NPTEL MOOC,JAN-FEB 2015
Week 3, Module 3

DESIGN AND ANALYSIS OF ALGORITHMS

Breadth first search (BFS)

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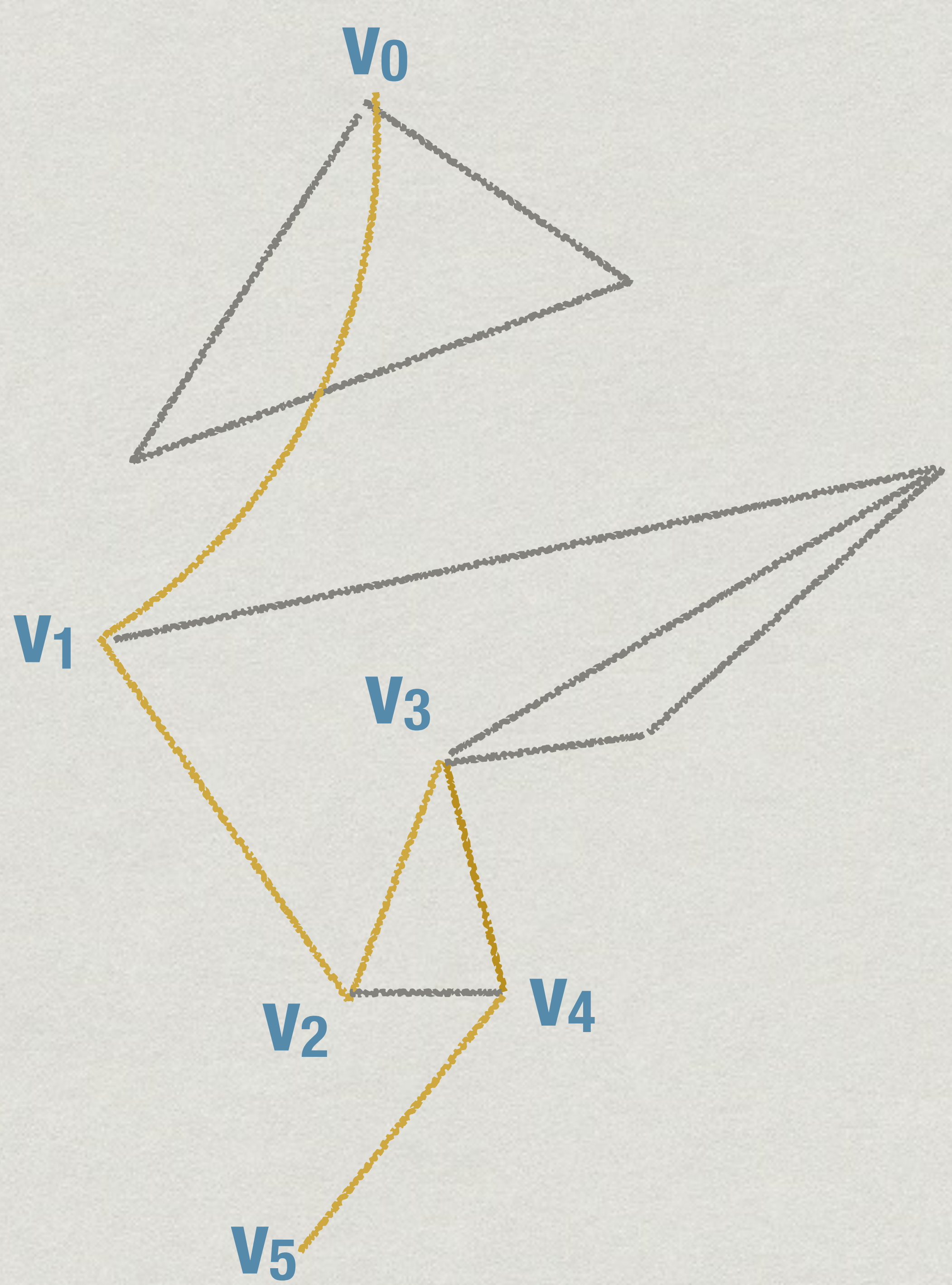
Graphs, formally

G = (V,E)

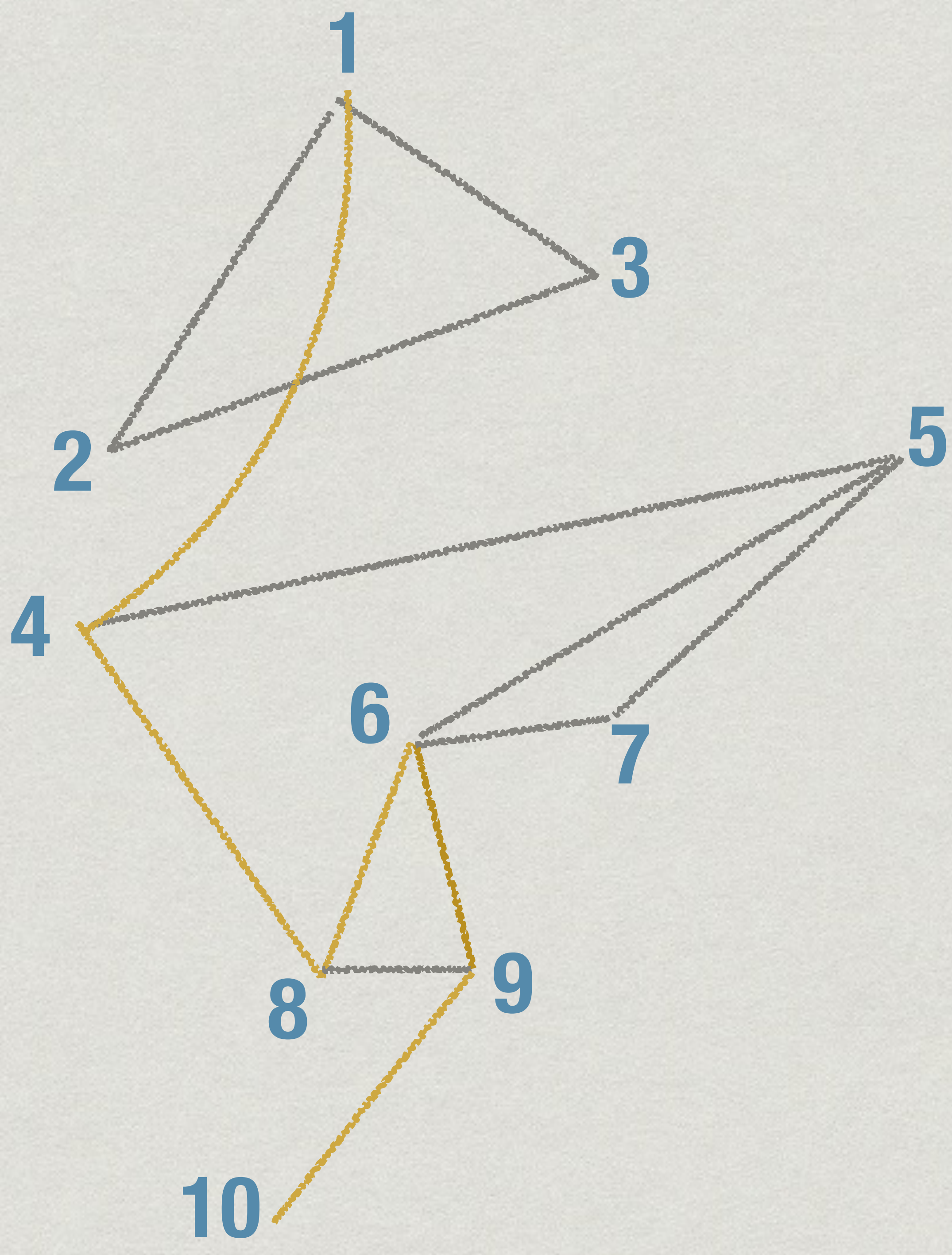
- Set of vertices V
- Set of edges E
    - E is a subset of pairs (v,v'): $E \subseteq V \times V$
    - Undirected graph: (v,v') and (v',v) are the same edge
    - Directed graph:
        - (v,v') is an edge from v to v'
        - Does not guarantee that (v',v) is also an edge

Finding a route

- Find a sequence of vertices v_0, v_1, …, v_k such that
  - v_0 is **source**
  - Each (v_i,v_{i+1}) is an edge in E
  - v_k is **target**

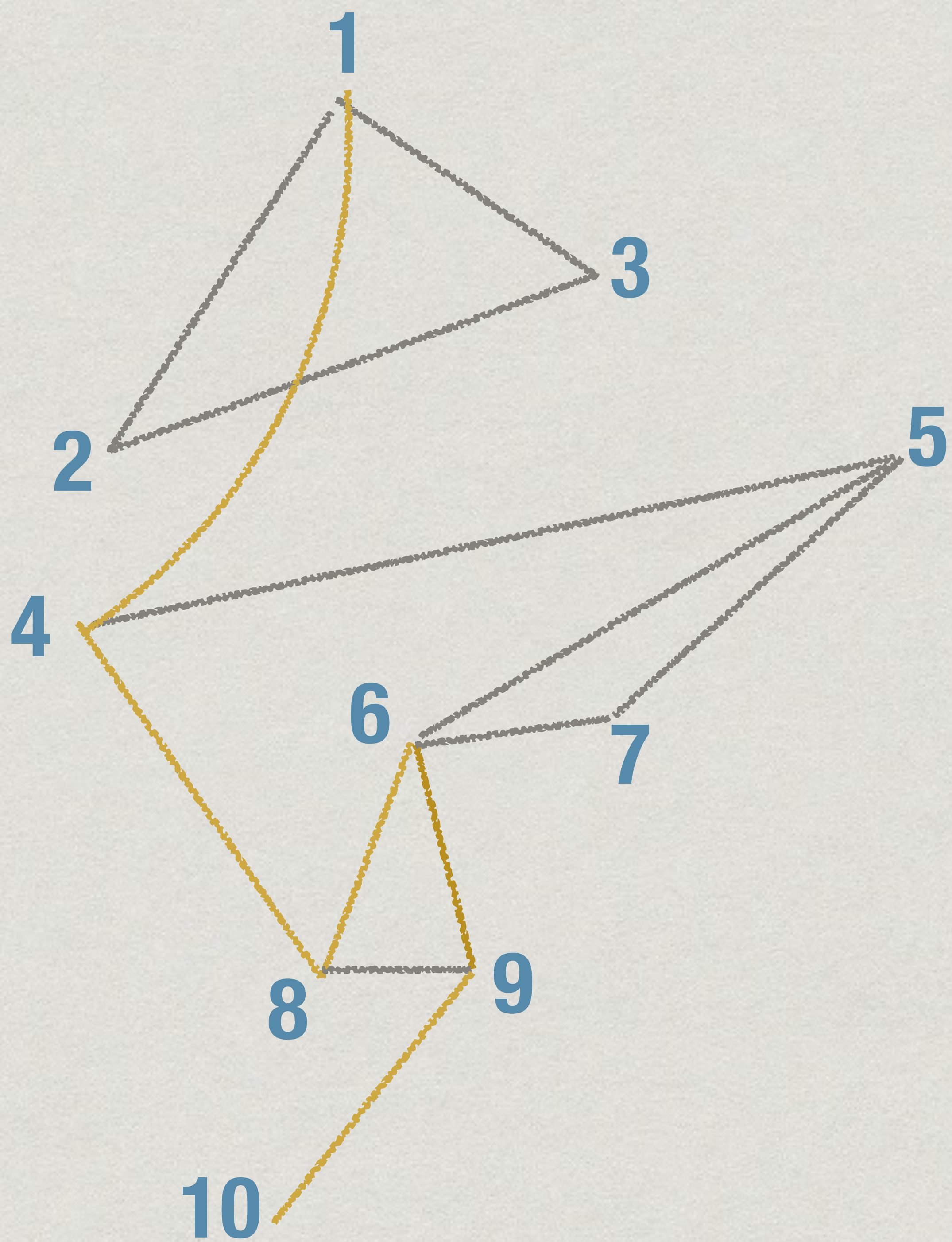


Adjacency matrix



	1	2	3	4	5	6	7	8	9	10
1	0	1	1	1	0	0	0	0	0	0
2	1	0	1	0	0	0	0	0	0	0
3	1	1	0	0	0	0	0	0	0	0
4	1	0	0	0	1	0	0	1	0	0
5	0	0	0	1	0	1	1	0	0	0
6	0	0	0	0	1	0	1	1	1	0
7	0	0	0	0	1	1	0	0	0	0
8	0	0	0	1	0	1	0	0	1	0
9	0	0	0	0	0	1	0	1	0	1
10	0	0	0	0	0	0	0	0	1	0

Adjacency list



- For each vertex, maintain a list of its neighbours

1	2,3,4
2	1,3
3	1,2
4	1,5,8
5	4,6,7
6	5,7,8,9
7	5,6
8	4,6,9
9	6,8,10
10	9

Finding a path

- Mark vertices that have been visited
- Keep track of vertices whose neighbours have already been explored
  - Avoid going round indefinitely in circles
- Two fundamental strategies: breadth first and depth first

Breadth first search

- Explore the graph level by level
    - First visit vertices one step away
    - Then two steps away
    - ...
- Remember which vertices have been **visited**
- Also keep track of vertices visited, but whose neighbours are yet to be **explored**

Breadth first search

- Recall that V = {1,2,…,n}
- Array visited[i] records whether i has been visited
- When a vertex is visited for the first time, add it to a **queue**
  - Explore vertices in the order they reach the queue

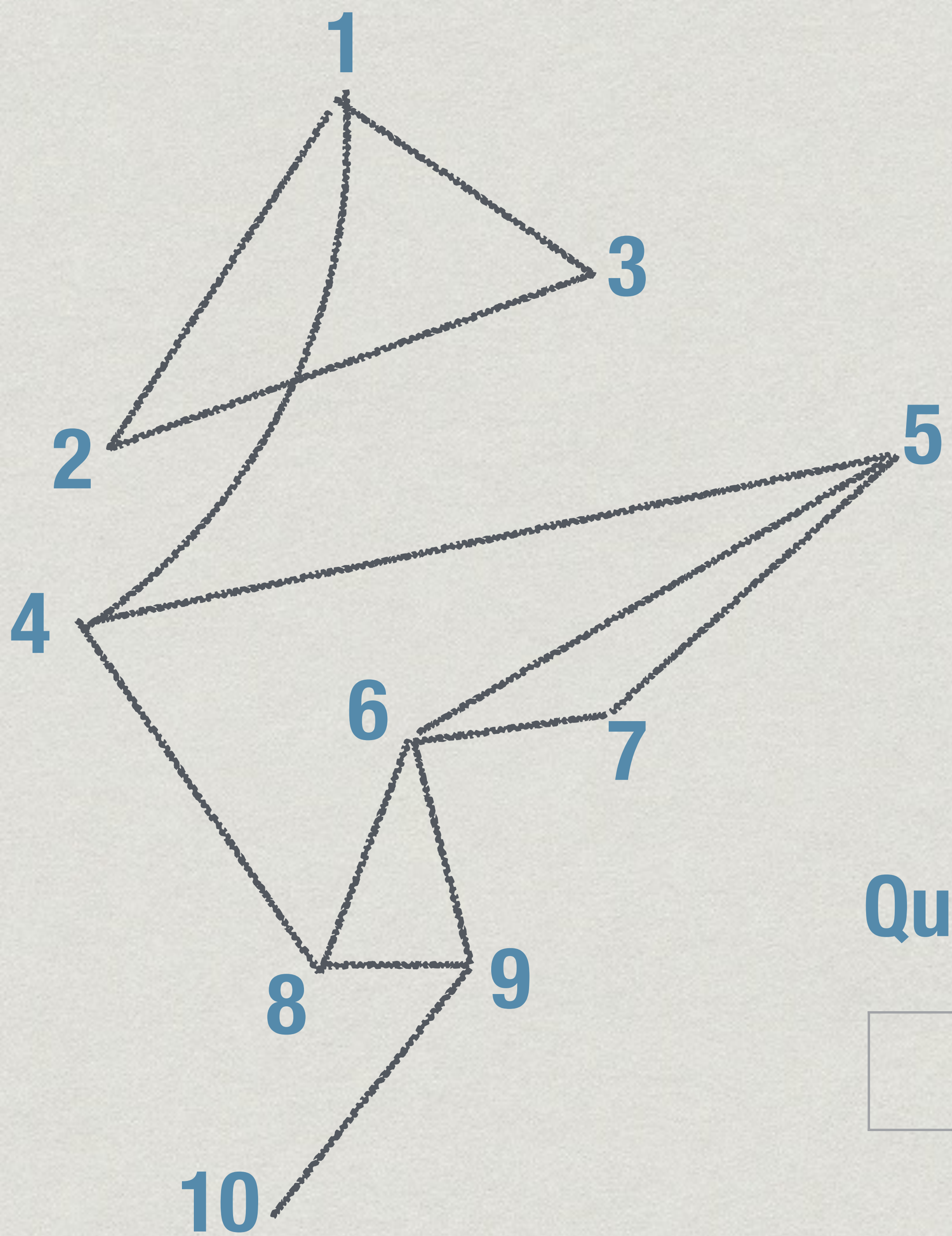
Breadth first search

- Exploring a vertex i:

```
for each edge (i,j)
  if visited[j] == 0
    visited[j] = 1
    append j to queue
```

- Initially, queue contains only source vertex
- At each stage, explore vertex at the head of the queue
- Stop when the queue becomes empty

