctness

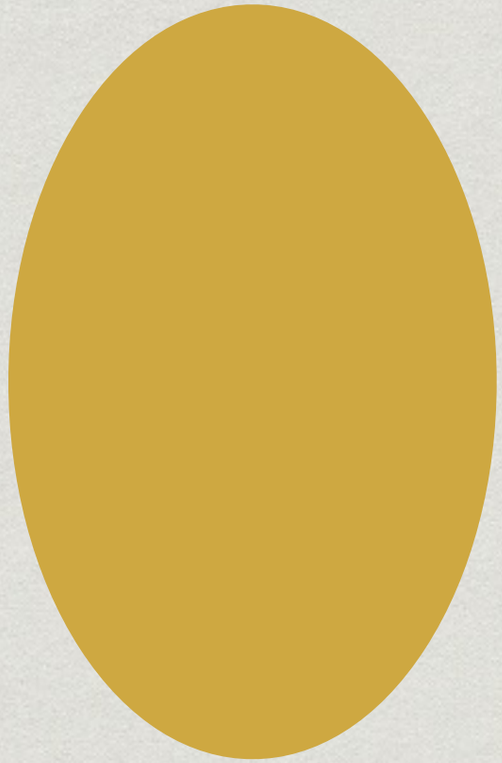
- Prim's algorithm is a greedy algorithm
  - Like Dijkstra's single source shortest path
- A local heuristic is used to decide which edge to add next to the tree
- Choices made are never reconsidered
- Why does this sequence of local choices achieve a global optimum?

Minimum separator lemma

- Let V be partitioned into two non-empty sets U and W = V - U
- Let e = (u,w) be minimum cost edge with u in U and w in W
  - Assume all edges have different weights (relax this condition later)
- Then every minimum cost spanning tree must include e

Minimum separator lemma

- Let T be a minimum cost spanning tree, e = (u,w) not in T
- u in U and w in W are connected by a path p in T
- p starts in U and ends in W
- Let f = (u’,w’) be the first edge on p such that u’ in U and w’ in W
- Drop f and add e to get a smaller spanning tree



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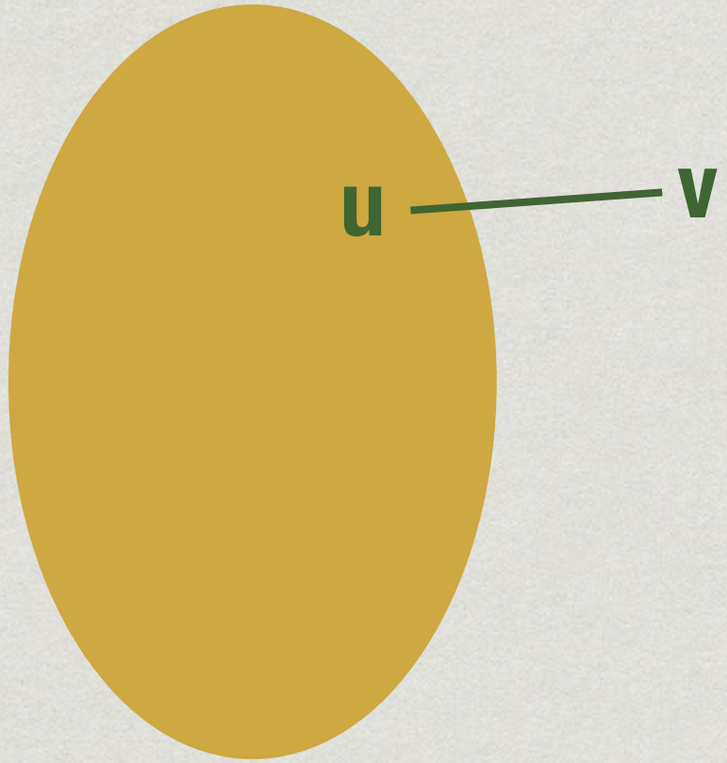

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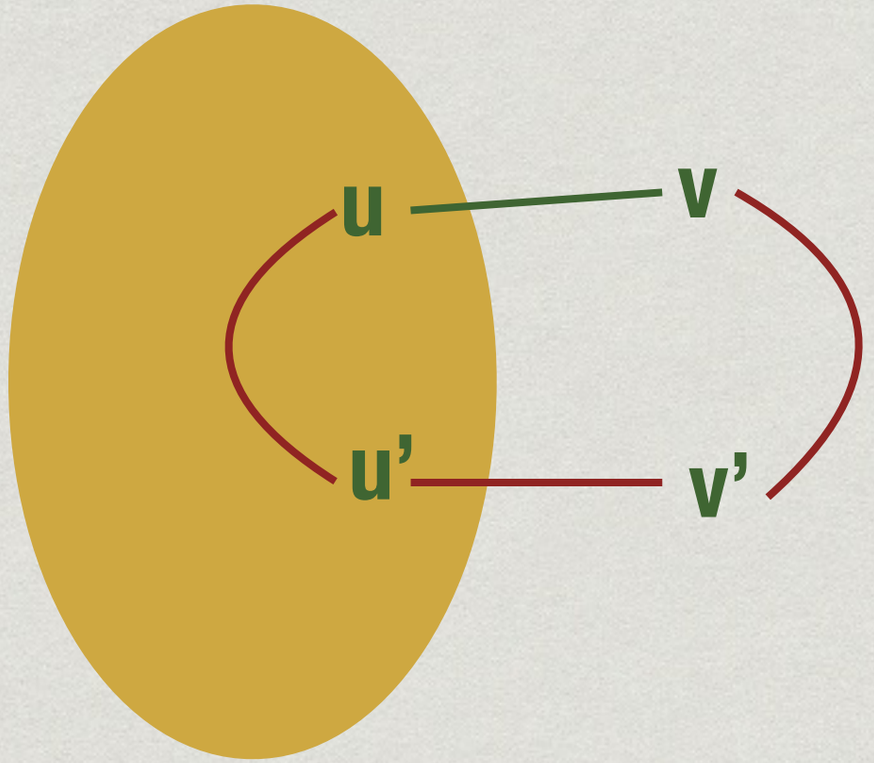
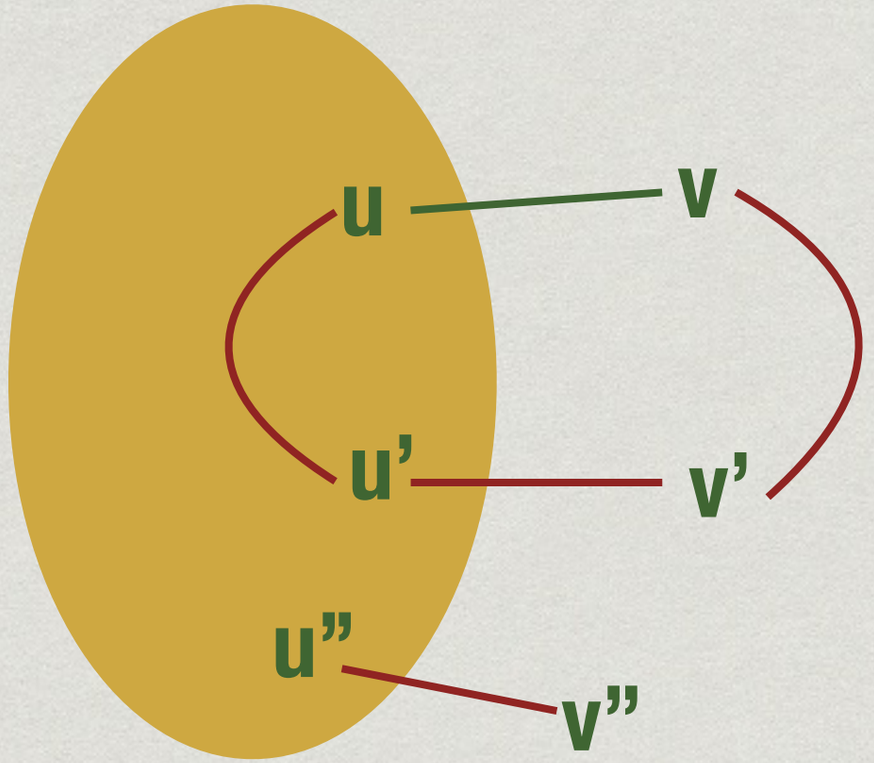


Minimum separator lemma

- Proof of the lemma is slightly subtle
- Not enough to replace any edge from U to W by e = (u,v)
- Need to identify such an edge on the path from u to v



Correctness of Prim's algorithm

- Correctness follows directly from minimum separator lemma
- At each stage, TV and (V-TV) form a non-trivial partition of V
- The smallest edge connecting TV to (V-TV) must belong to every minimum cost spanning tree
  - This is the edge that the algorithm picks

Further observations

- Need not start with smallest edge overall
  - For any vertex v, smallest edge attached to v must be in the minimum cost spanning tree
    - Consider the partition {v}, V-{v}
- Can start with any such edge

Prim's algorithm revisited

- Start with TV = {s} for any vertex s
- For each vertex v outside TV, maintain
  - Distance_TV(v), smallest edge weight from v to TV
  - Neighbour_TV(v), nearest neighbour of v in TV
- At each stage, add to TV ("burn") vertex u with smallest Distance_TV(u)
  - Update Distance_TV(v), Neighbour_TV(v) for each neighbour of u
- Very similar to Dijkstra's algorithm!