Breadth first search



Visited

1	
2	
3	
4	
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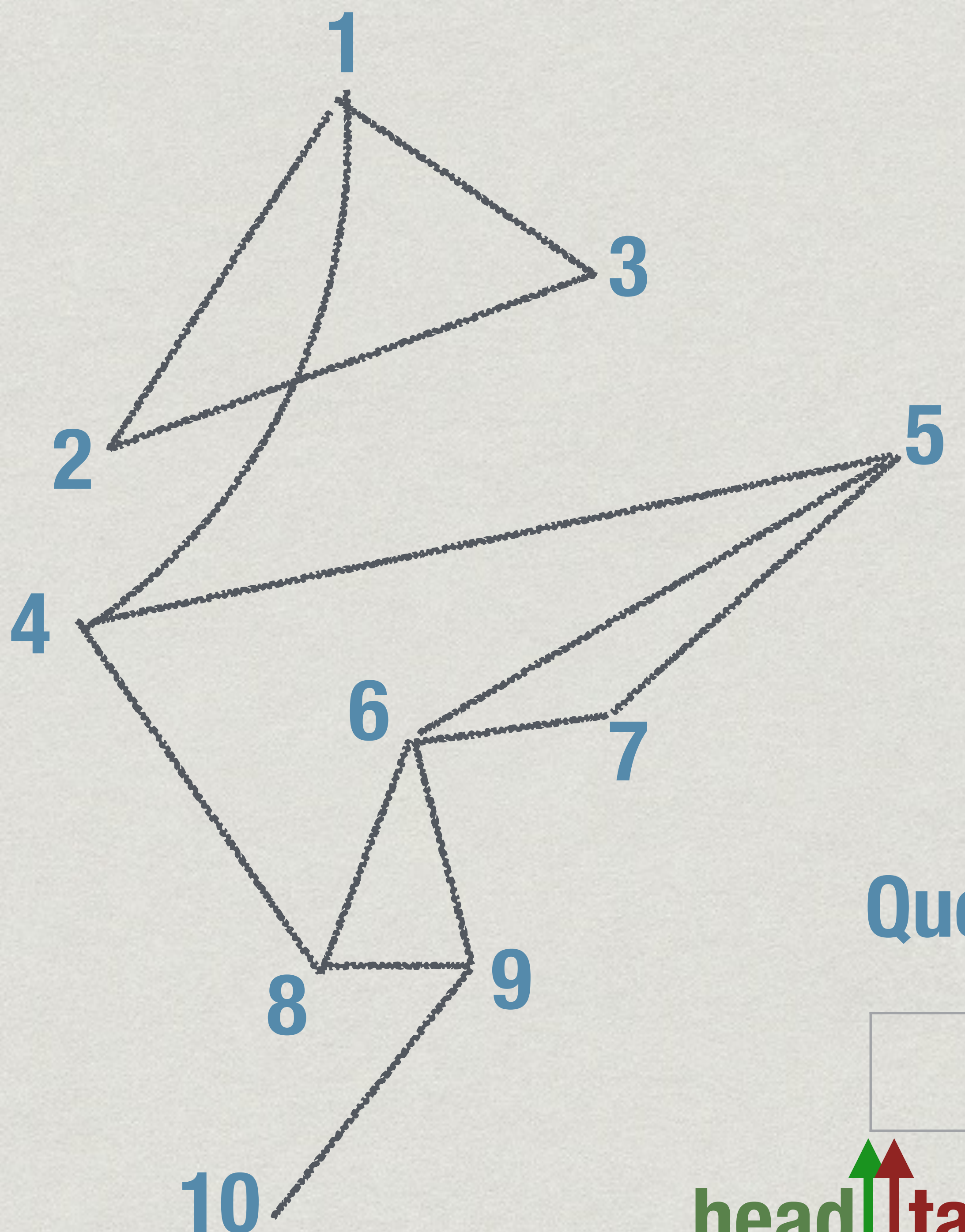
7	
8	
9	
10	

Queue

Breadth first search



Visited

1	
2	
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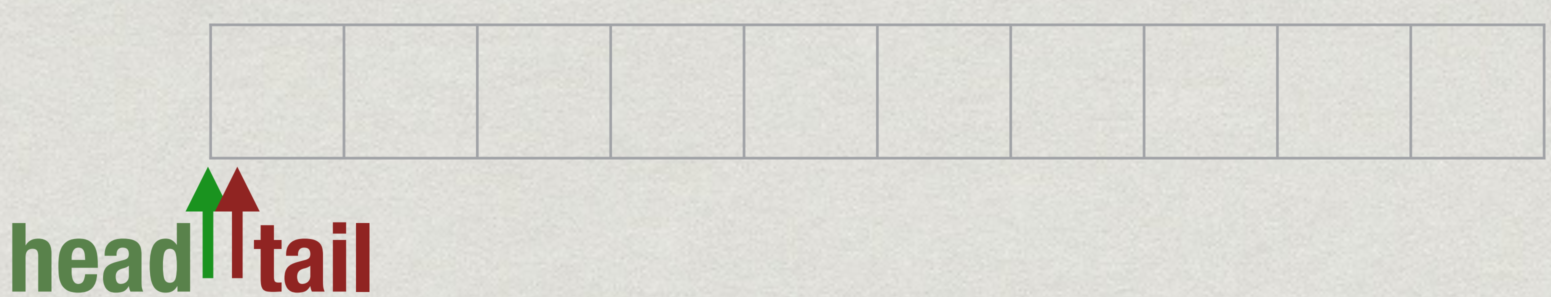
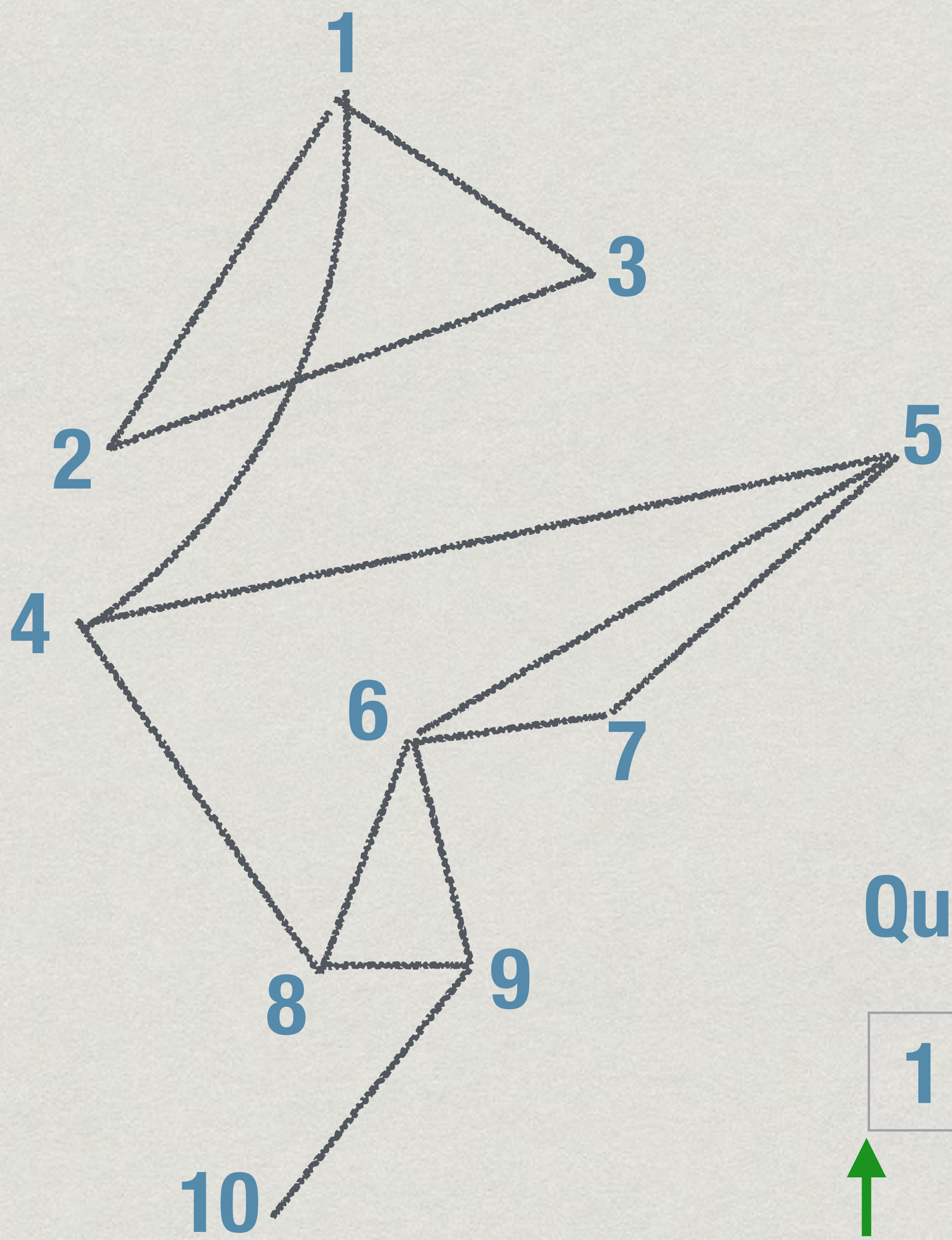
10	

Queue



Breadth first search



Visited

1	1
2	
3	
4	
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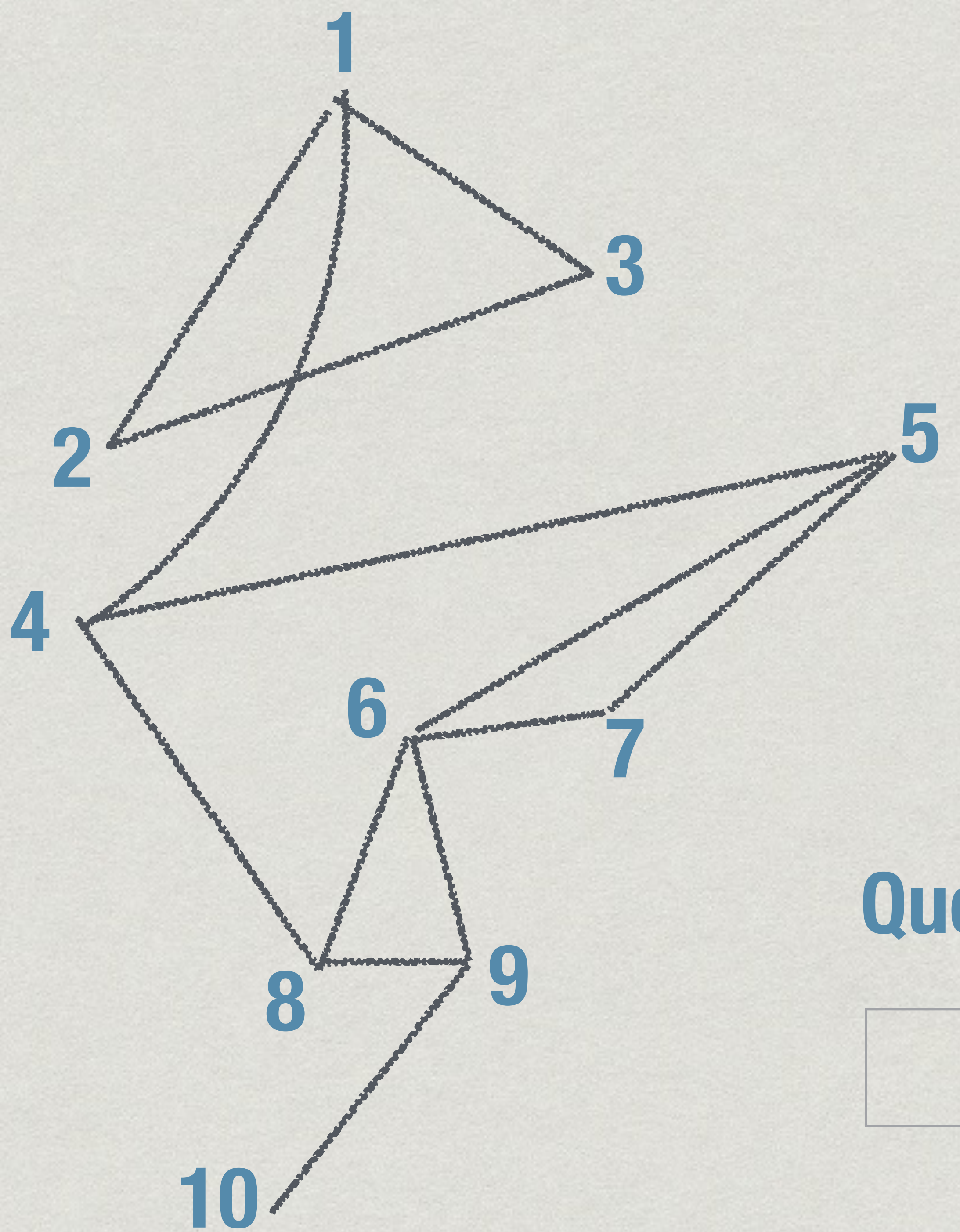
9	
10	

Queue

1									

Breadth first search



Visited

1	1
2	
3	
4	
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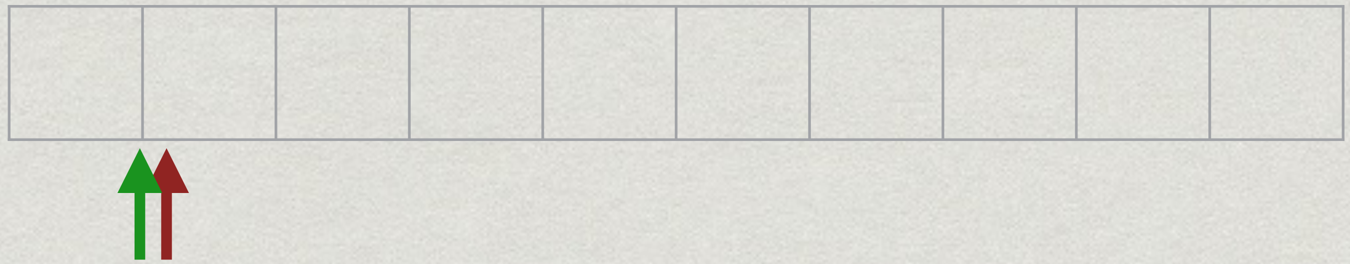
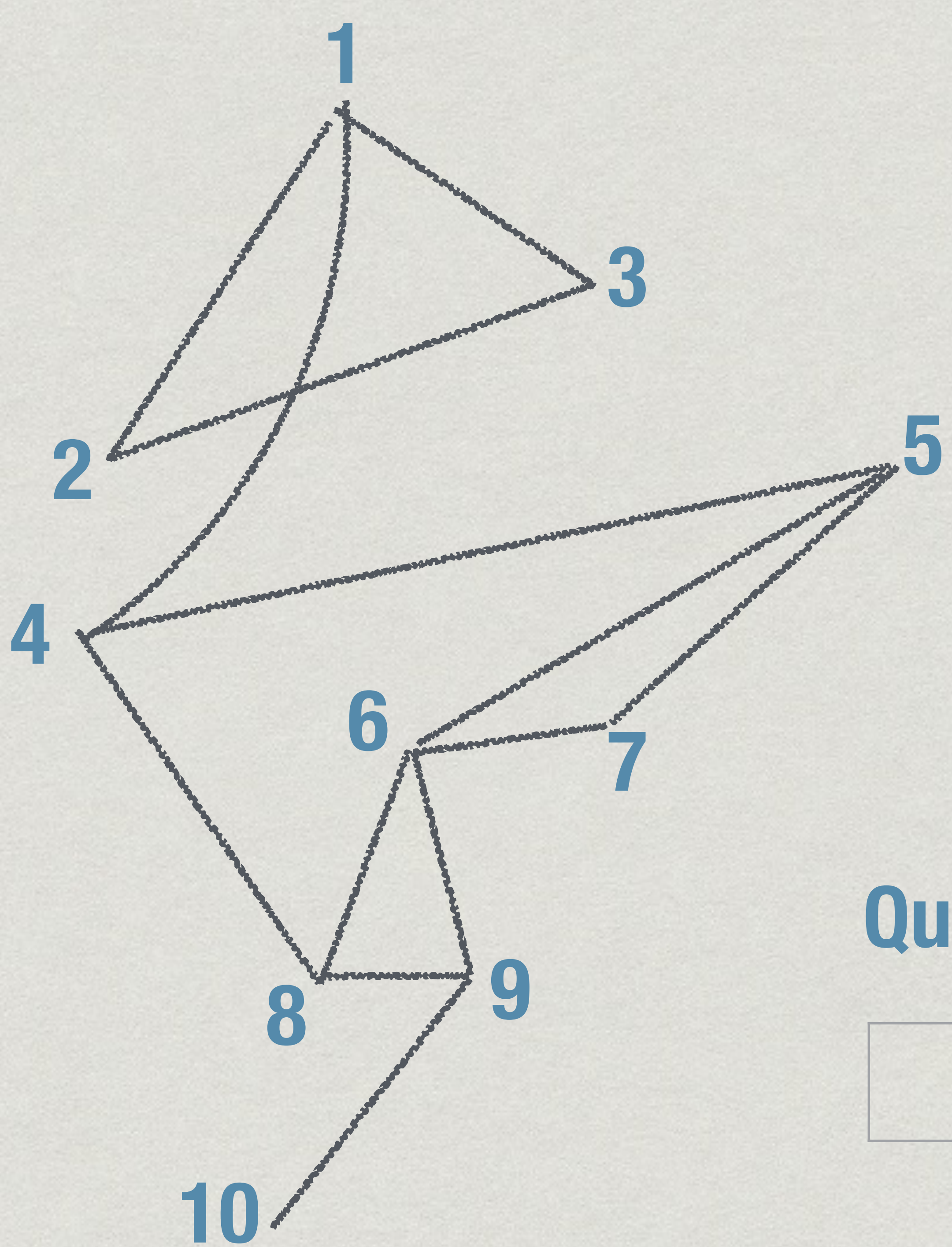
8	
9	
10	

Queue



Breadth first search



Visited

1	1
2	1
3	
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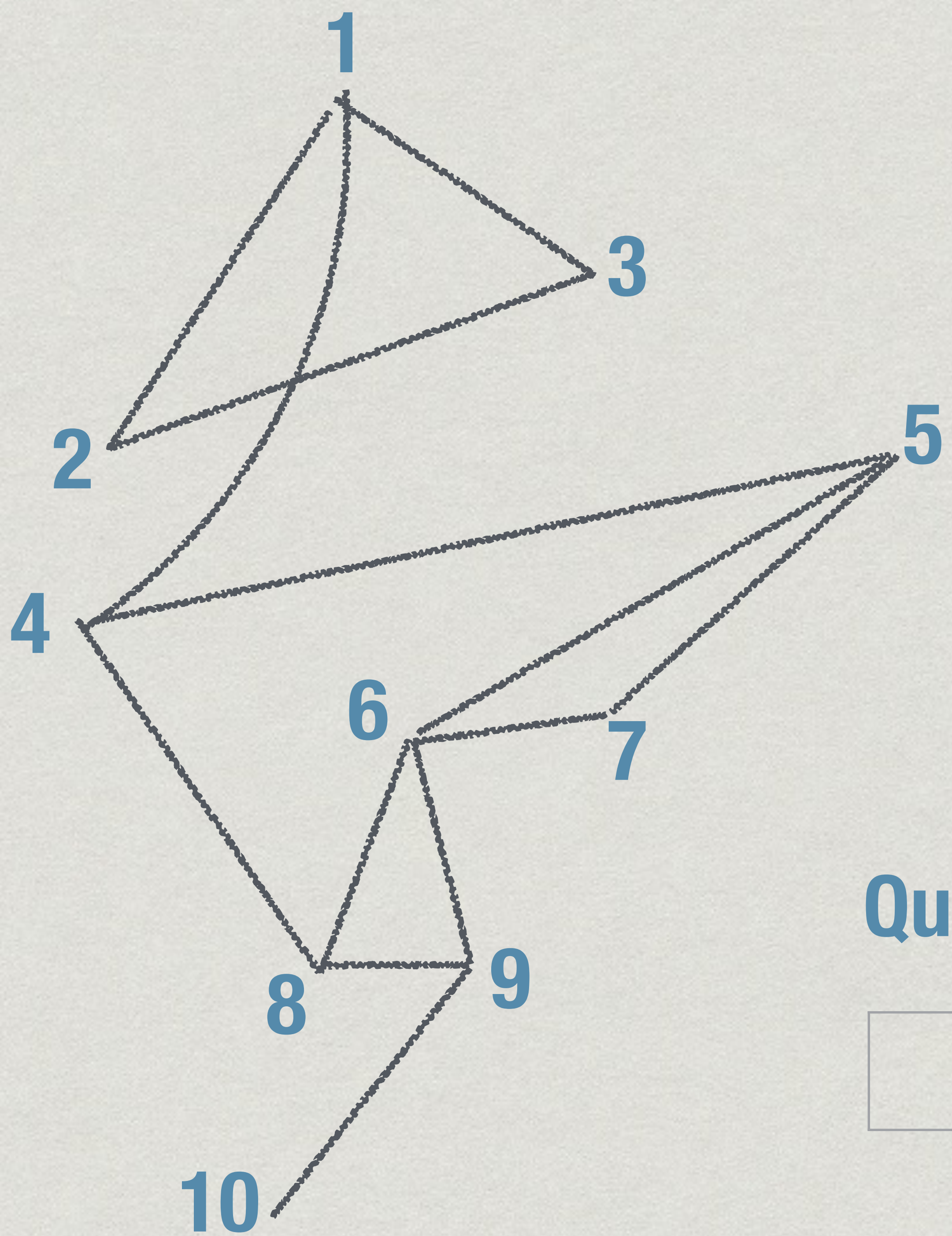
9	
10	

Queue

	2								

Breadth first search



Visited

1	1
2	1
3	1
4	
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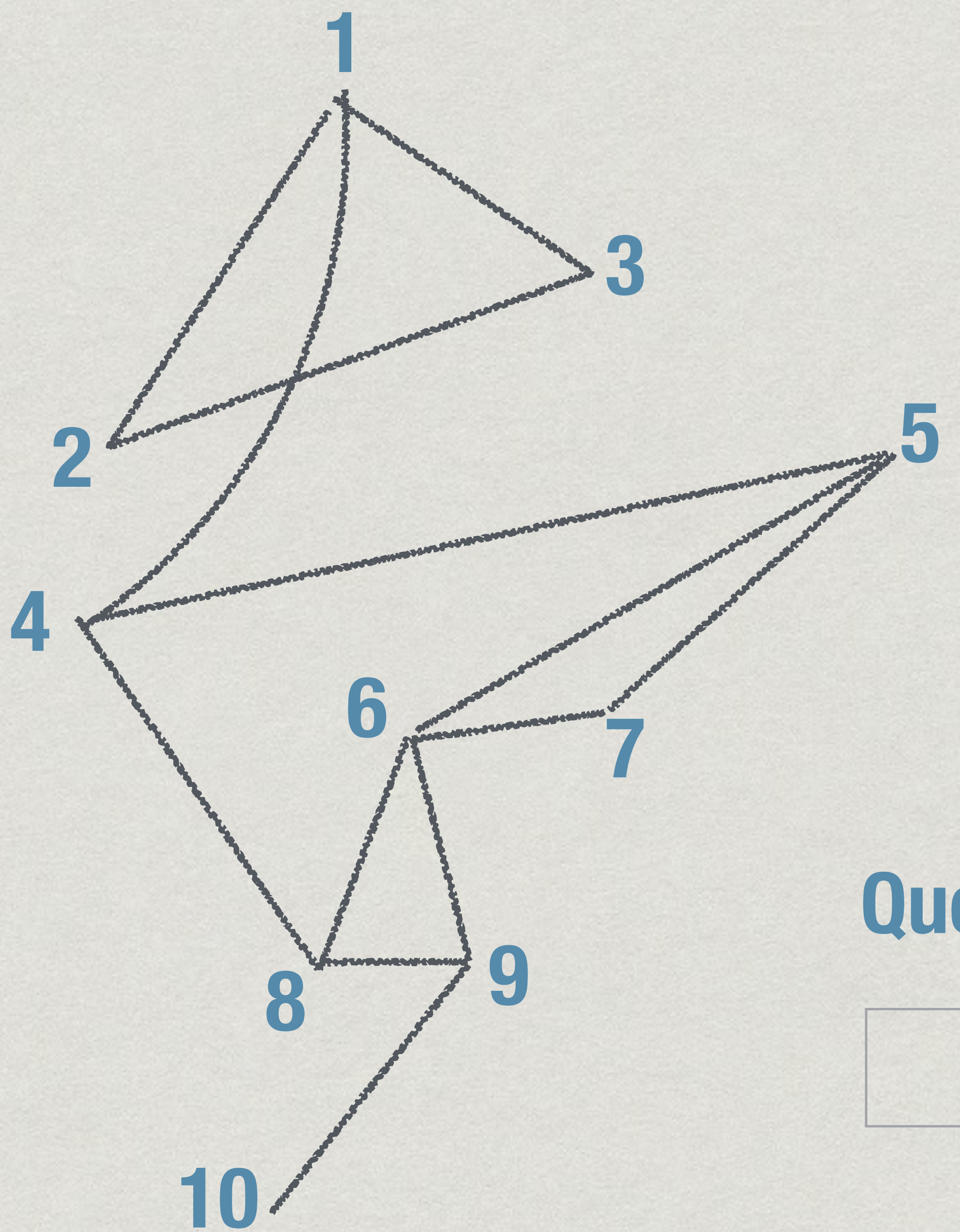
8	
9	
10	

Queue

	2	3							

Breadth first search



Visited

1	1
2	1
3	1
4	1
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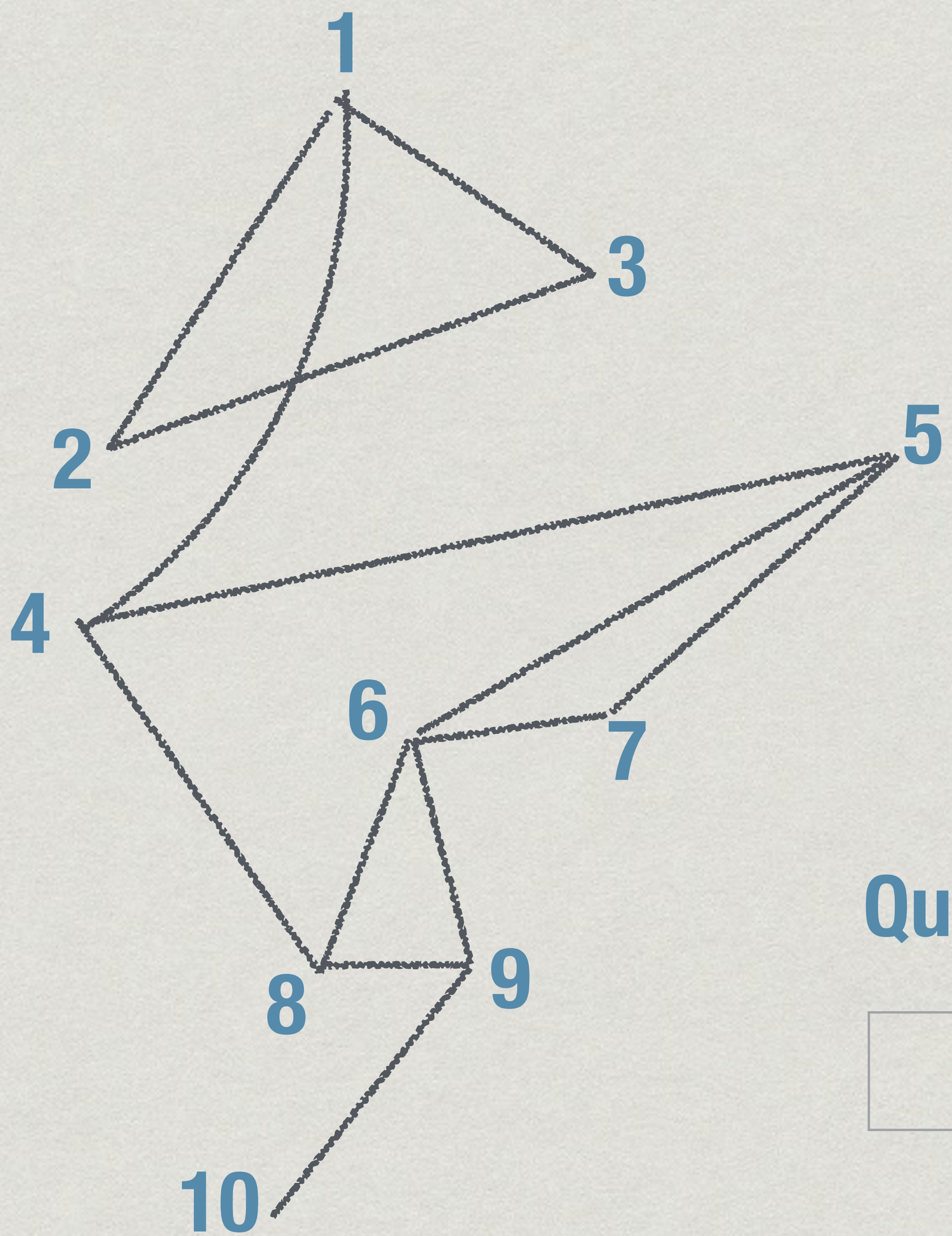
8	
9	
10	