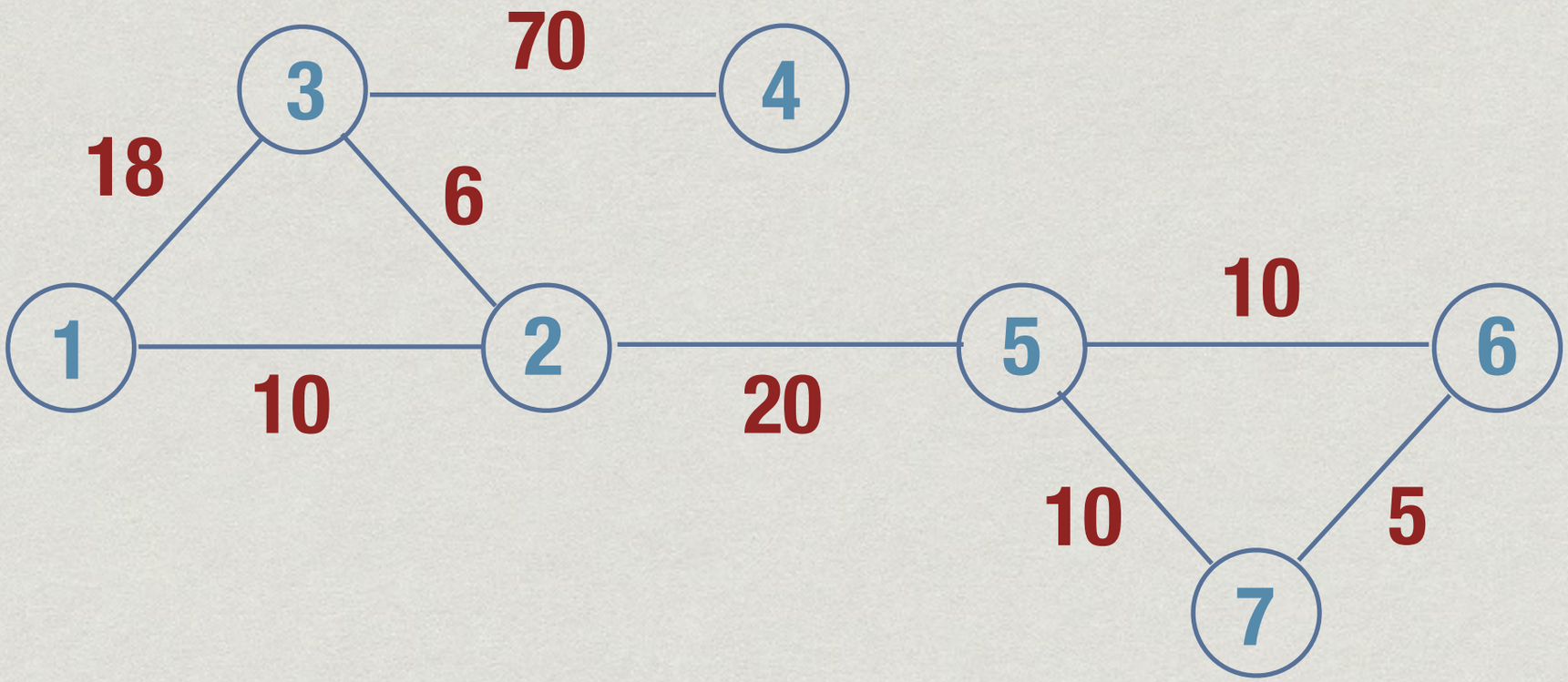
Prim's algorithm, refined

```
function Prim
  for i = 1 to n
    visited[i] = False; Nbr_TV[i] = -1; Dist_TV[i] = infinity

  TE = []   //List of spanning tree edges
  visited[1] = True
  for each edge (1,j)
    Nbr_TV[j] = 1; Dist_TV[j] = weight(1,j)

  for i = 2 to n
    Choose u such that Visited[u] == False
                           and Dist_TV[u] is minimum
    Visited[u] = True
    TE.append{(u,Nbr_TV[u])}
    for each edge (u,v) with Visited[v] == False
      if Dist_TV[v] > weight(u,v)
        Dist_TV[v] = weight(u,v); Nbr_TV[i] = u
```

Prim's algorithm



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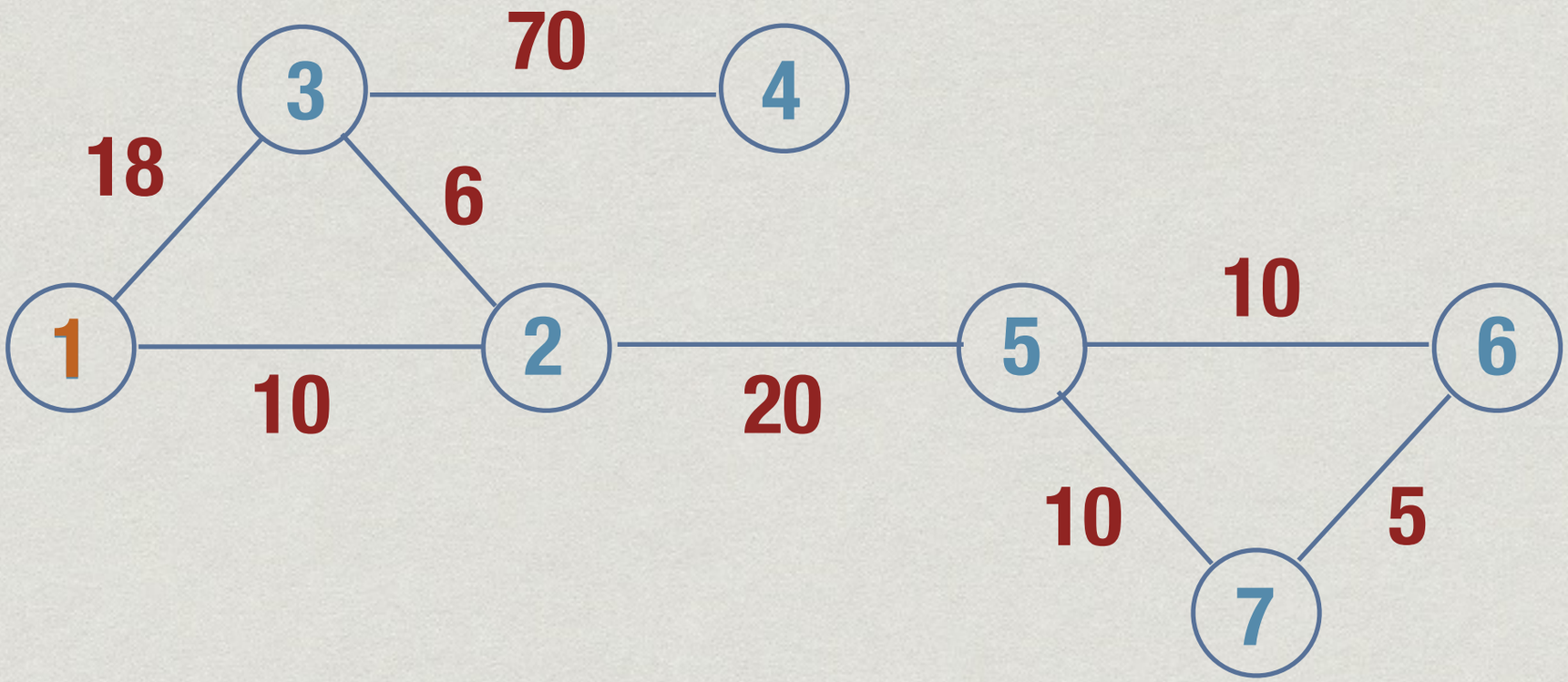

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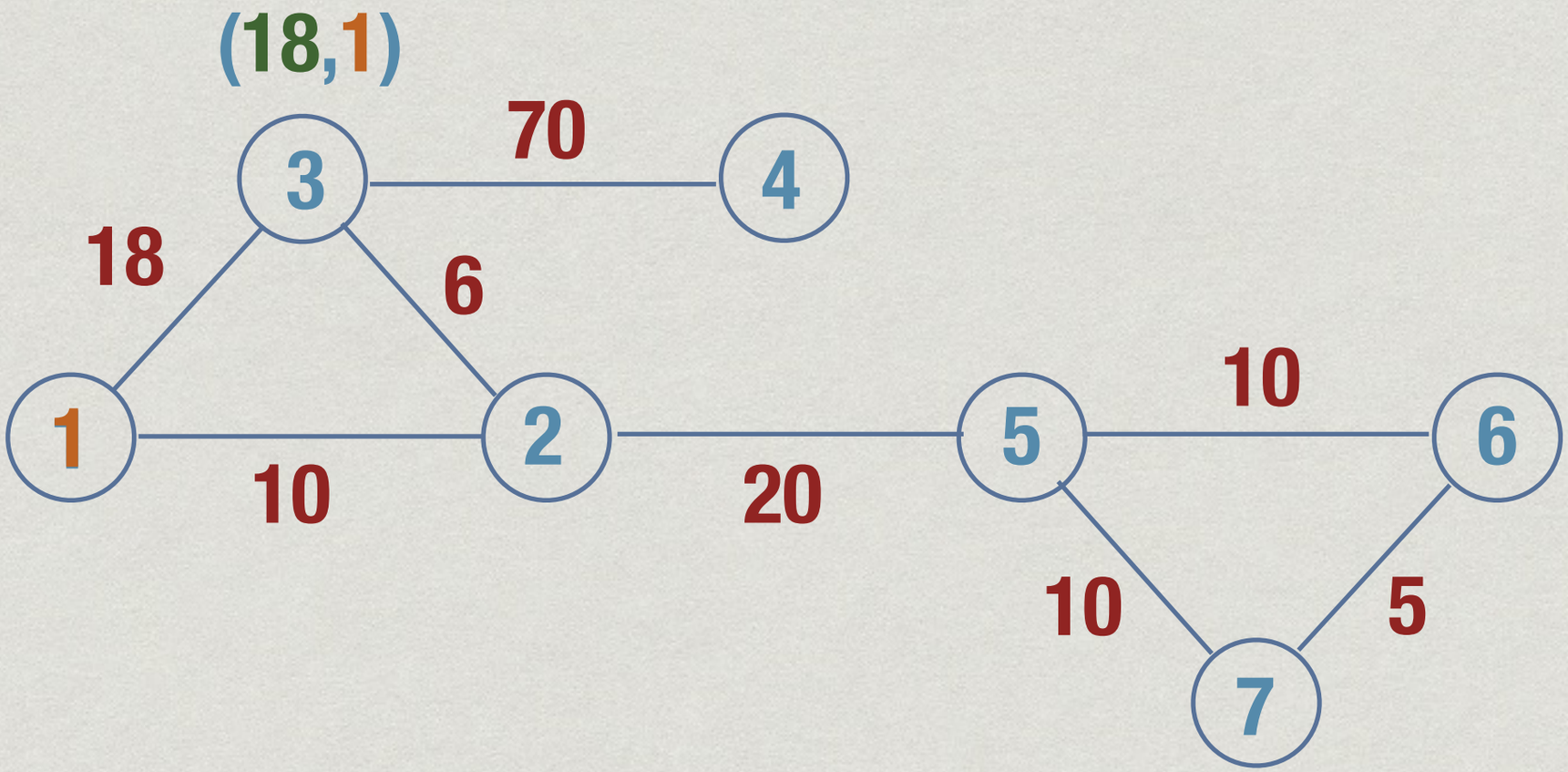

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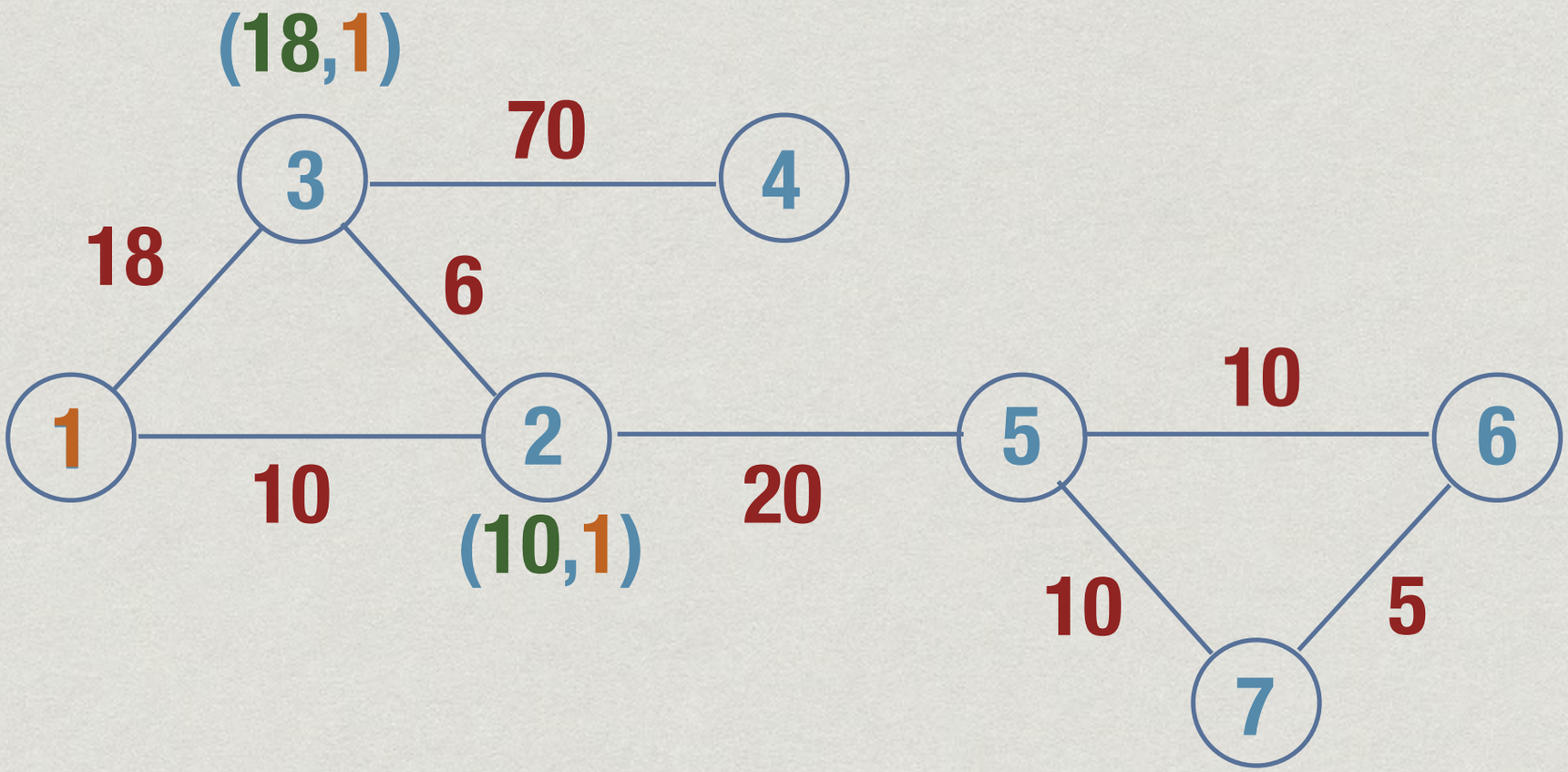

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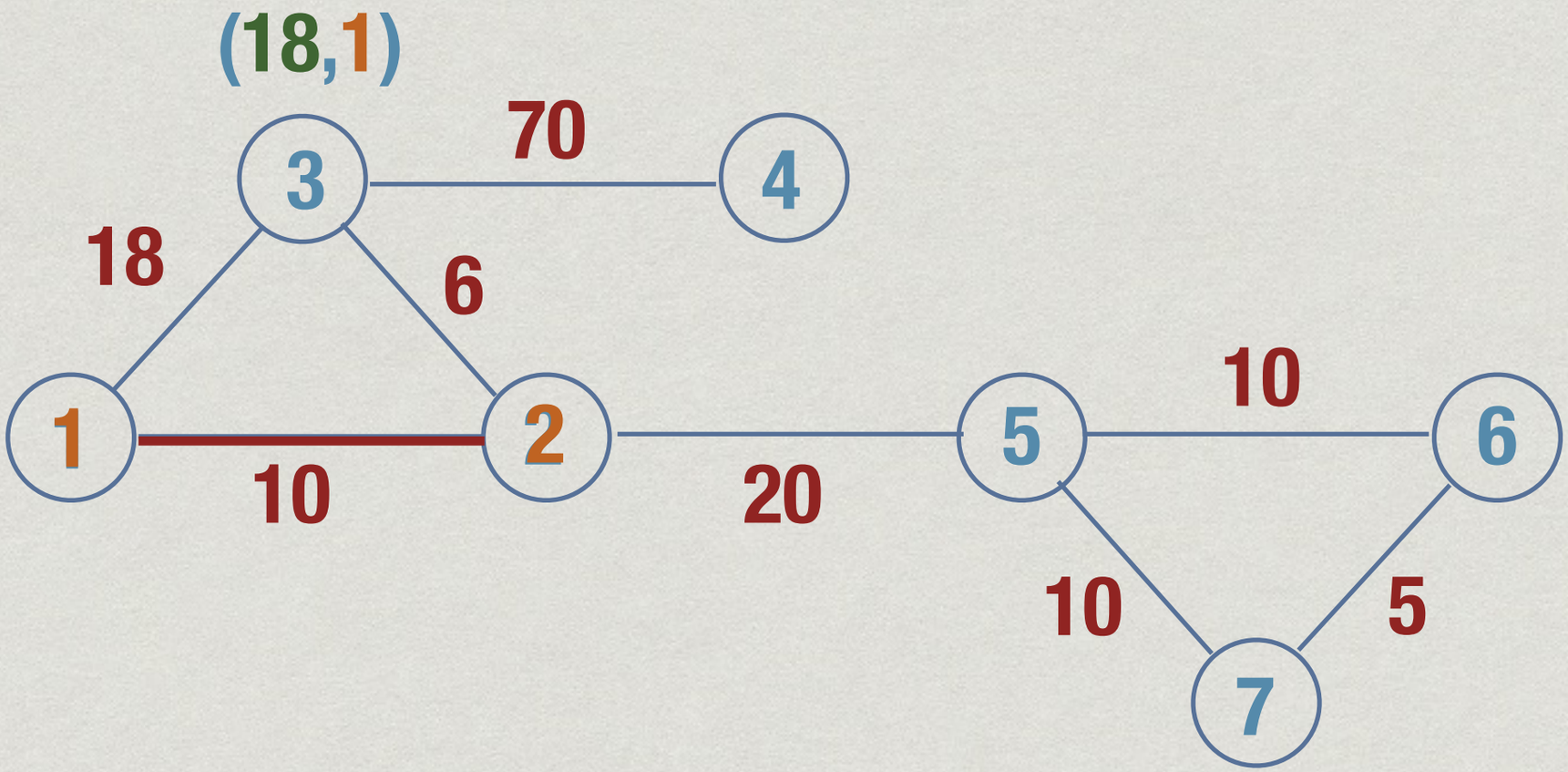