Queue

	2	3	4						

Breadth first search



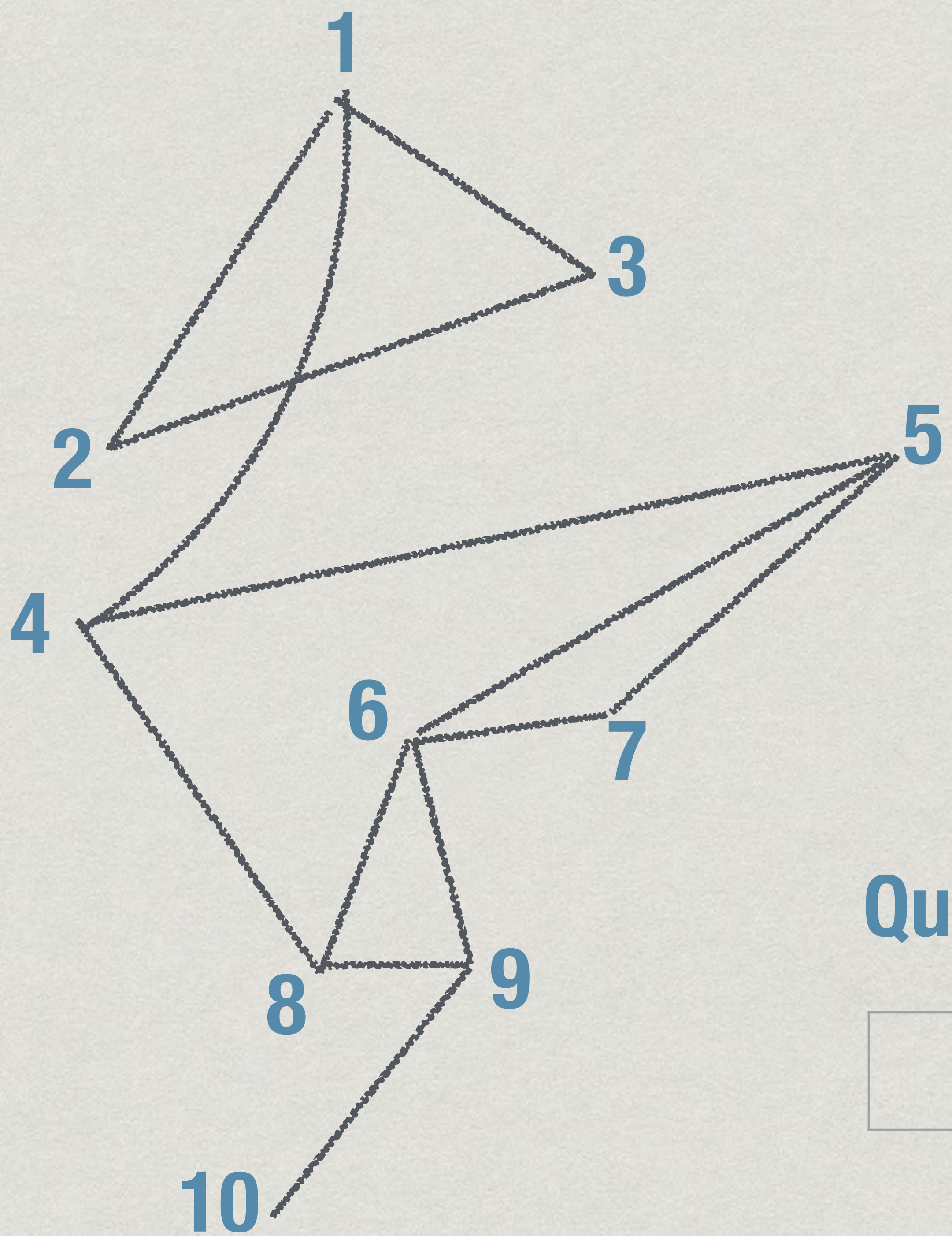
Visited

1	1
2	1
3	1
4	1
5	
6	
7	
8	
9	
10	

Queue

		3	4						

Breadth first search



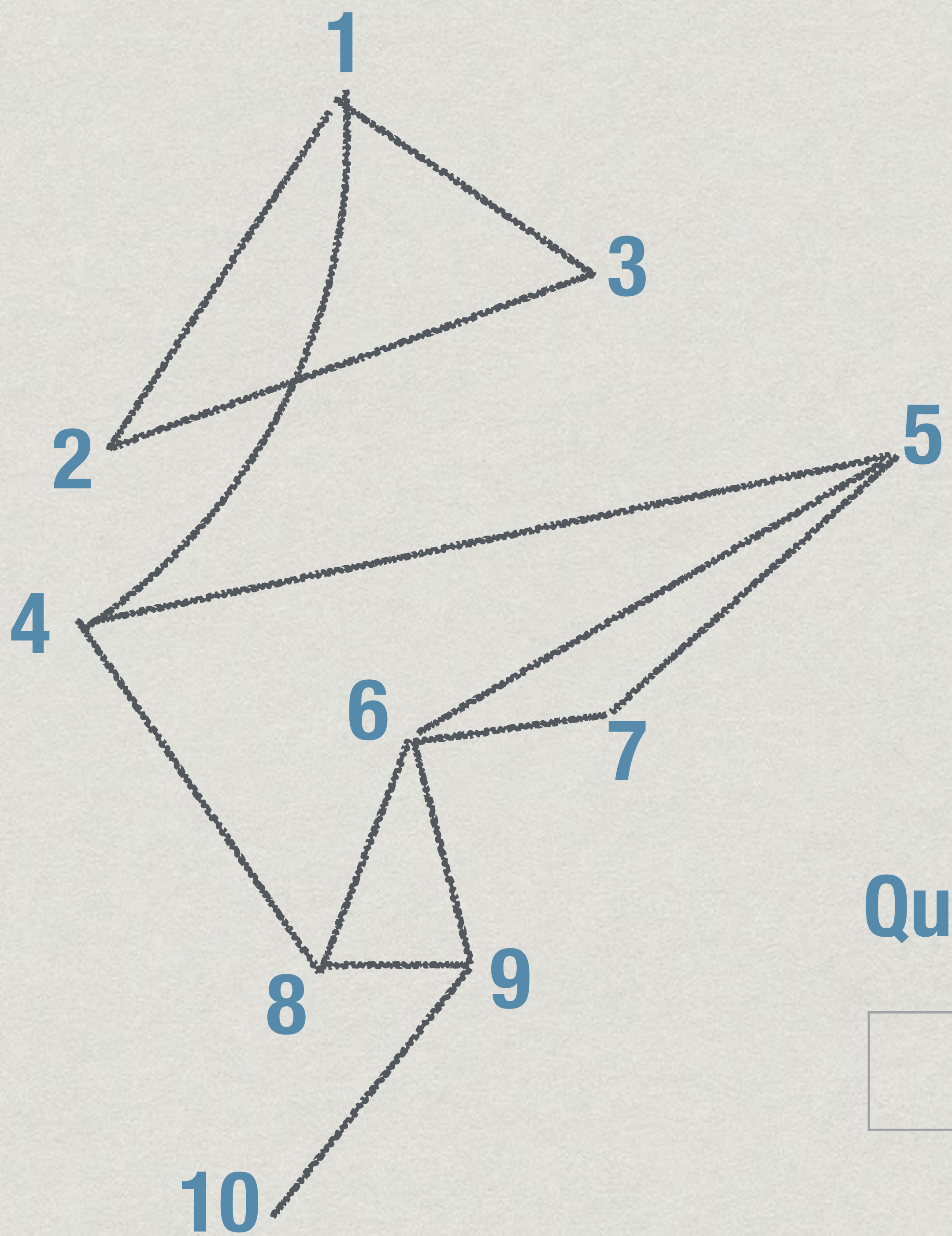
Visited

1	1
2	1
3	1
4	1
5	
6	
7	
8	
9	
10	

Queue

			4						

Breadth first search



Visited

1	1
2	1
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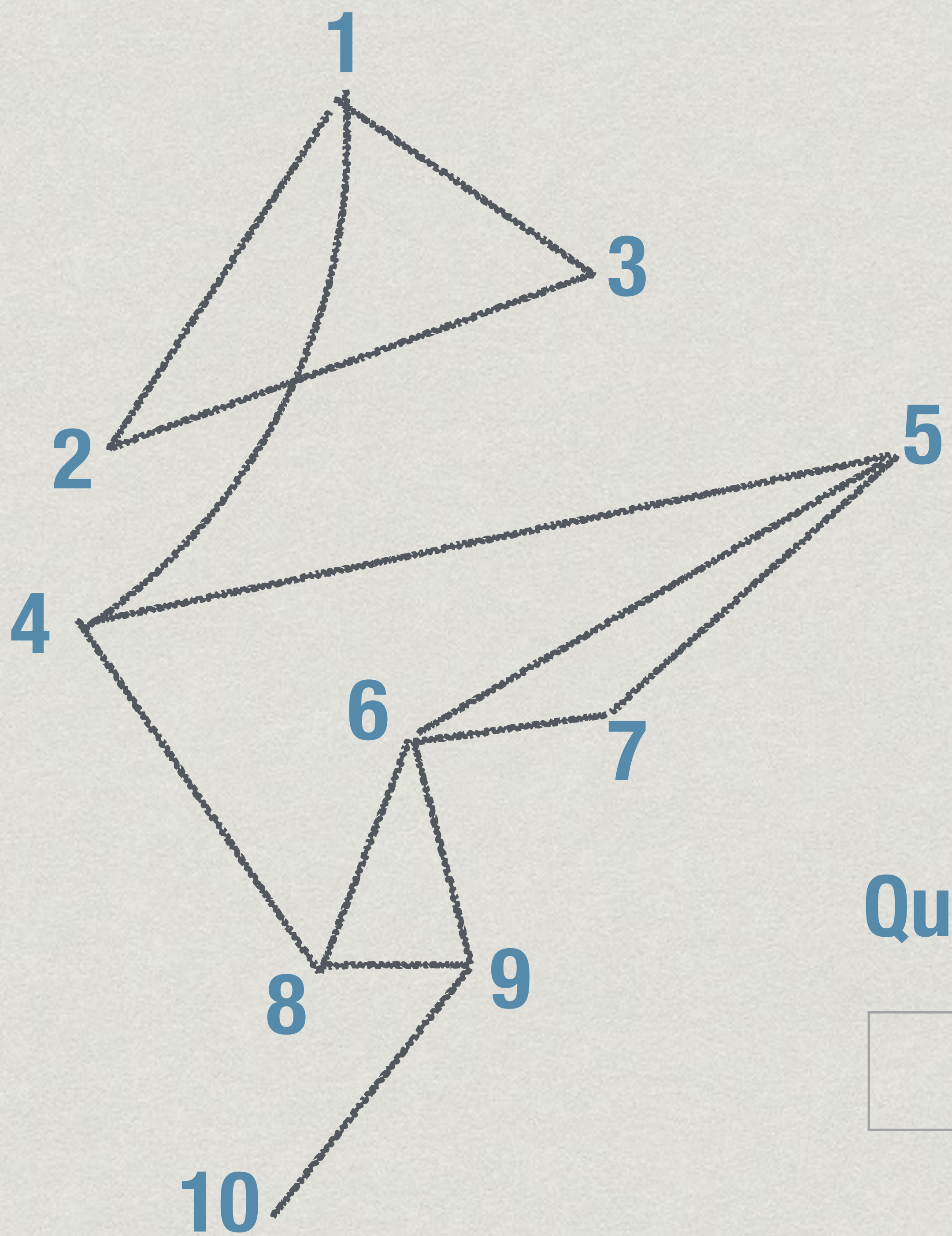
10	

Queue

Breadth first search



Visited

1	1
2	1
3	1
4	1
5	1
6	
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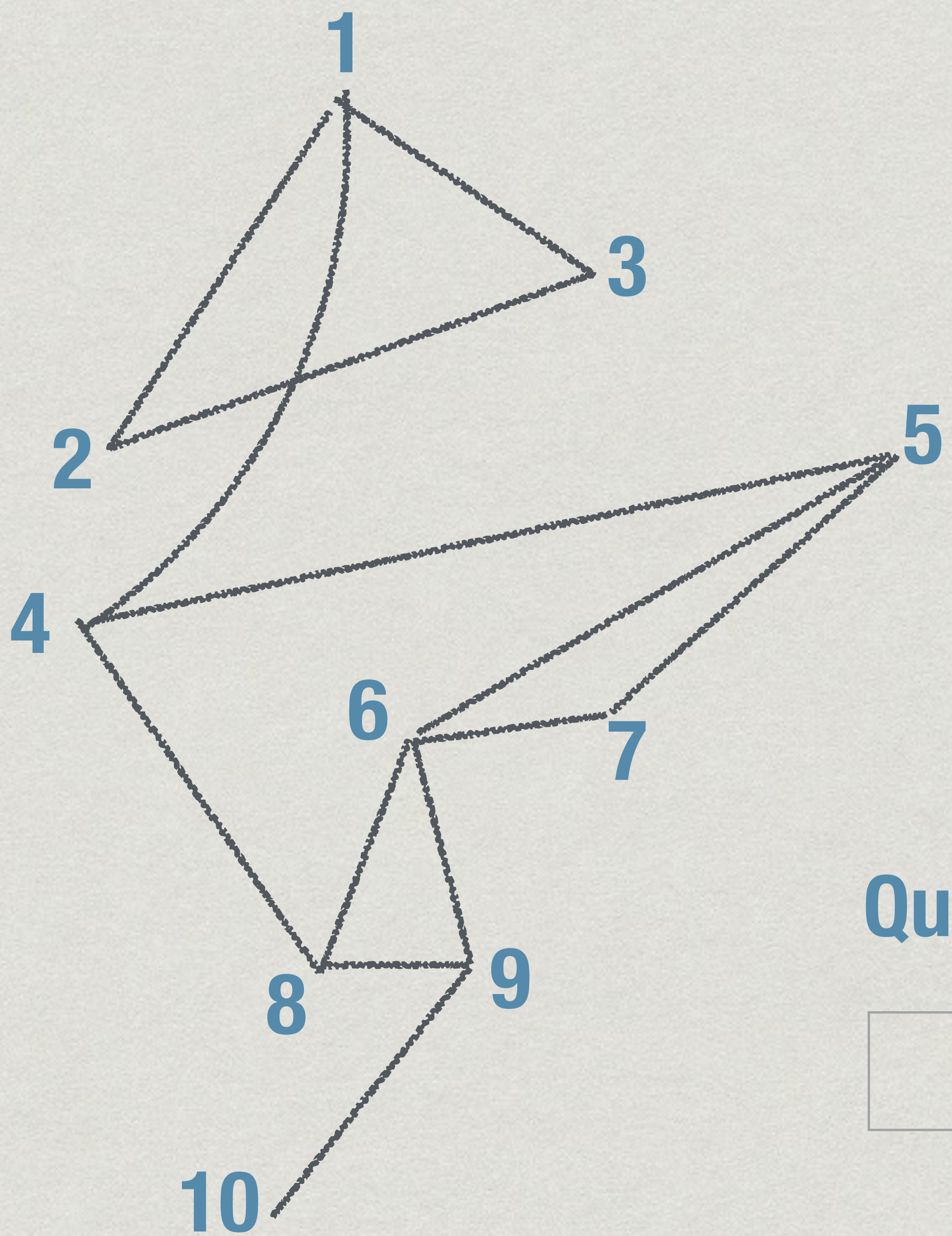
8	
9	
10	

Queue

				5					

Breadth first search



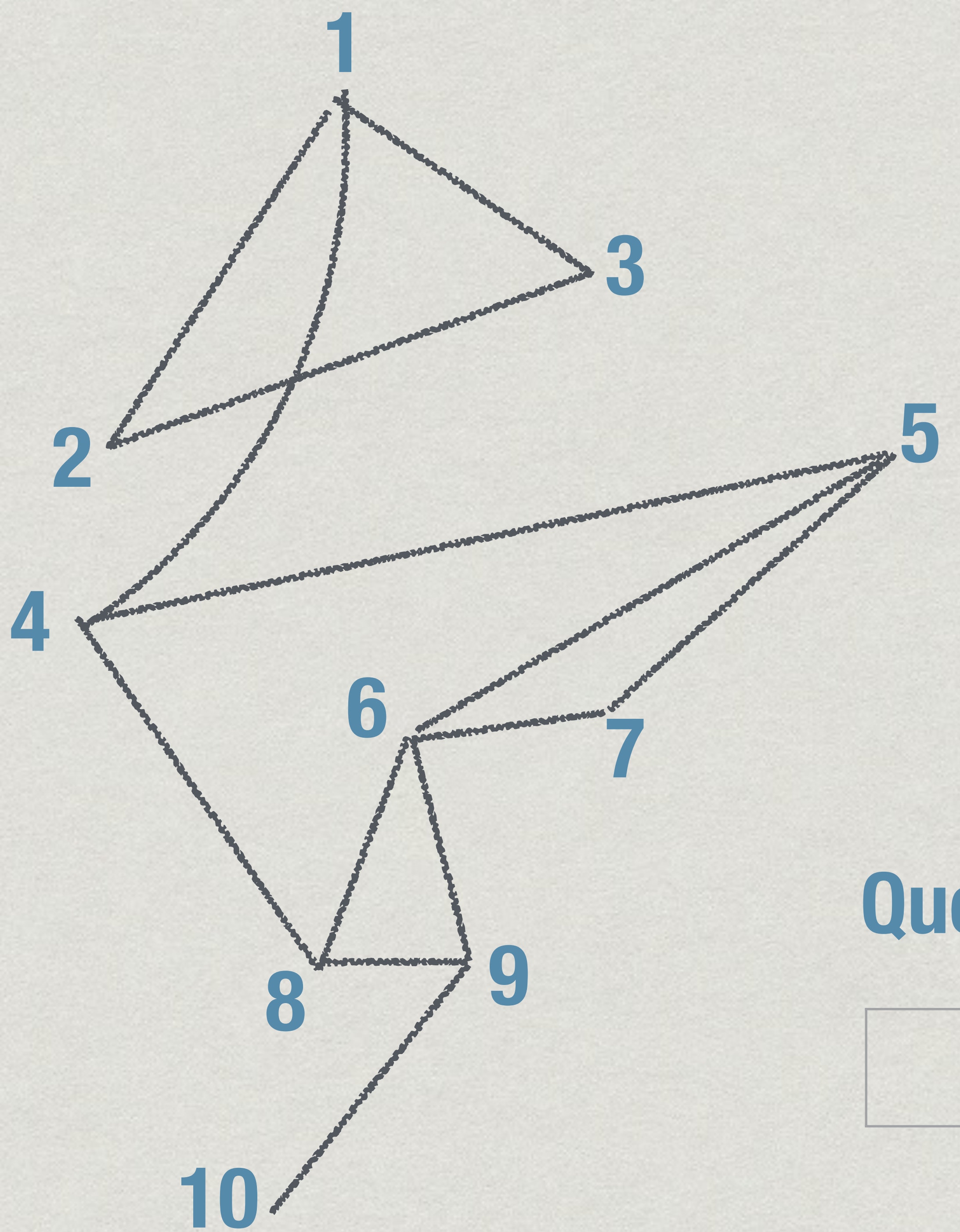
Visited

1	1
2	1
3	1
4	1
5	1
6	
7	
8	1
9	
10	

Queue

				5	8				

Breadth first search



Visited

1	1
2	1
3	1
4	1
5	1
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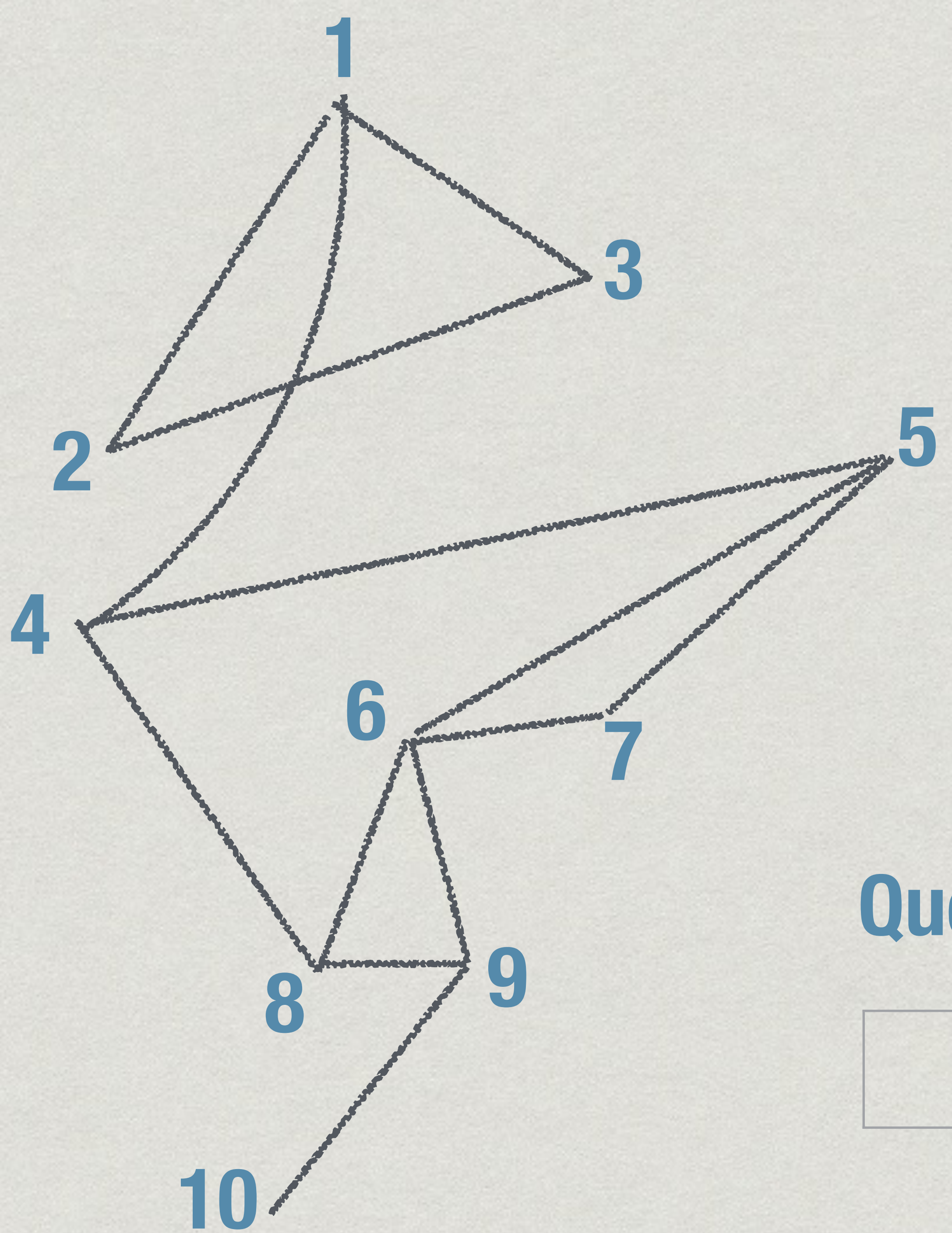
8	1
9	
10	