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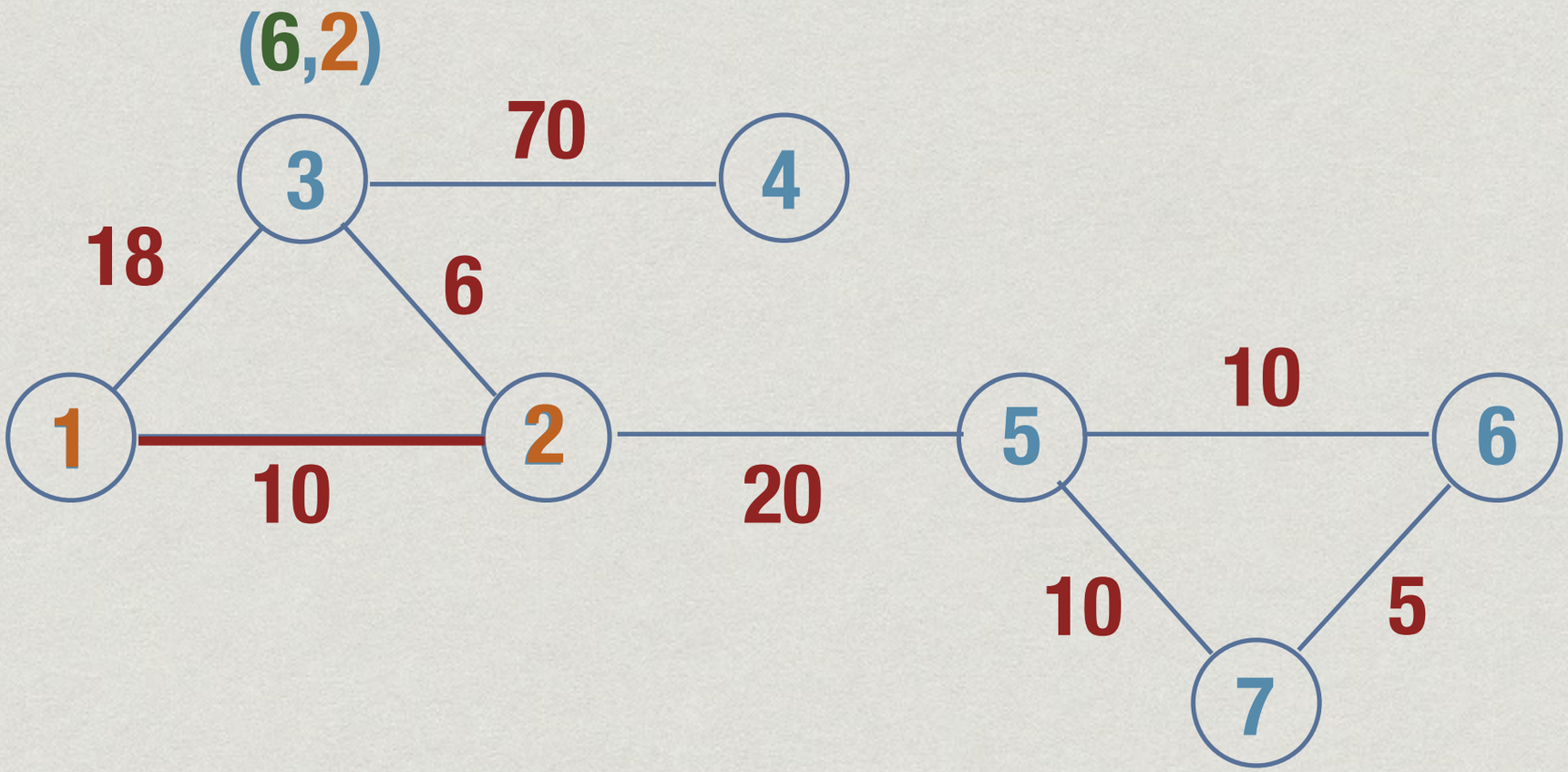

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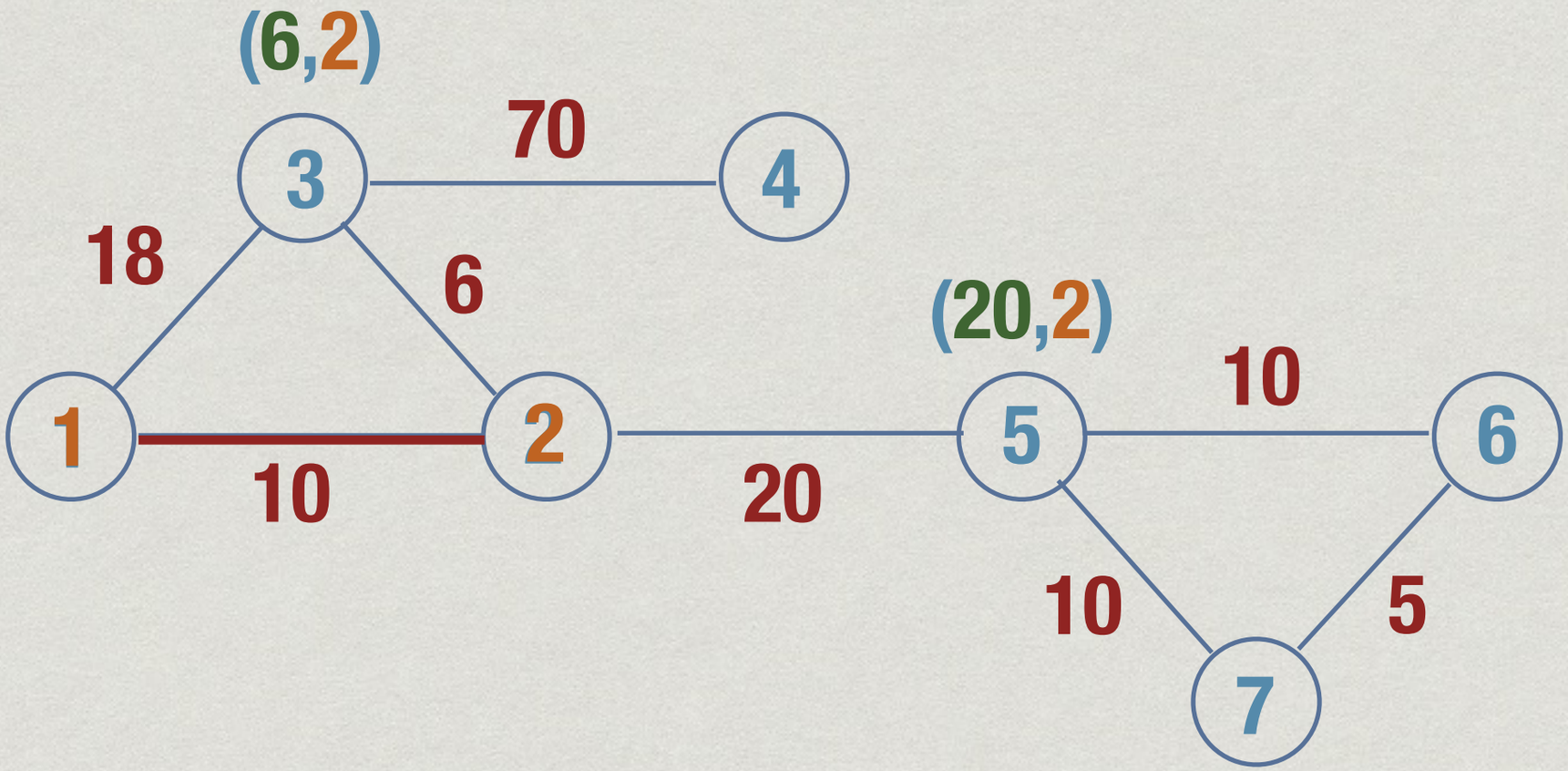

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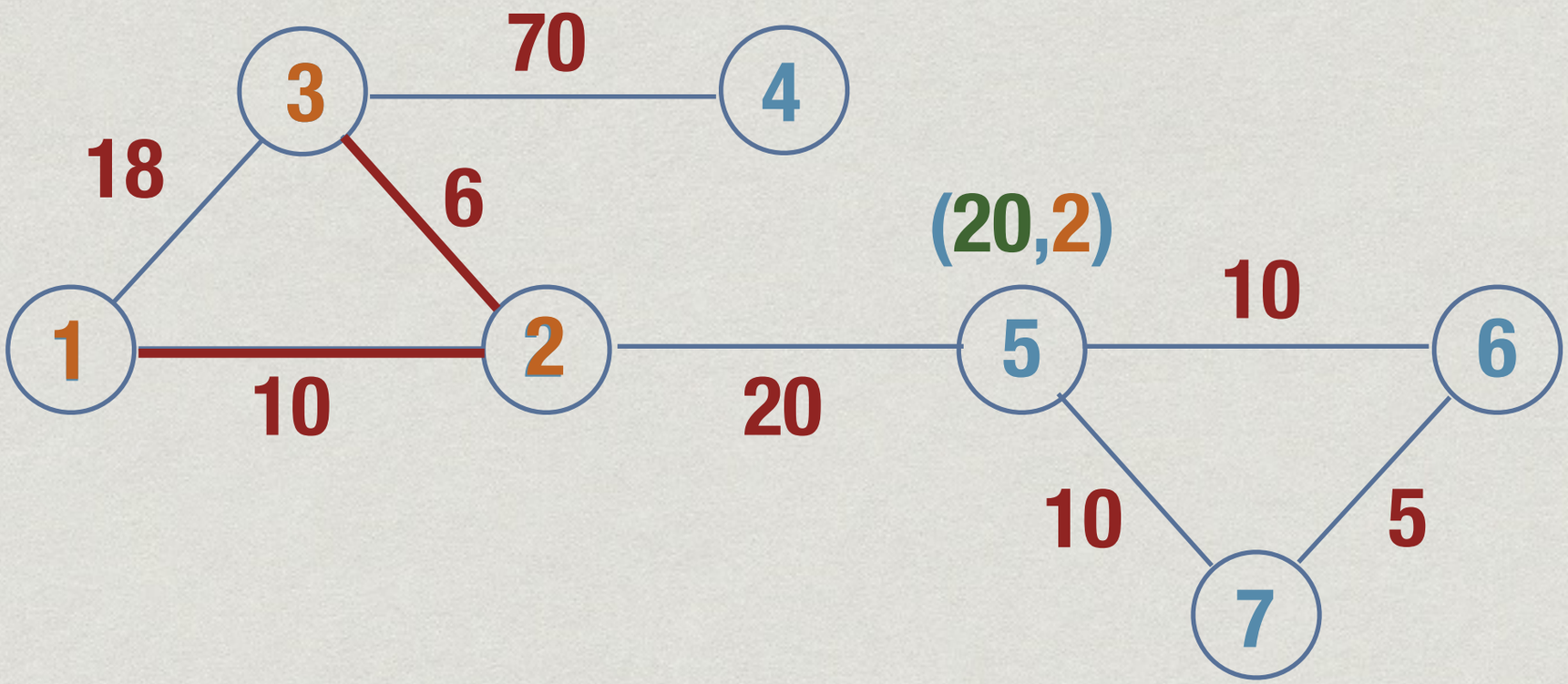

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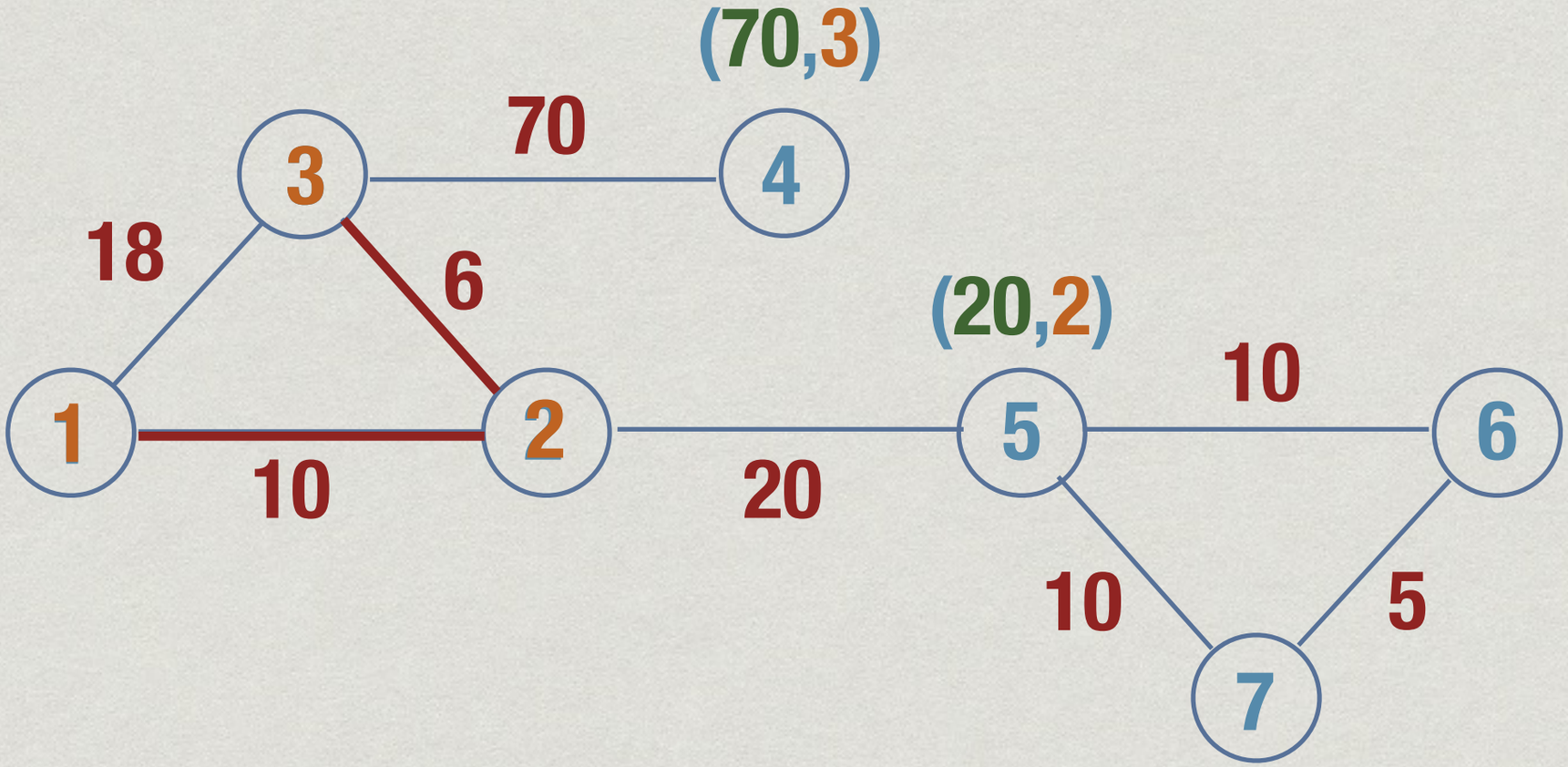