Queue

					8				

Breadth first search



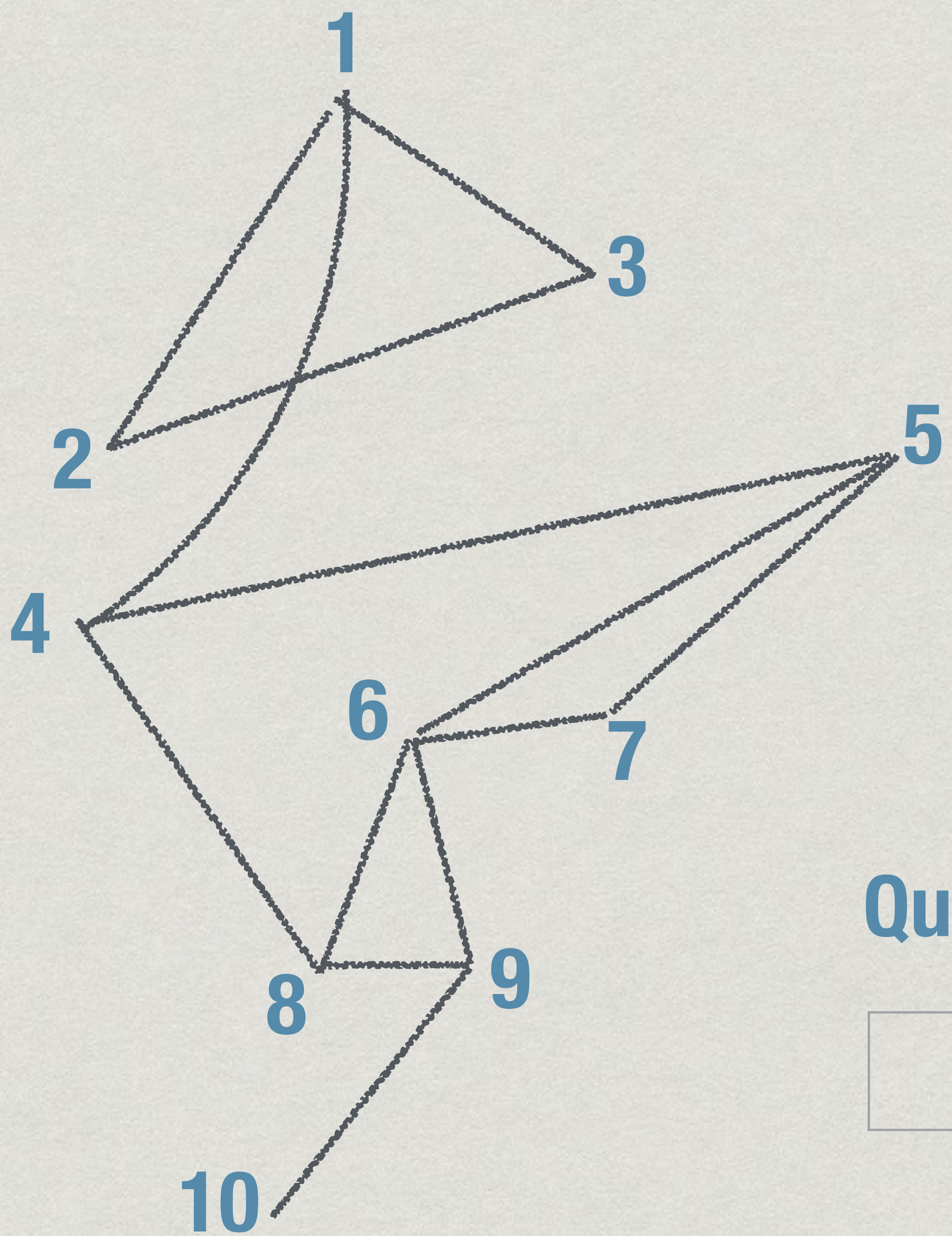
Visited

1	1
2	1
3	1
4	1
5	1
6	1
7	
8	1
9	
10	

Queue

					8	6			

Breadth first search



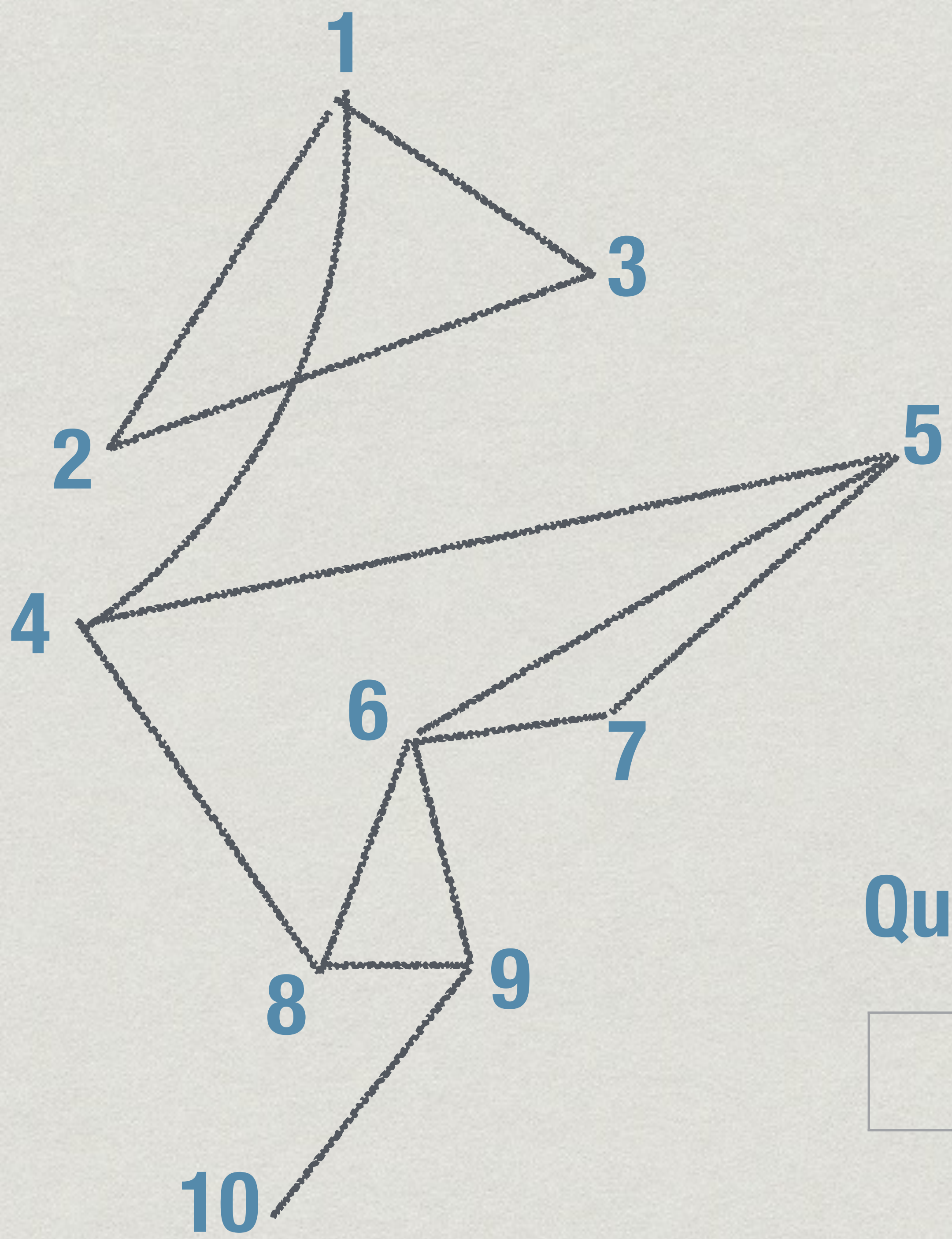
Visited

1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	1
9	
10	

Queue

					8	6	7		

Breadth first search



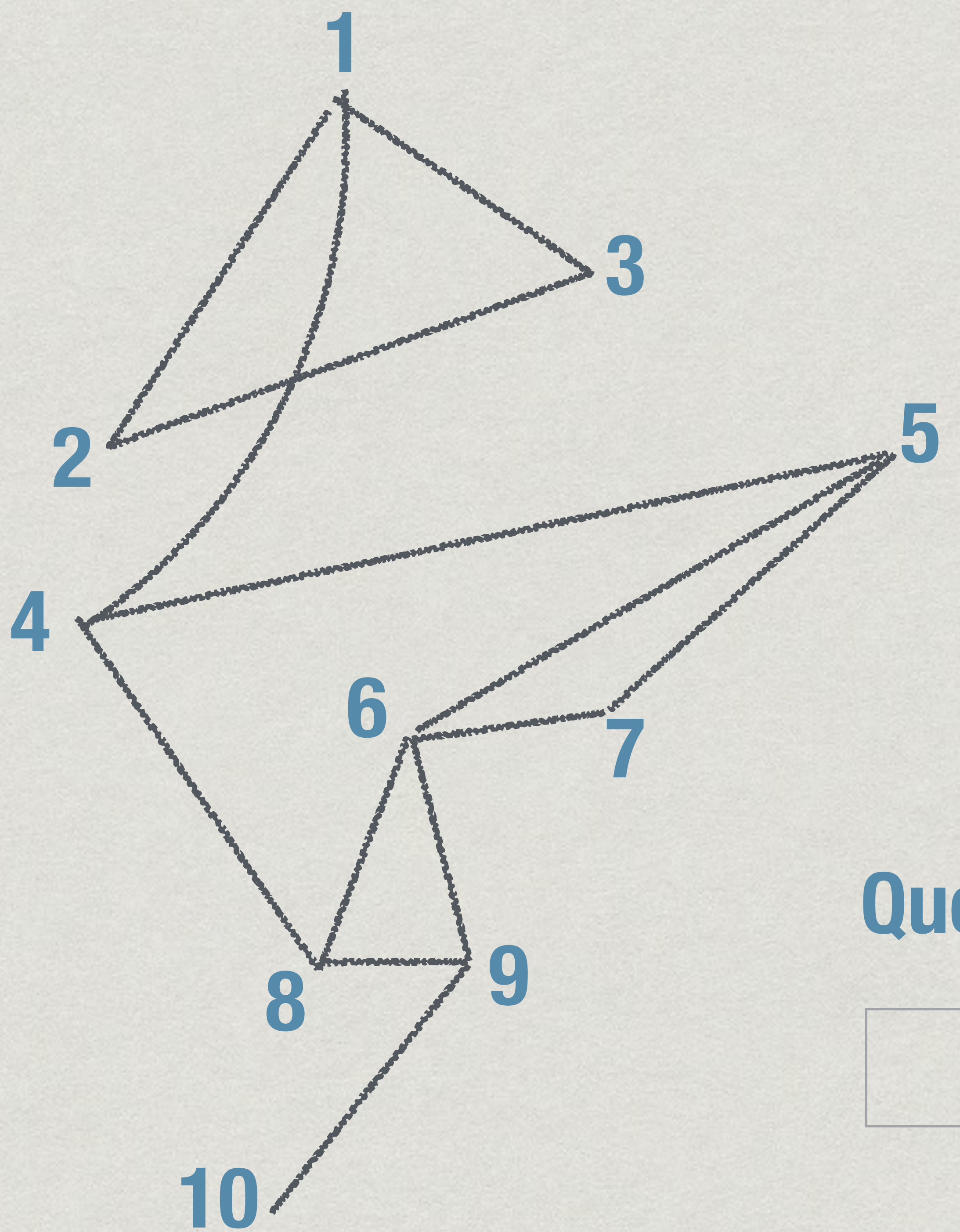
Visited

1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	1
9	
10	

Queue

						6	7		

Breadth first search



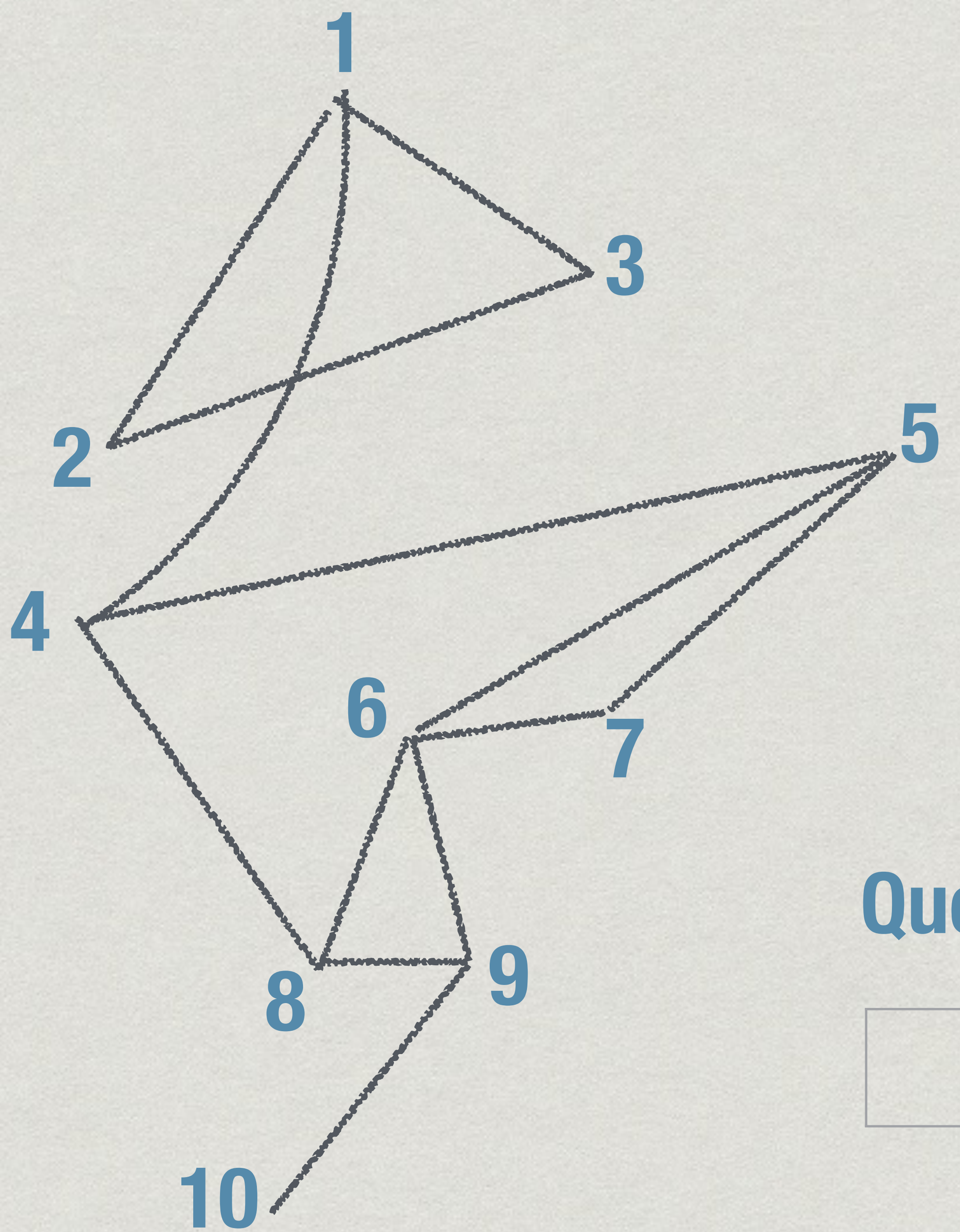
Visited

1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	1
9	1
10	

Queue

						6	7	9	

Breadth first search



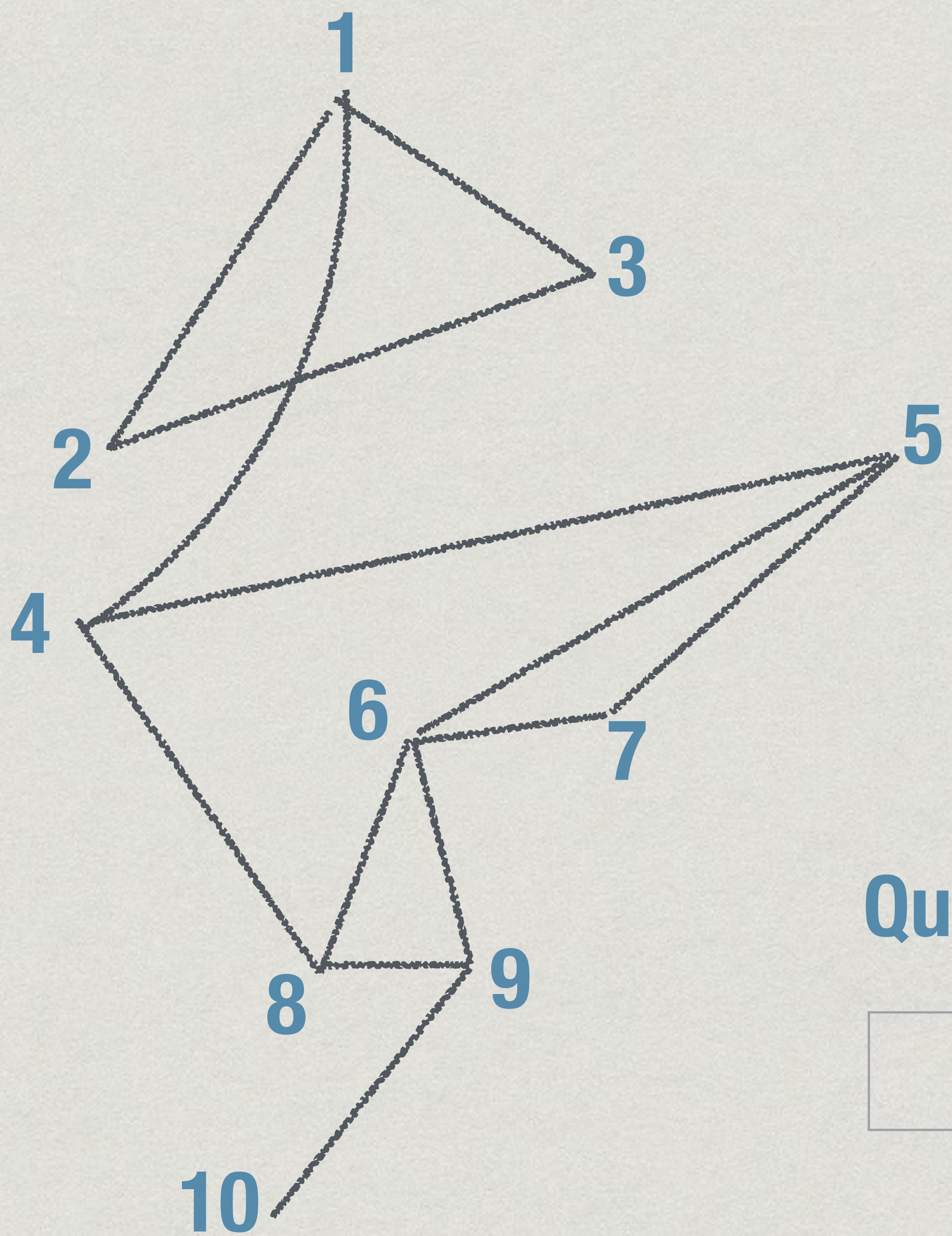
Visited

1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	1
9	1
10	

Queue

							7	9	

Breadth first search



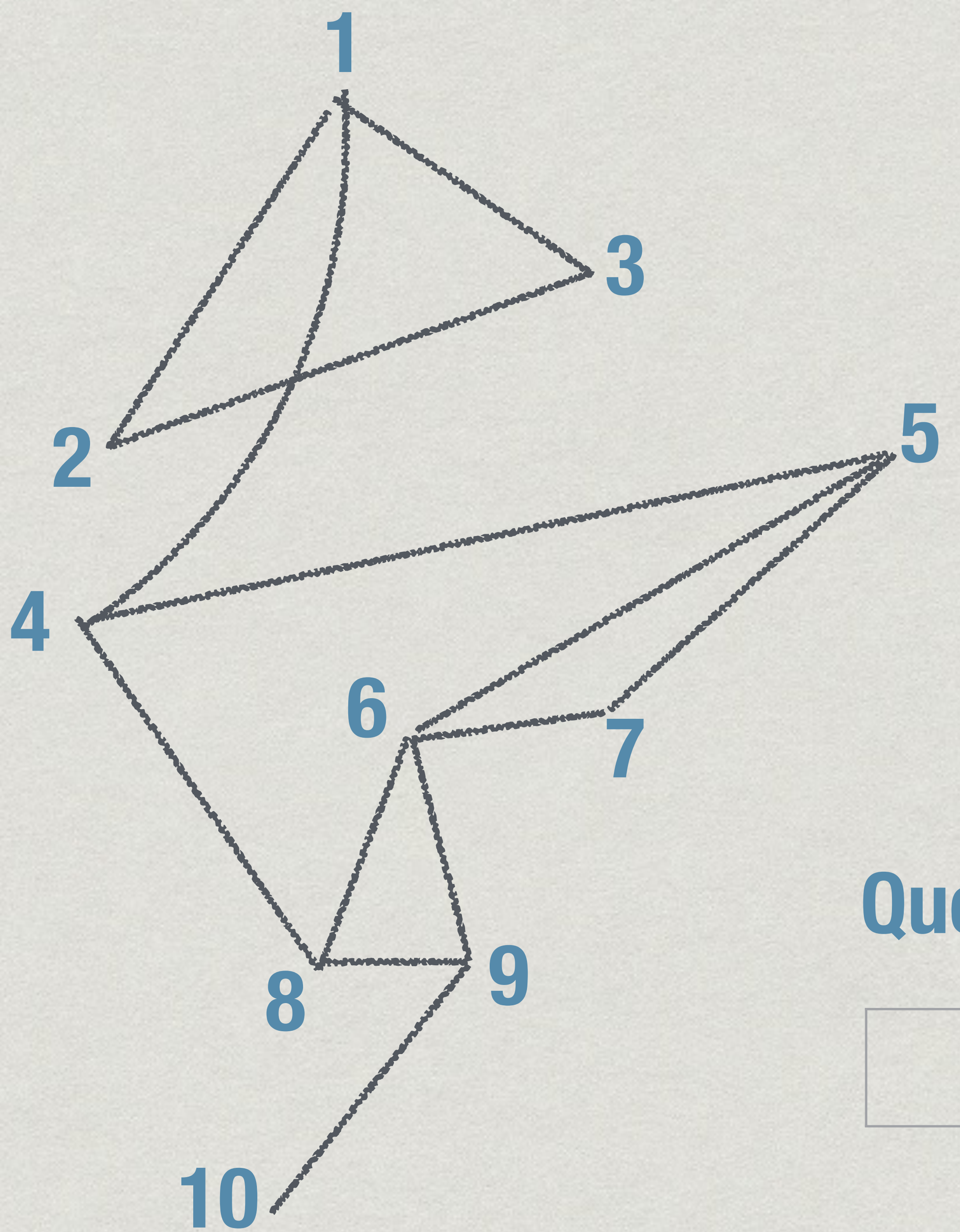
Visited

1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	1
9	1
10	

Queue

								9	

Breadth first search

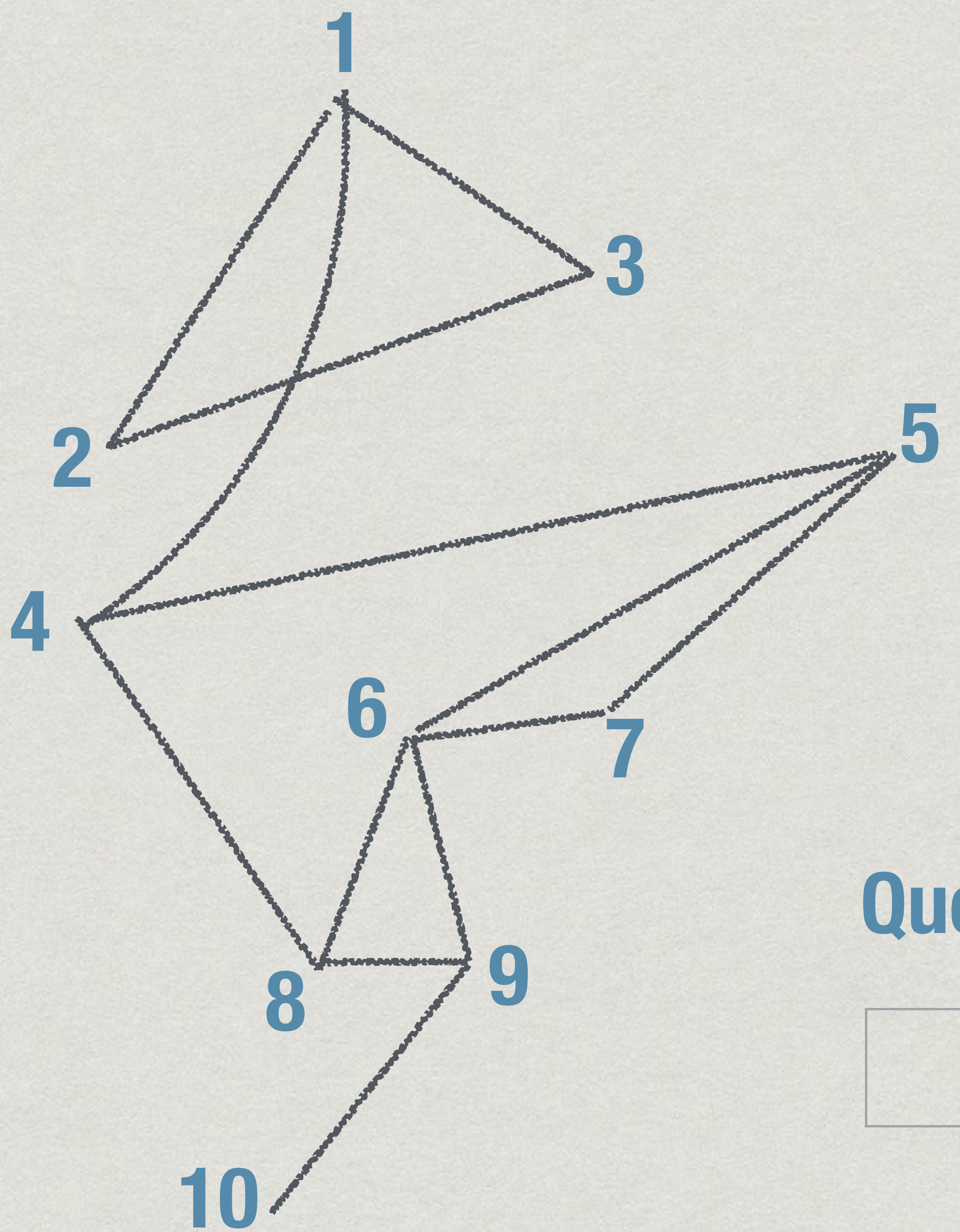


Visited

1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	1
9	1
10	

Queue

Breadth first search



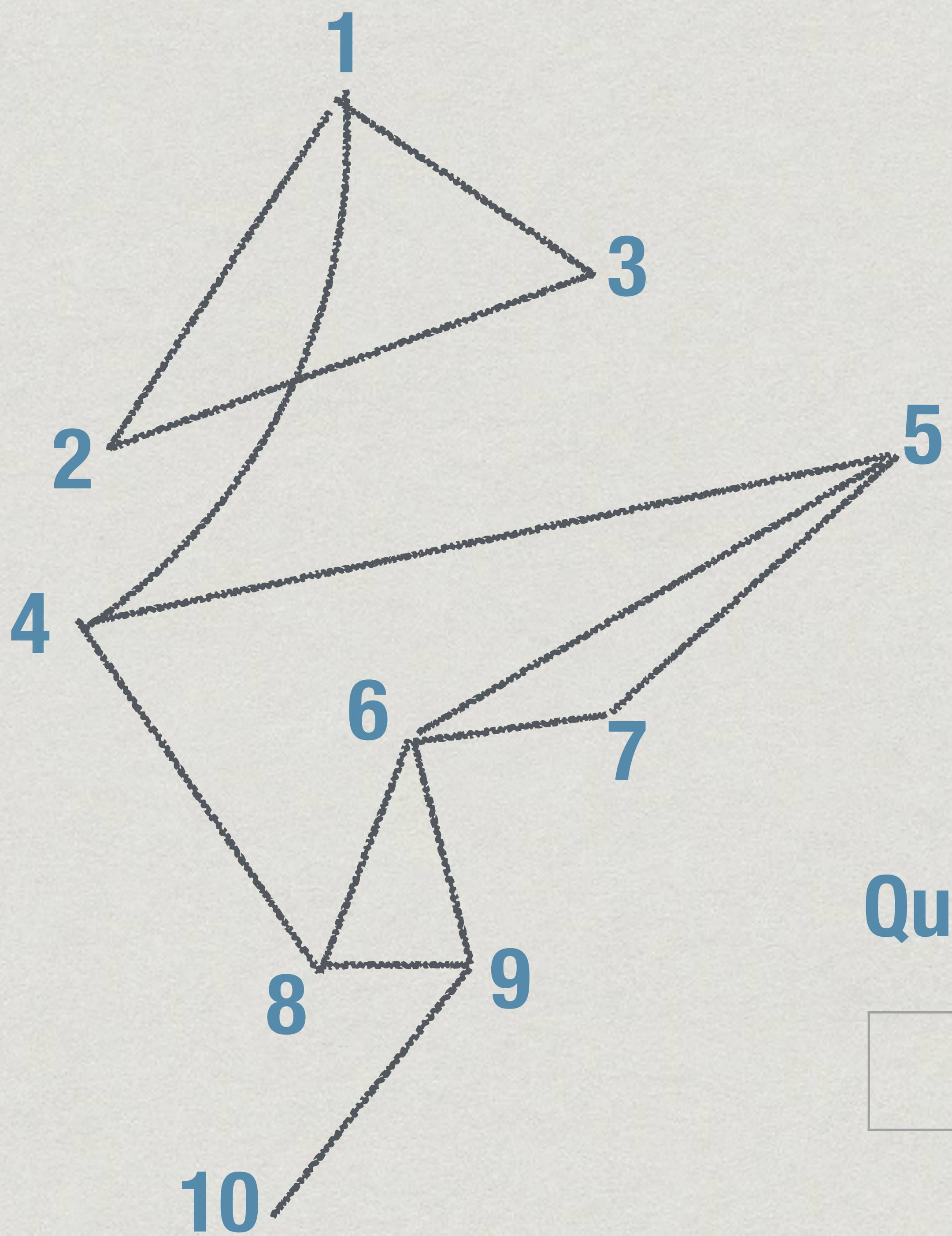
Visited

1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	1
9	1
10	1

Queue

									10

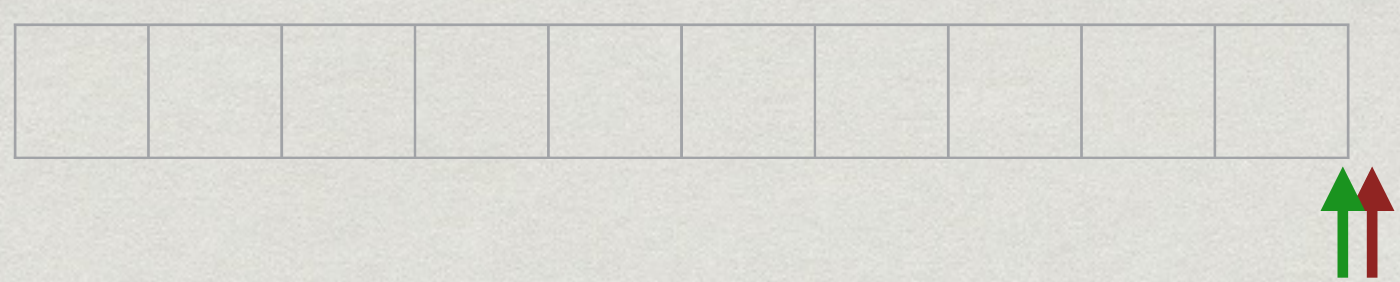
Breadth first search



Visited

1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	1
9	1
10	1

Queue



Breadth first search

```
function BFS(i) // BFS starting from vertex i

  //Initialization
  for j = 1..n {visited[j] = 0}; Q = []

  //Start the exploration at i
  visited[i] = 1; append(Q,i)

  //Explore each vertex in Q
  while Q is not empty
     j = extract_head(Q)
     for each (j,k) in E
        if visited[k] == 0
           visited[k] = 1; append(Q,k)
```

Complexity of BFS

- Each vertex enters Q exactly once
- If graph is connected, loop to process Q iterated n times
  - For each j extracted from Q, need to examine all neighbours of j
  - In adjacency matrix, scan row j: n entries
- Hence, overall $O(n^2)$

Complexity of BFS

- Let m be the number of edges in E. What if m << n^2?