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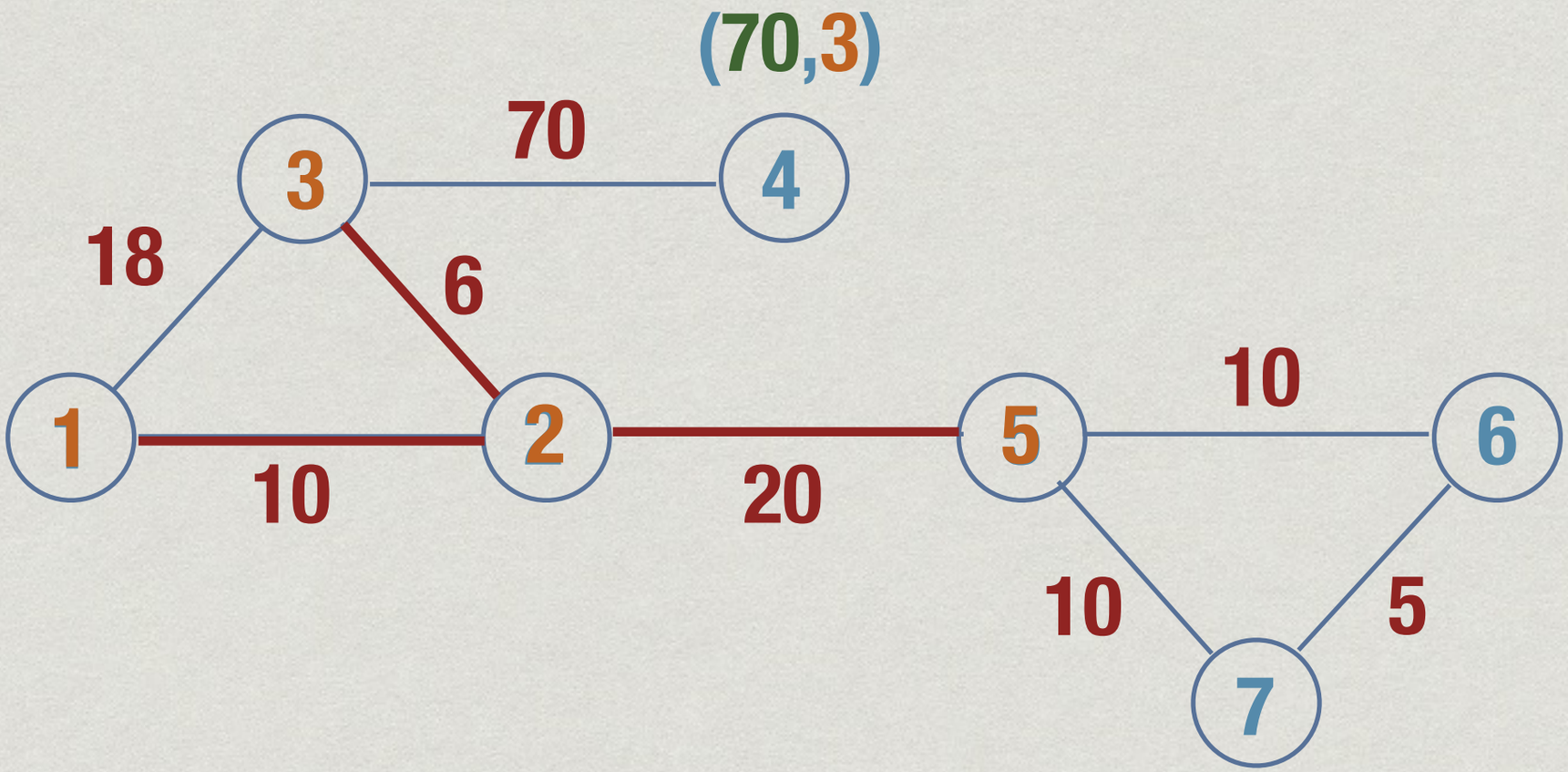

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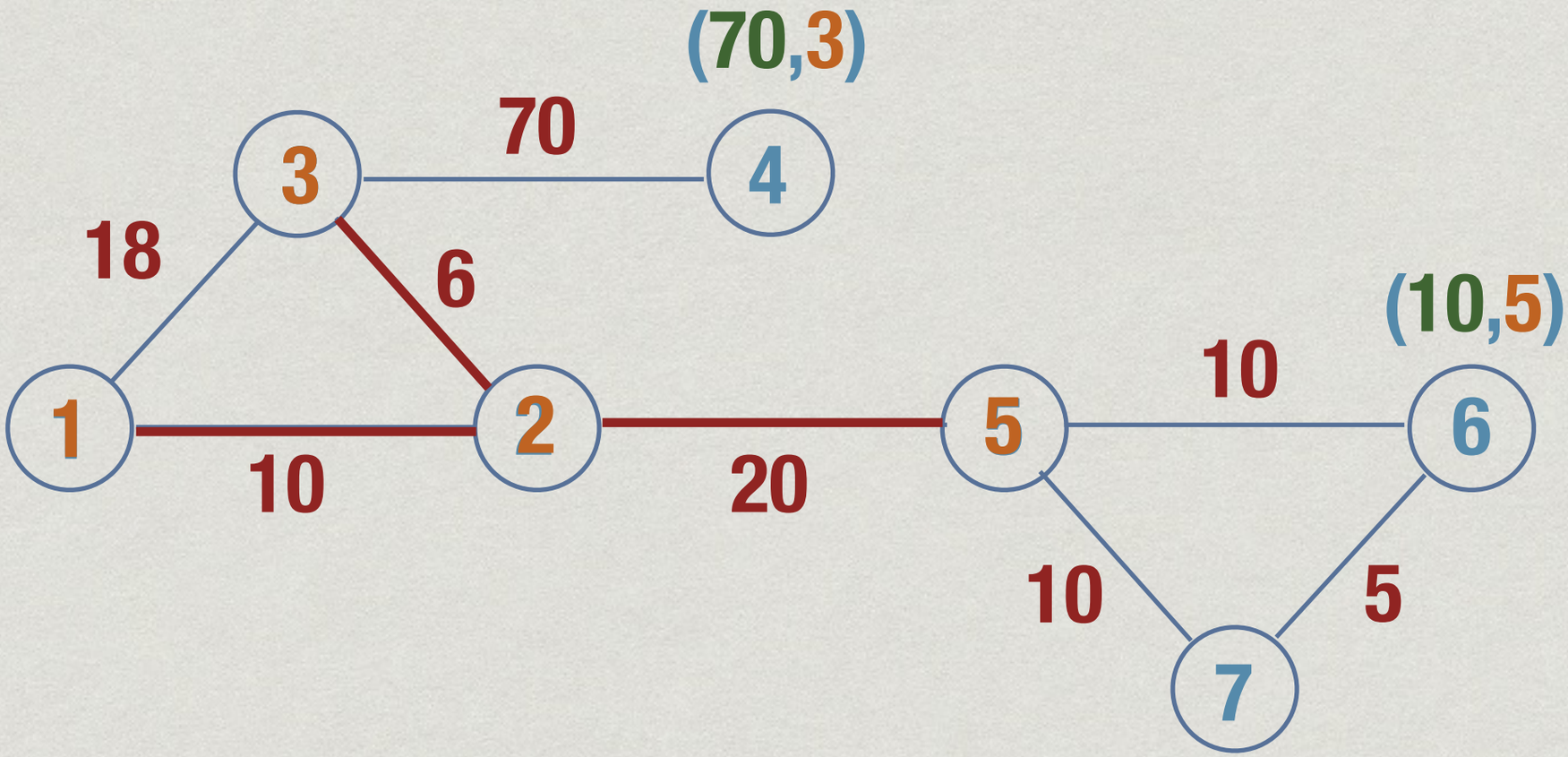

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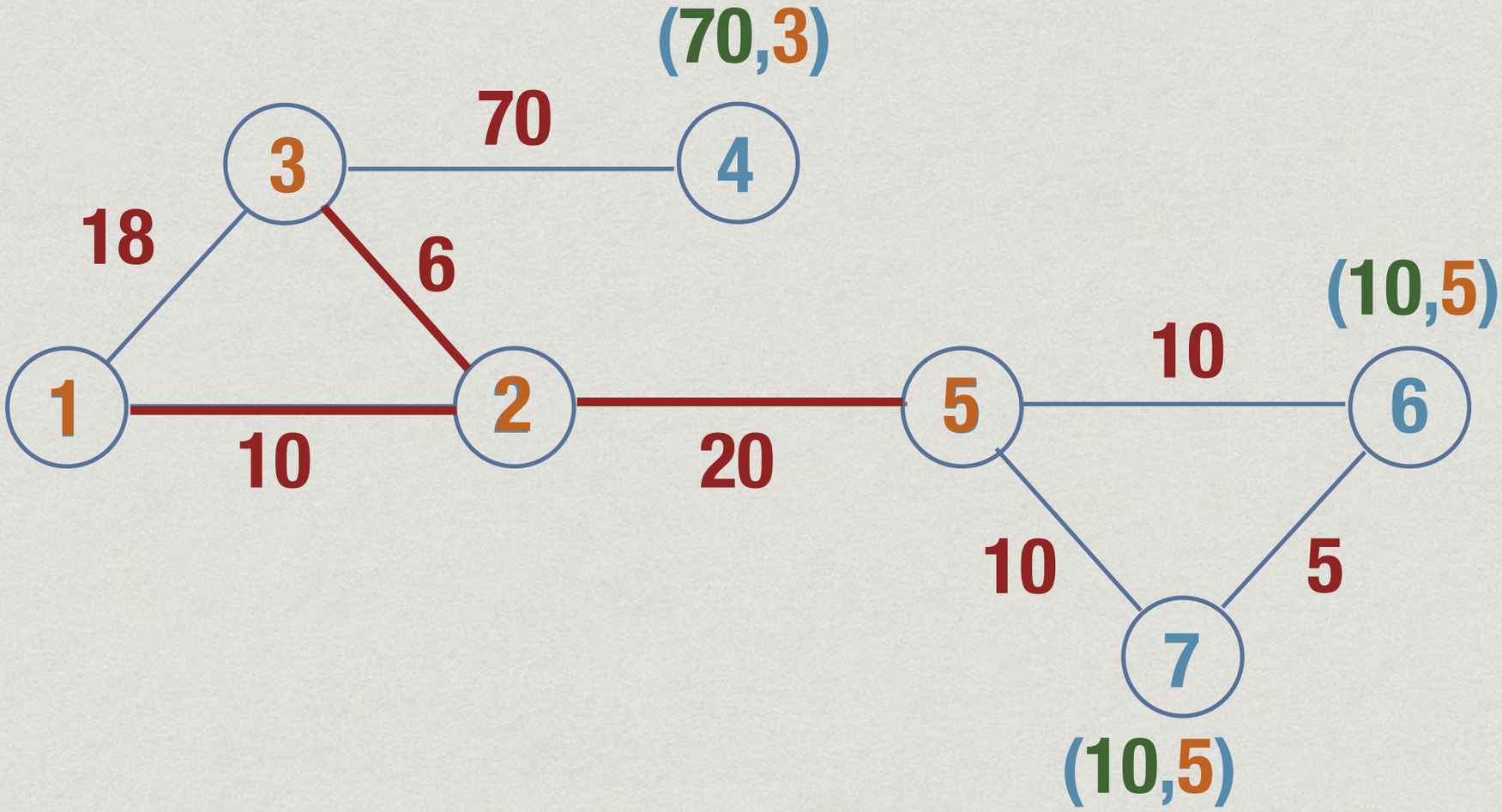

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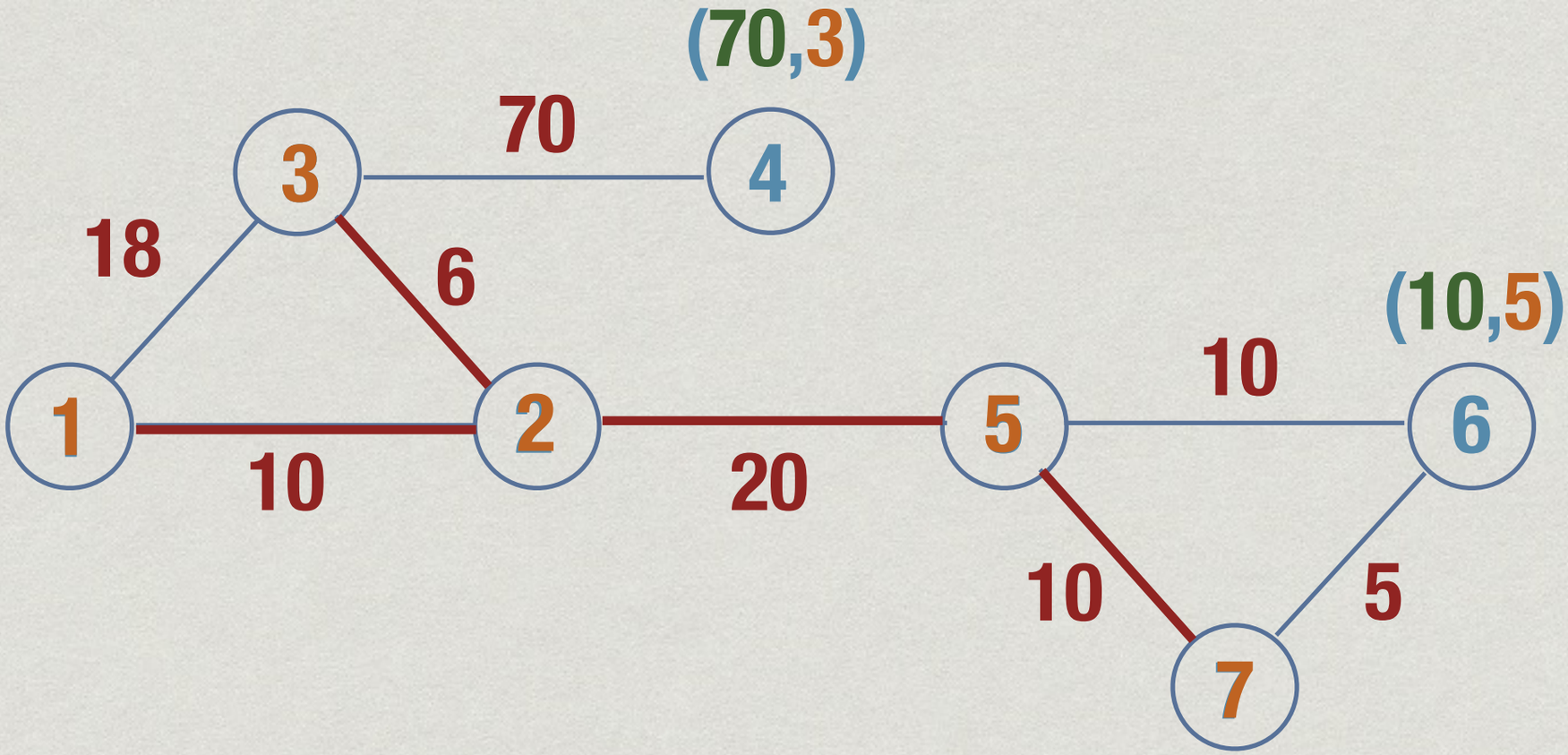

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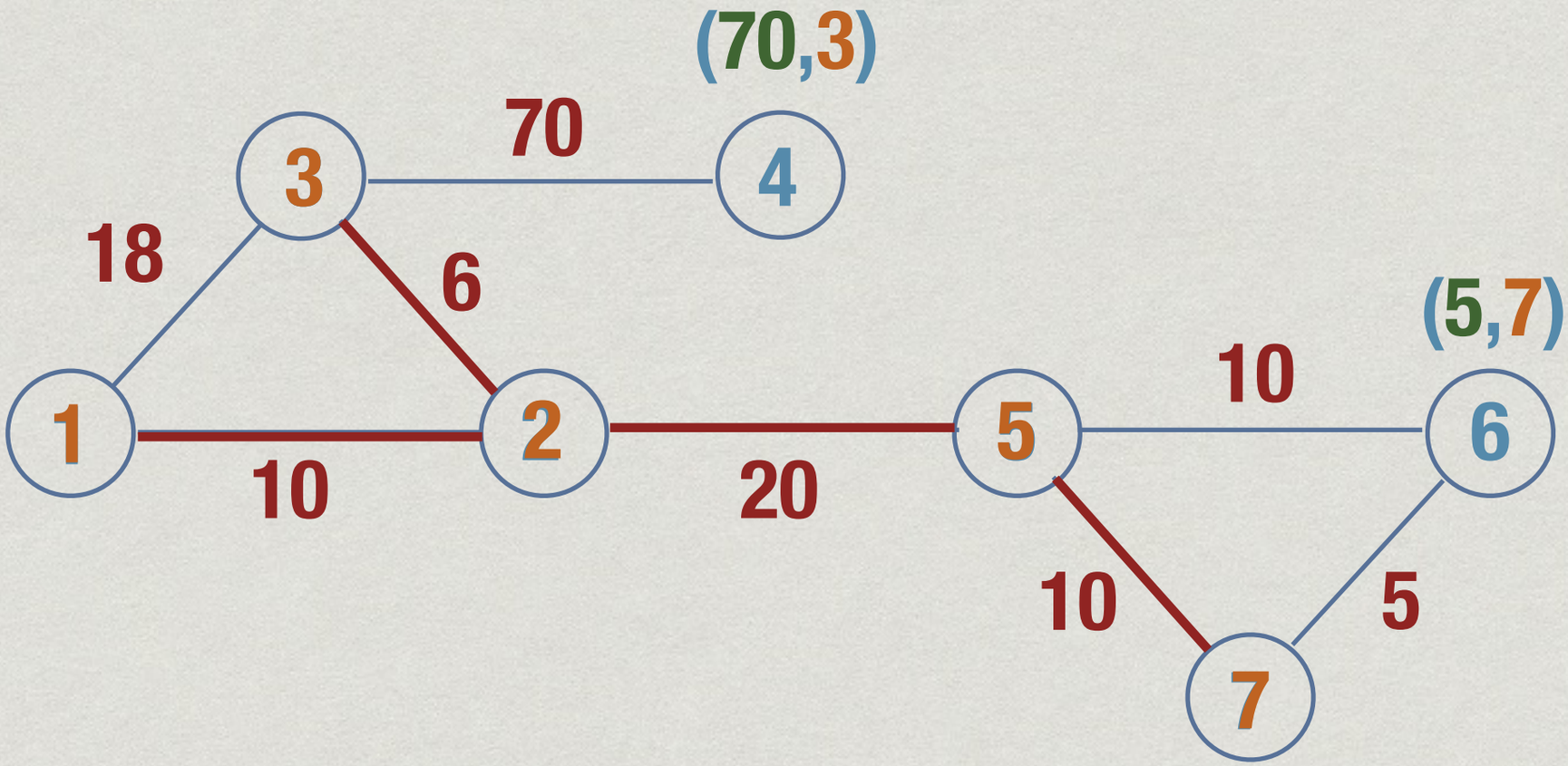