- Adjacency list: scanning neighbours of j takes time proportional to number of neighbours (**degree** of j)
- Across the loop, each edge (i,j) is scanned twice, once when exploring i and again when exploring j
  - Overall, exploring neighbours takes time O(m)
- Marking n vertices visited still takes O(n)
- Overall, O(n+m)

Complexity of BFS

- For graphs, O(m+n) is considered the best possible
  - Need to see each edge and vertex at least once
- O(m+n) is considered to be **linear** in the size of the graph

Enhancements to BFS

- If BFS(i) sets visited[j] = 1, we know that i and j are connected
- How do we identify a path from i to j
- When we mark visited[k] = 1, remember the neighbour from which we marked it
  - If exploring edge (j,k) visits k, set parent[k] = j

Breadth first search

```
function BFS(i) // BFS starting from vertex i

    //Initialization
    for j = 1..n {visited[j] = 0; parent[j] = -1}
    Q = []

    //Start the exploration at i
    visited[i] = 1; append(Q,i)

    //Explore each vertex in Q
    while Q is not empty
        j = extract_head(Q)
        for each (j,k) in E
            if visited[k] == 0
                visited[k] = 1; parent[k] = j; append(Q,k);
```

Reconstructing the path

- BFS(i) sets visited[j] = 1
- visited[j] = 1, so parent[j] = j’ for some j’
- visited[j’] = 1, so parent[j’] = j” for some j”
- …
- Eventually, trace back path to k with parent[k] = i

Recording distances

- BFS can record how long the path is to each vertex
- Instead of binary array visited[ ], keep integer array level[ ]
- level[j] = -1 initially
- level[j] = p means j is reached in p steps from i

Breadth first search

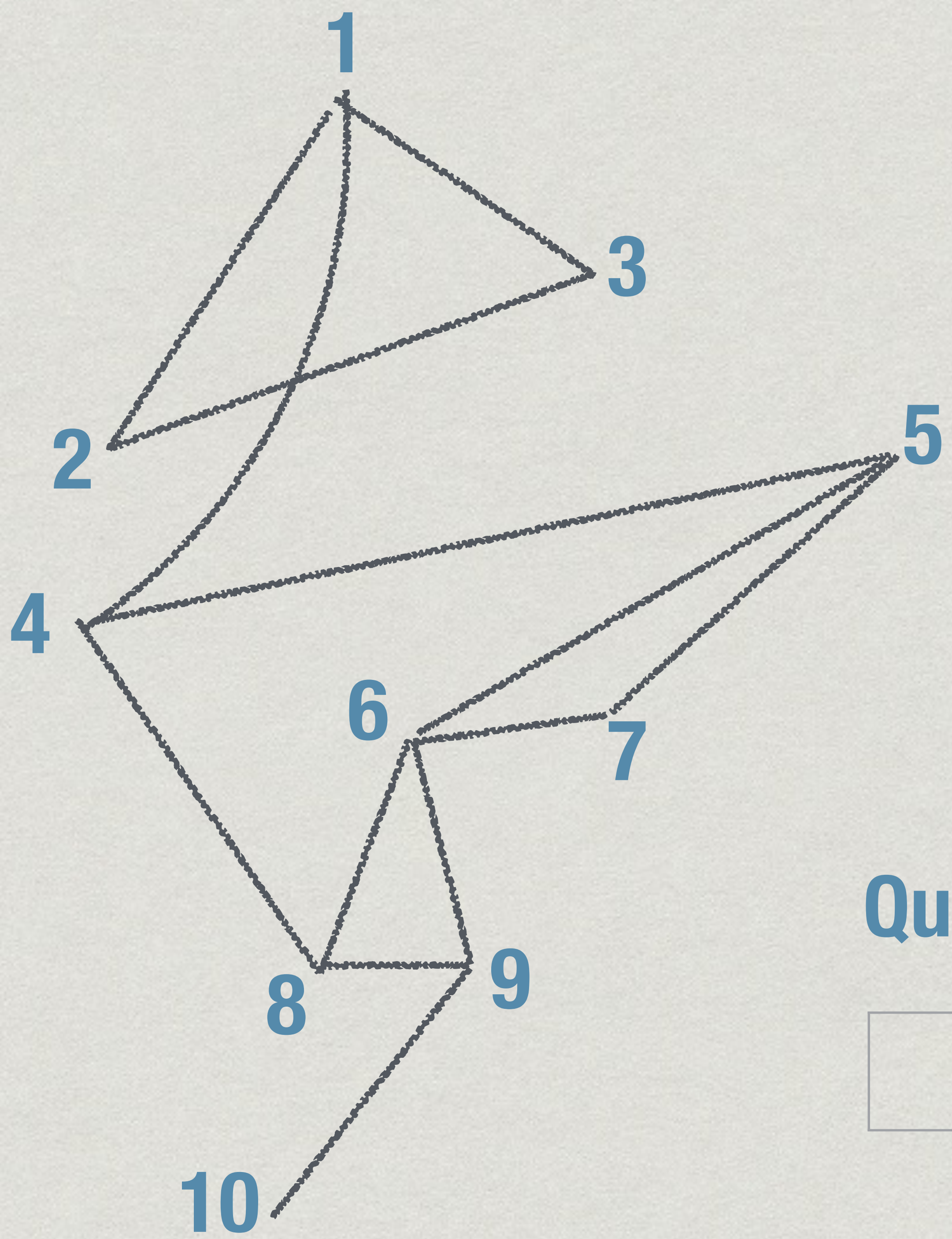
```
function BFS(i) // BFS starting from vertex i

    //Initialization
    for j = 1..n {level[j] = -1; parent[j] = -1}
    Q = []

    //Start the exploration at i, level[i] set to 0
    level[i] = 0; append(Q,i)

    //Explore each vertex in Q, increment level for each new vertex
    while Q is not empty
        j = extract_head(Q)
        for each (j,k) in E
            if level[k] == -1
                level[k] = 1+level[j]; parent[k] = j;
                append(Q,k);
```