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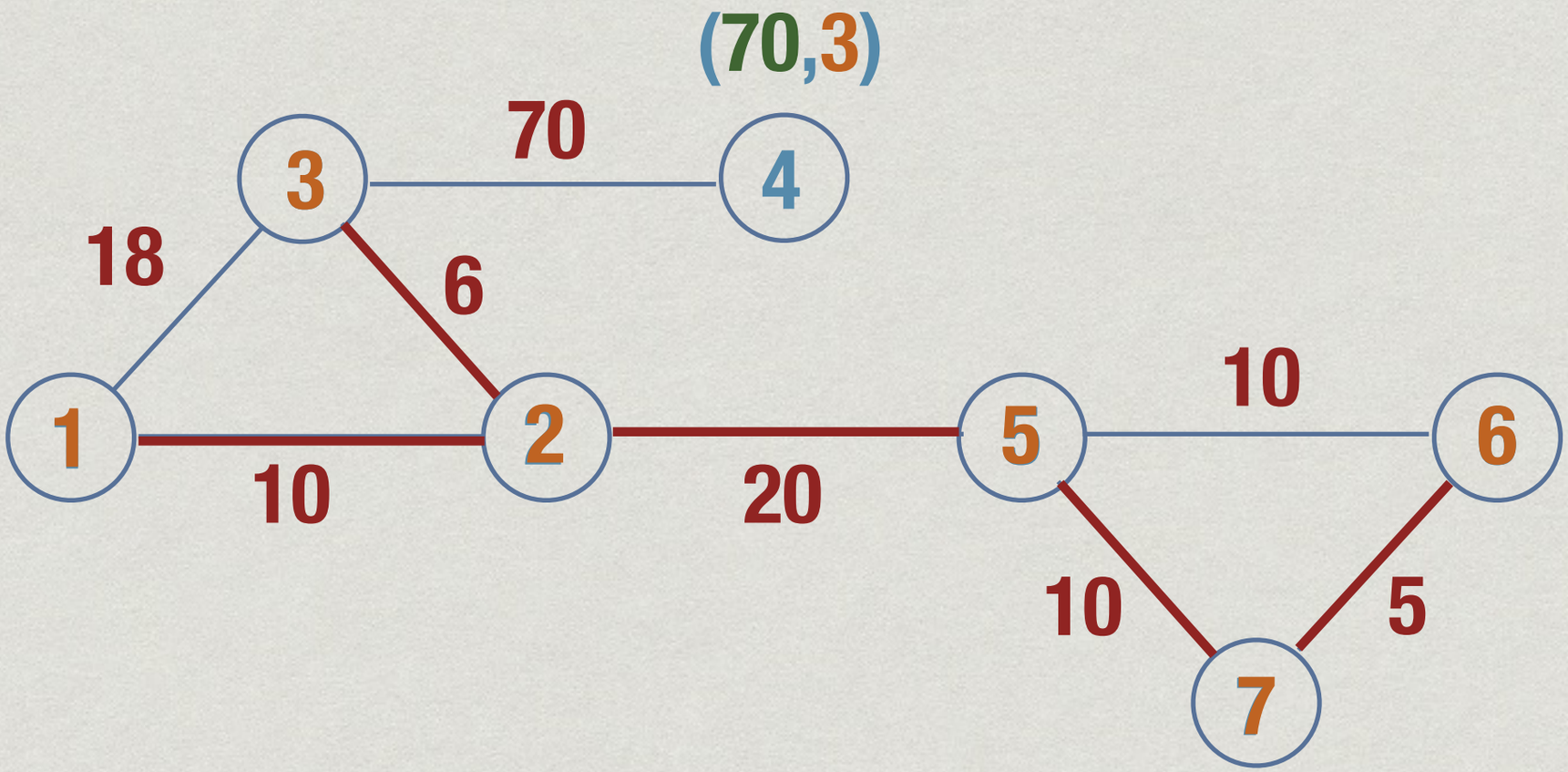

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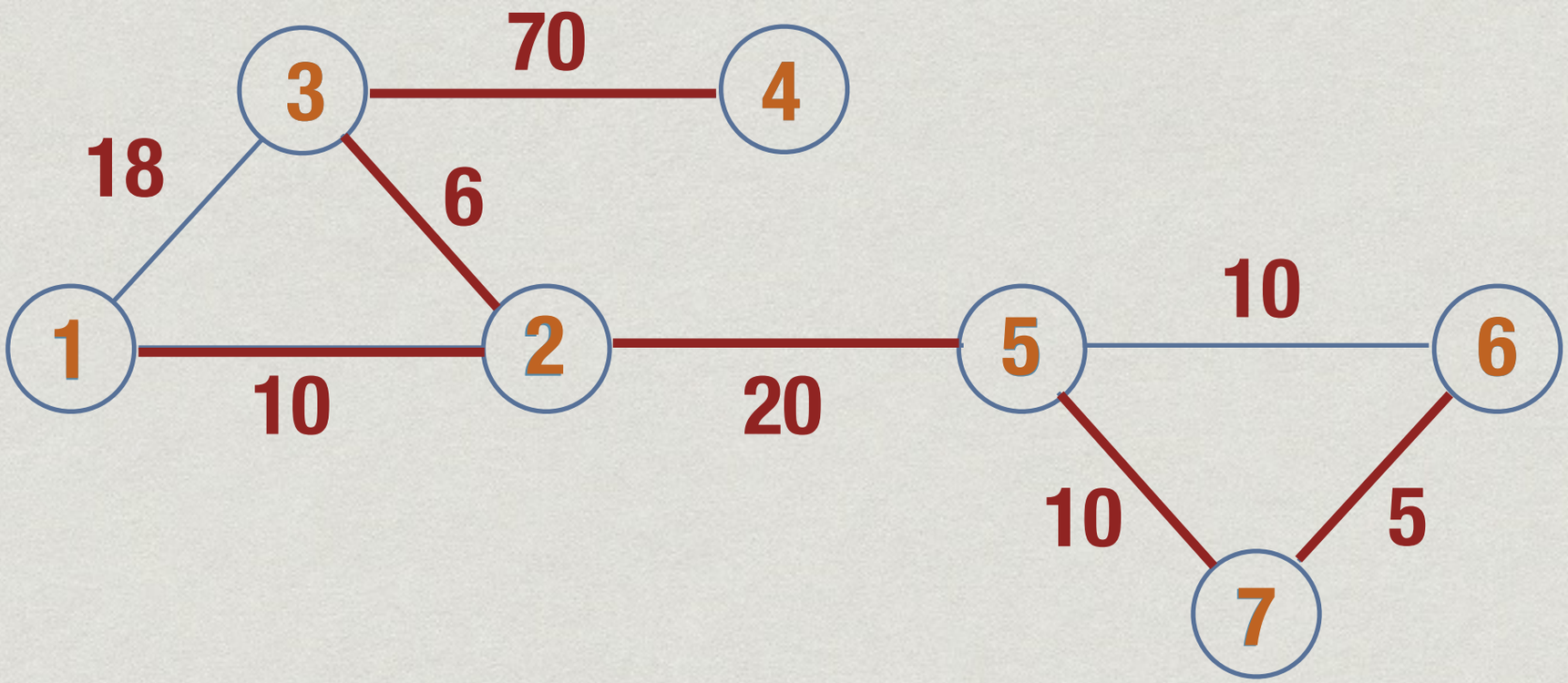

Complexity

- Similar to Dijkstra's algorithm
- Outer loop runs n times
  - In each iteration, we add one vertex to the tree
    - O(n) scan to find nearest vertex to add
- Each time we add a vertex v, we have to scan all its neighbours to update distances
  - O(n) scan of adjacency matrix to find all neighbours
- Overall $O(n^2)$

Complexity

- Moving from adjacency matrix to adjacency list
  - Across n iterations, O(m) to update neighbours
- Maintain distance information in a heap
  - Finding minimum and updating is O(log n)
- Overall O(n log n + m log n) = O((m+n) log n)

Minimum separator lemma

- We assumed edge weights are distinct
- Duplicate edge weights?
  - Fix an overall ordering {1,2,…,m} of edges
  - Edge e = ((u,v),i) is smaller than f = ((u’,v’),j) if
    - weight(e) < weight(f)
    - weight(e) = weight(f) and i < j

Multiple spanning trees

- If edge weights repeat, the minimum cost spanning tree is not unique
  - "Choose u such that Dist_TV(u) is minimum"
- Different choices generate different trees
  - Different ways of ordering edges {1,2,…,m}
- In general, number of possible minimum cost spanning trees is exponential
  - Greedy algorithm efficiently picks out one of them