Breadth first search



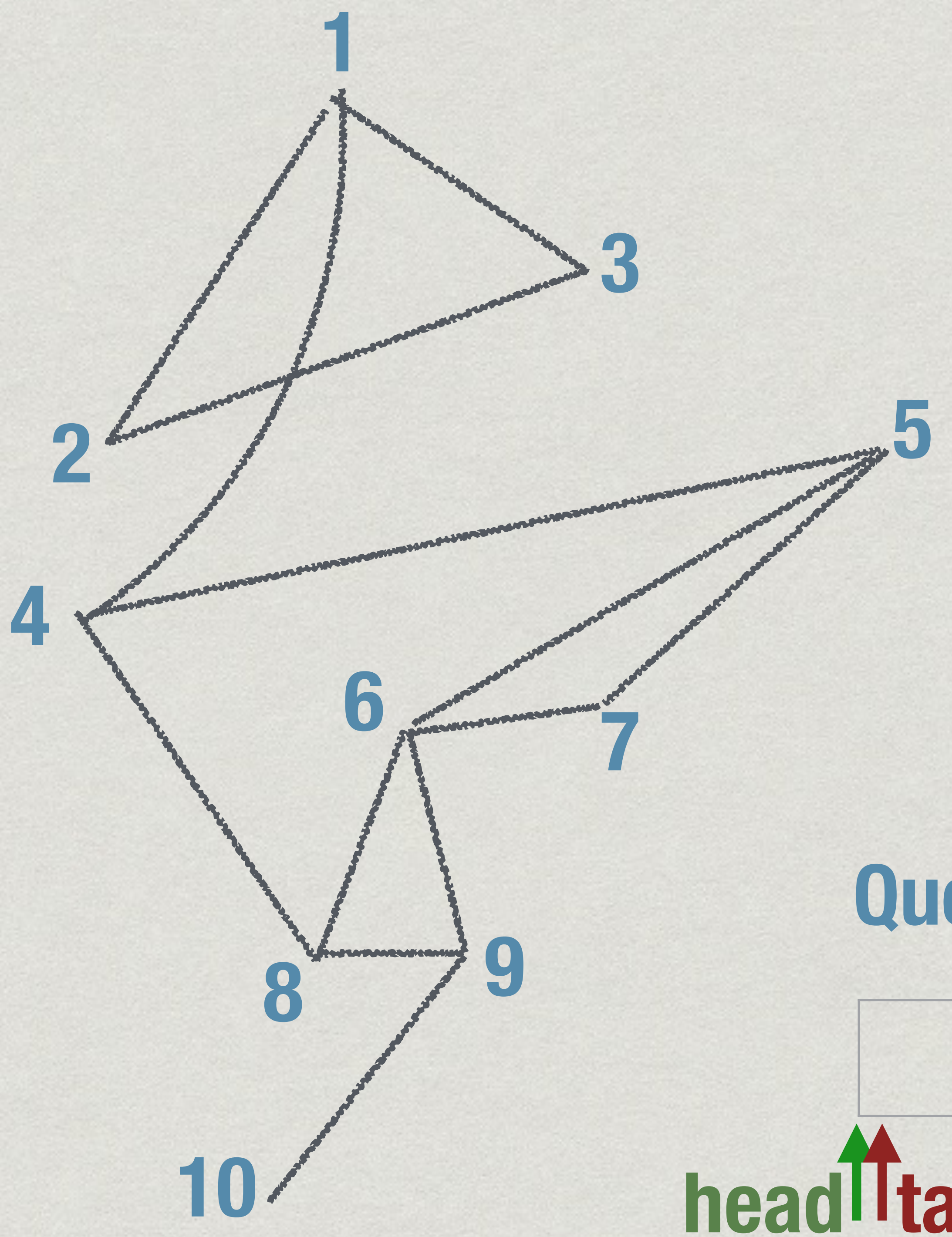
L : Level

P : Parent

	L	P
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		

Queue

Breadth first search

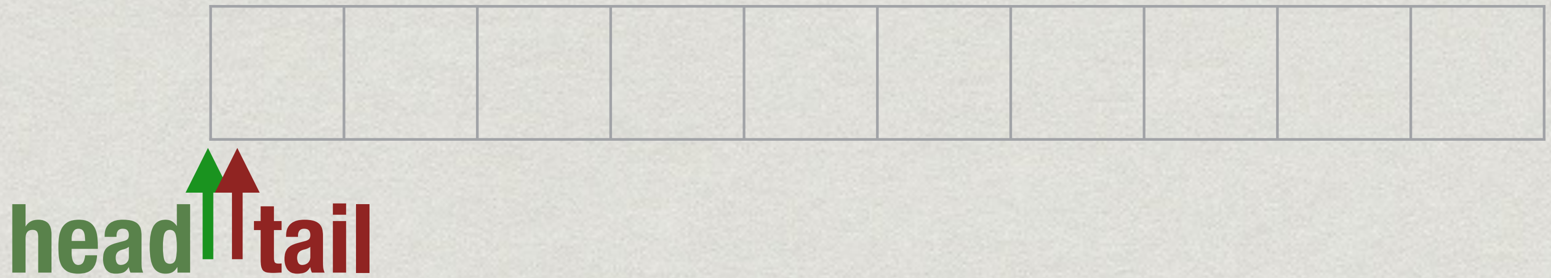


L : Level

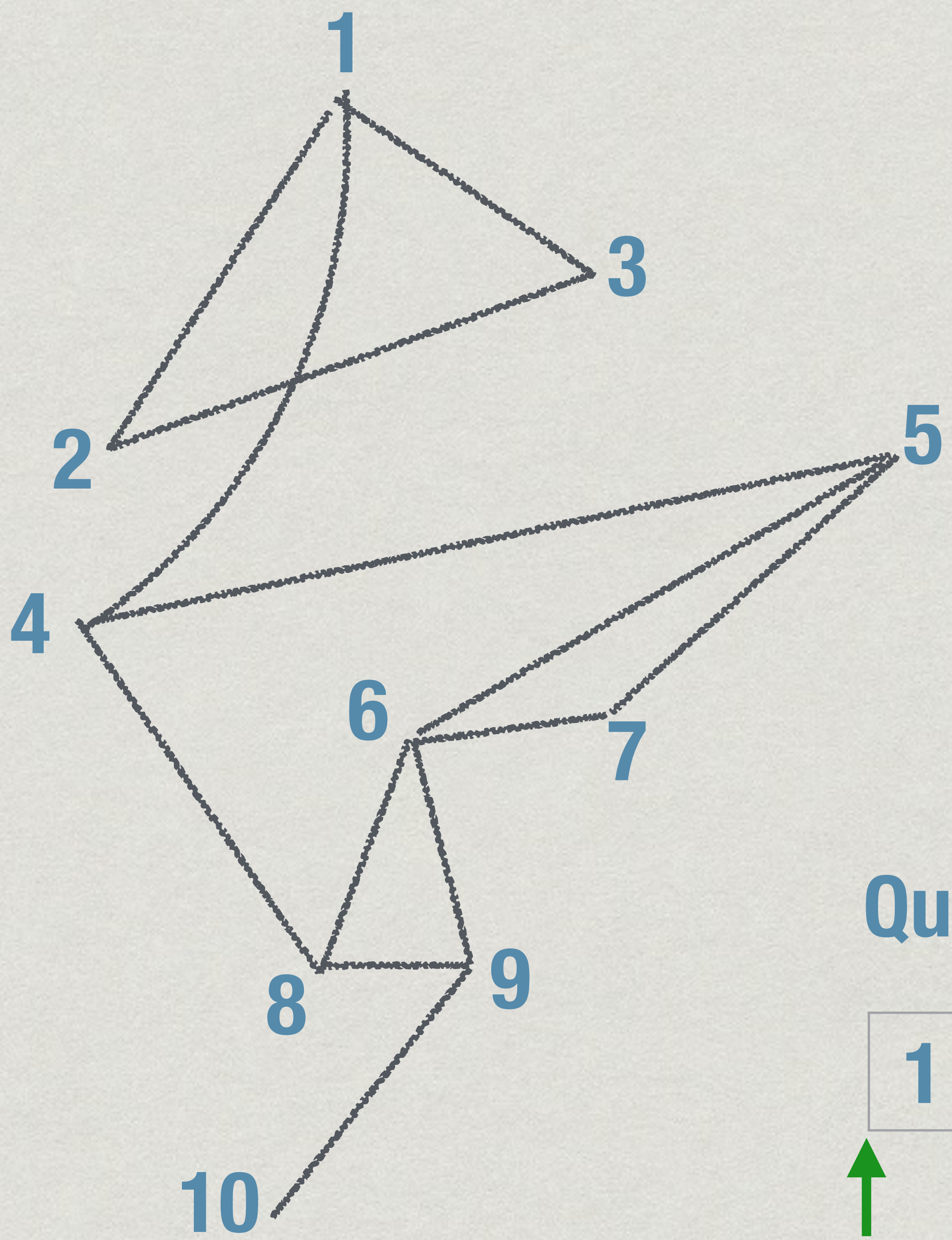
P : Parent

	L	P
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		

Queue



Breadth first search



L : Level

P : Parent

	L	P
1	0	-
2		
3		
4		
5		
6		
7		
8		
9		
10		

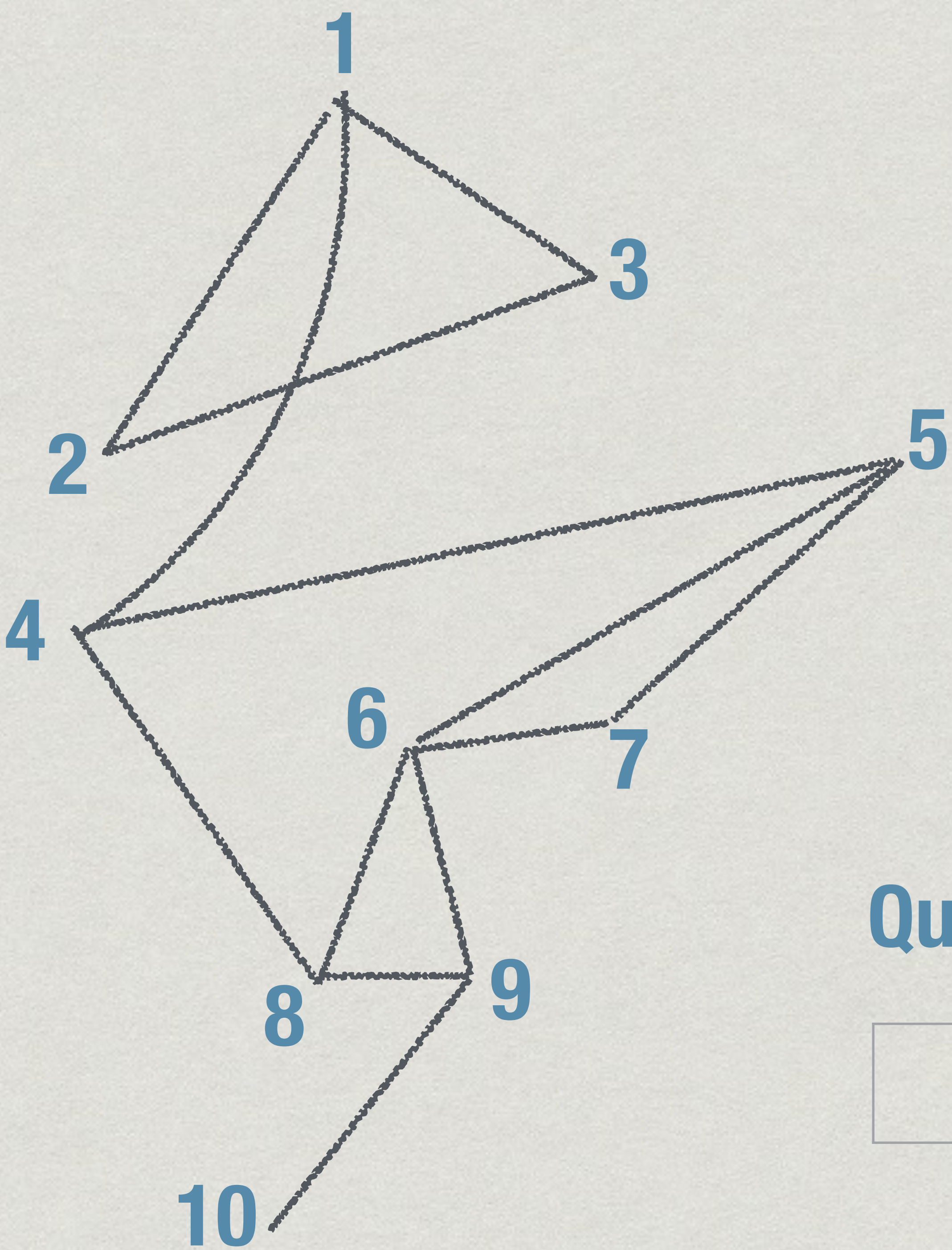
Queue

1									

Breadth first search

L : Level

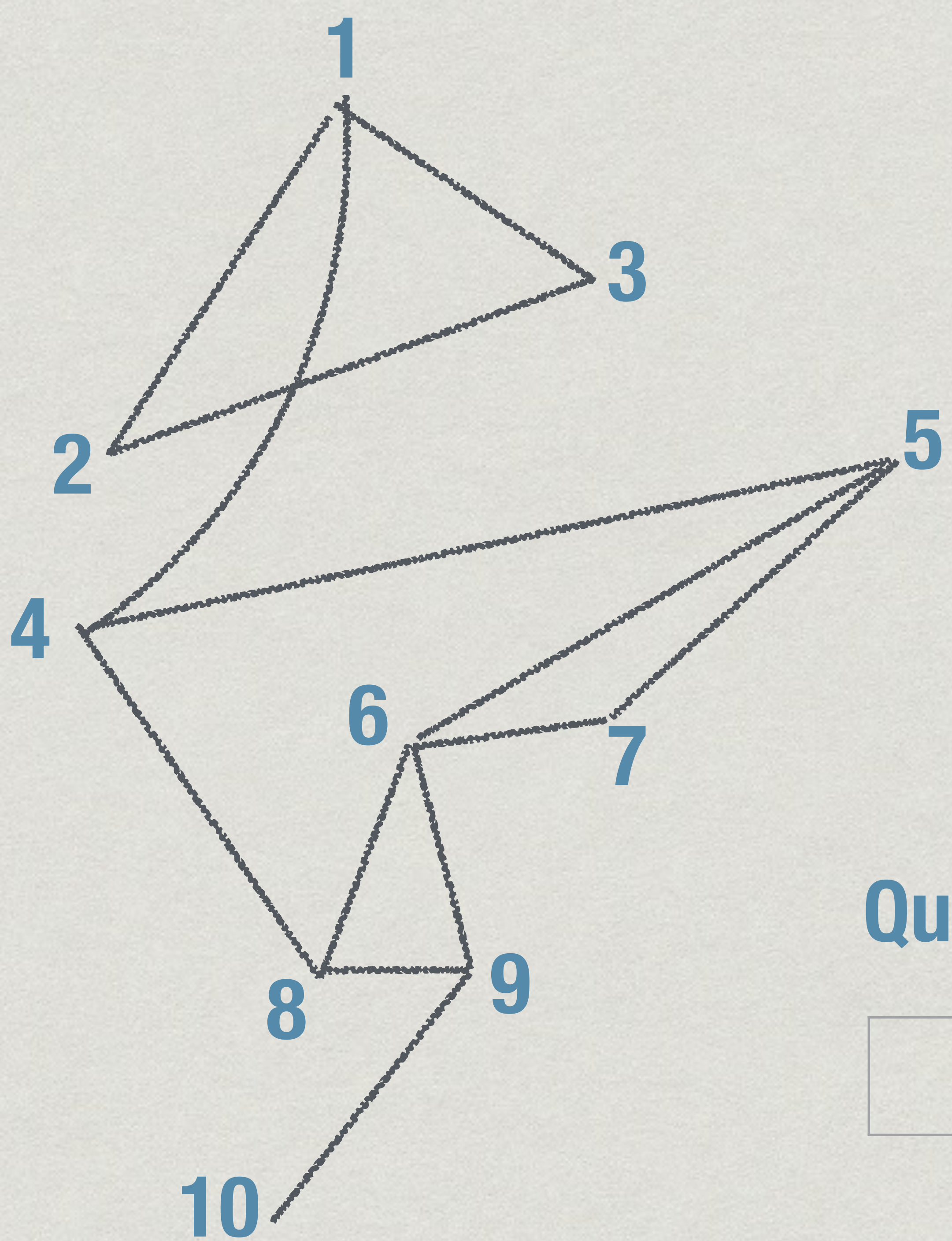
P : Parent



	L	P
1	0	-
2		
3		
4		
5		
6		
7		
8		
9		
10		

Queue

Breadth first search

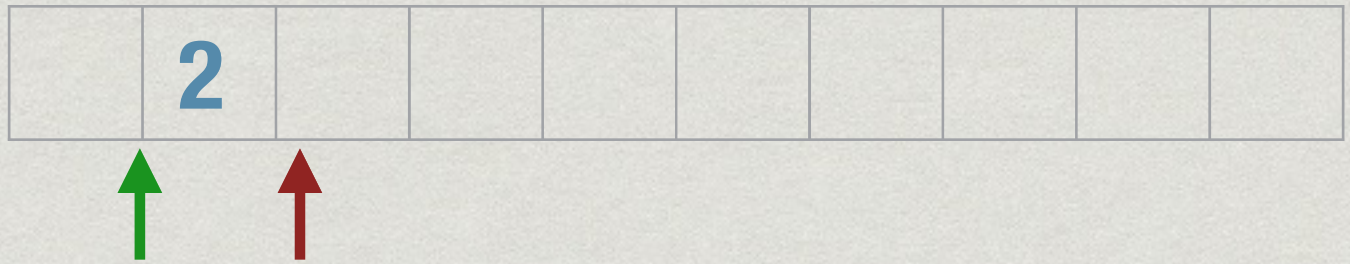


L : Level

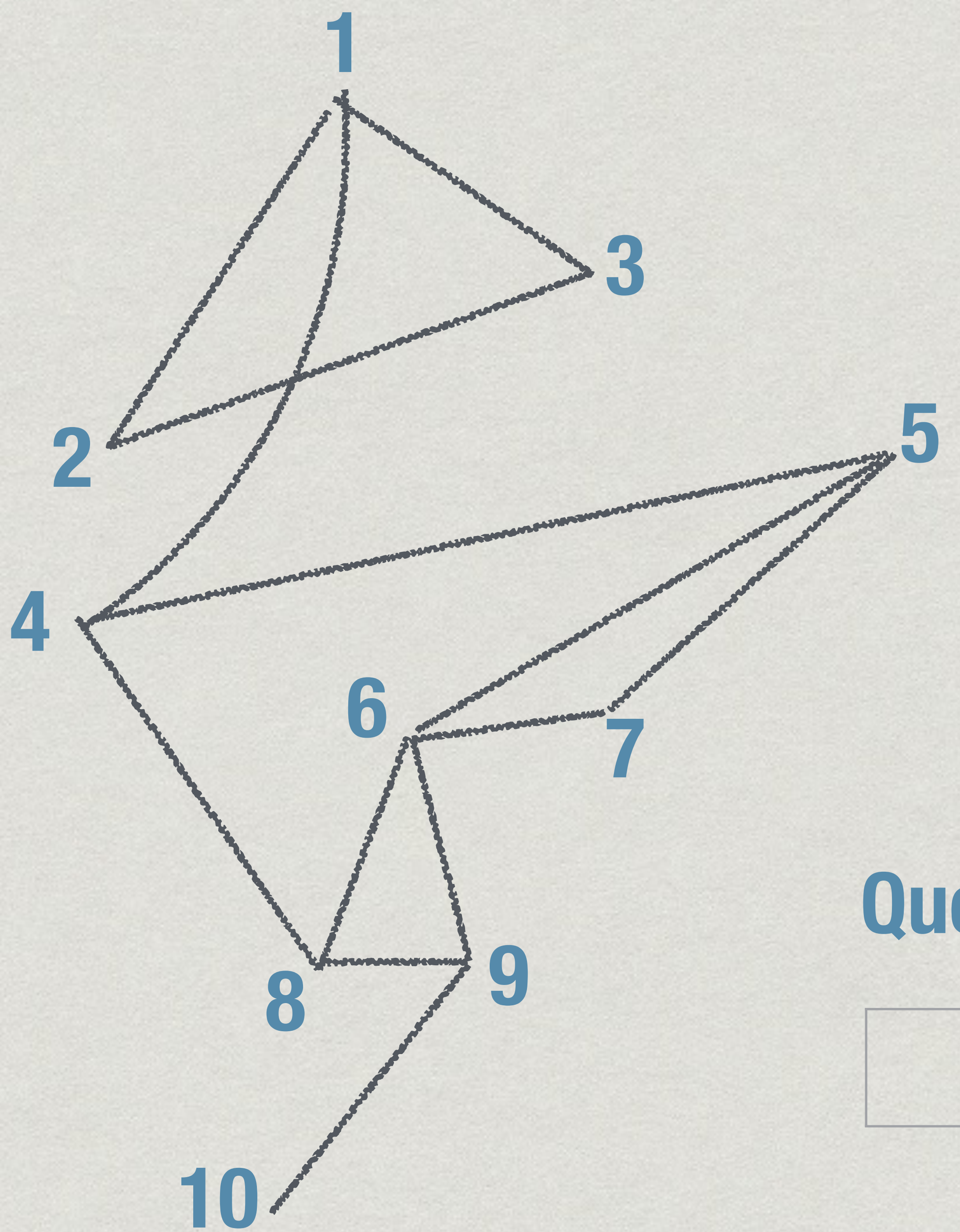
P : Parent

	L	P
1	0	-
2	1	1
3		
4		
5		
6		
7		
8		
9		
10		

Queue



Breadth first search



L : Level

P : Parent

	L	P
1	0	-
2	1	1
3	1	1
4		
5		
6		
7		
8		
9		
10		

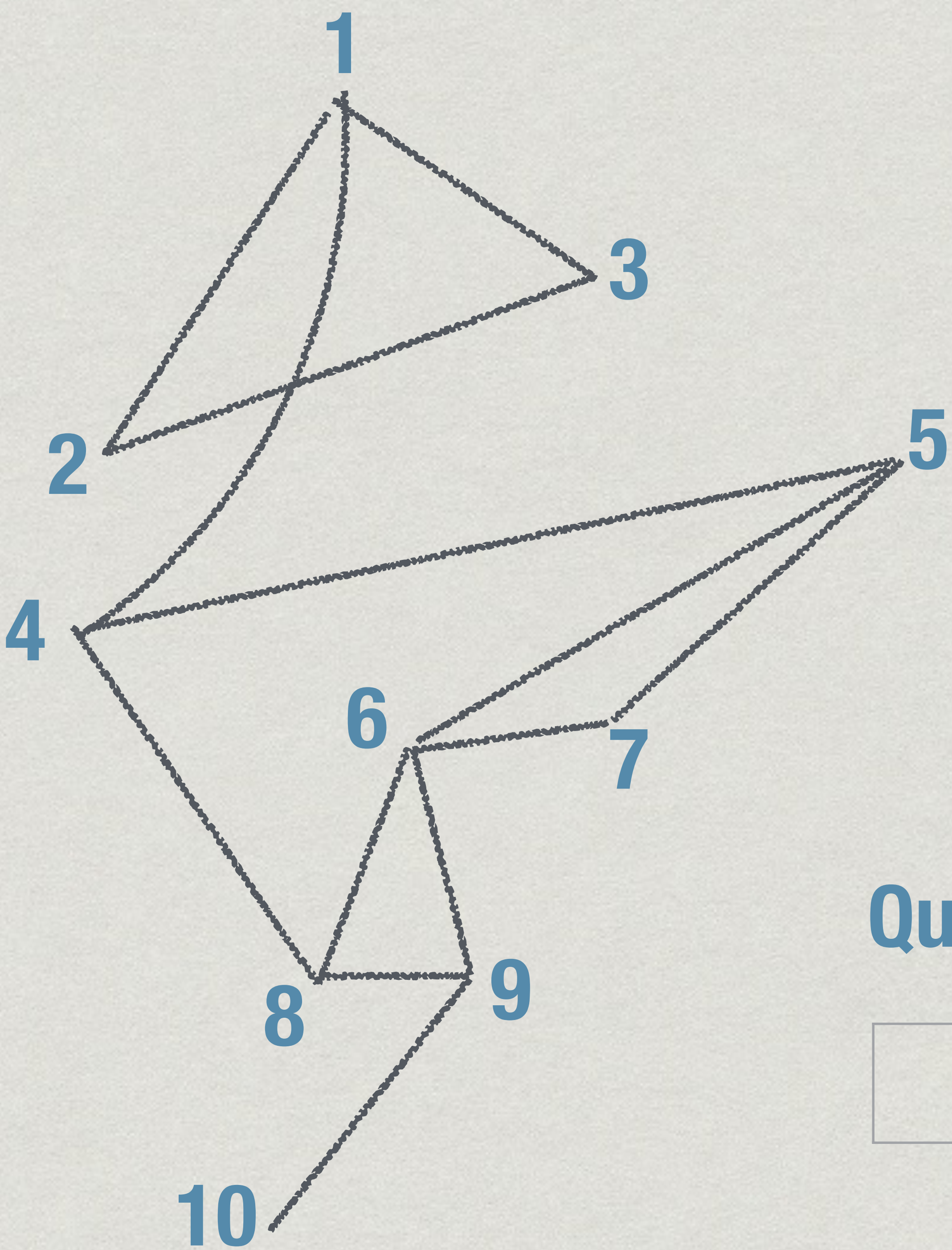
Queue

	2	3							

Breadth first search

L : Level

P : Parent

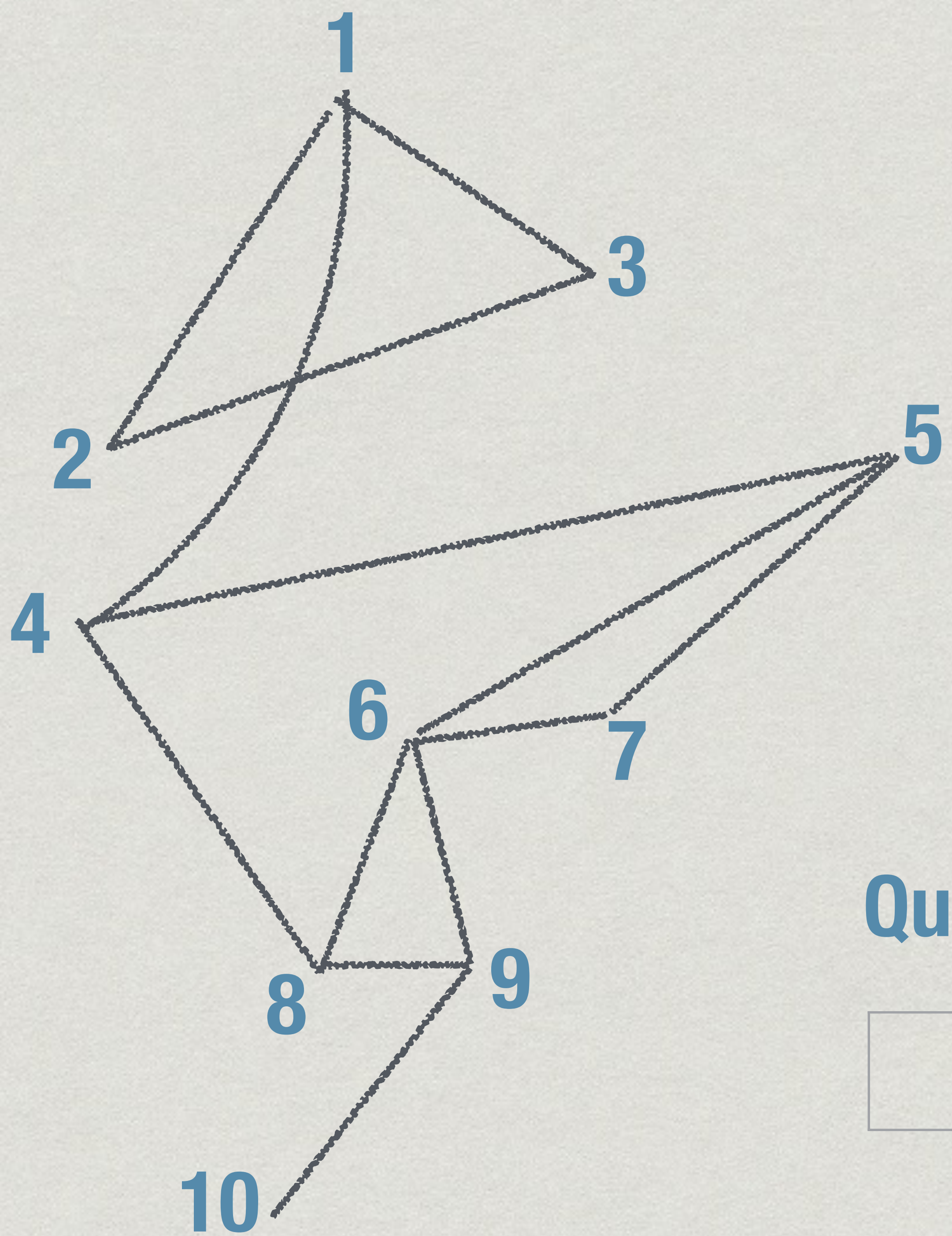


	L	P
1	0	-
2	1	1
3	1	1
4	1	1
5		
6		
7		
8		
9		
10		

Queue

	2	3	4						

Breadth first search



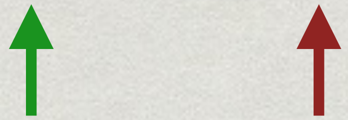
L : Level

P : Parent

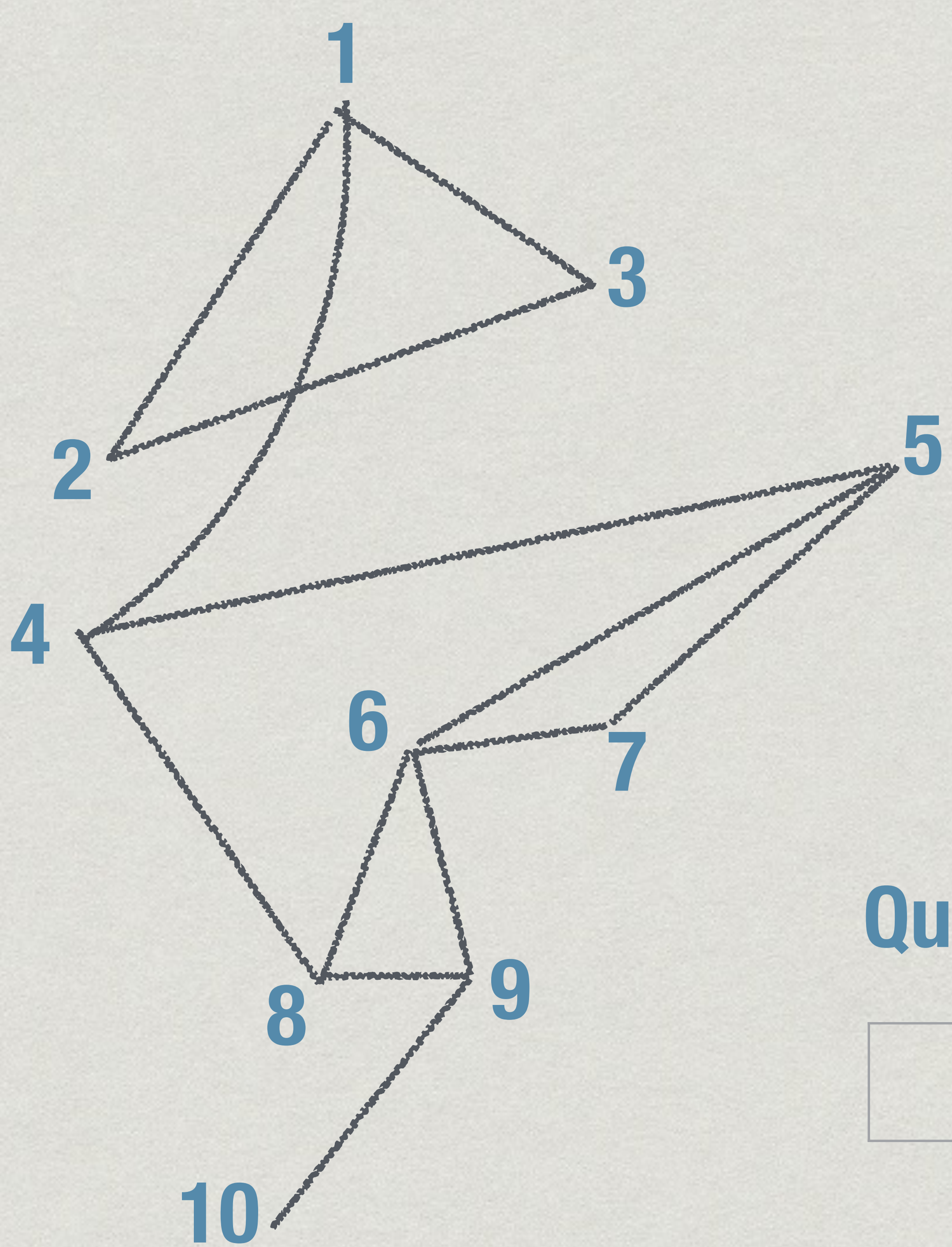
	L	P
1	0	-
2	1	1
3	1	1
4	1	1
5		
6		
7		
8		
9		
10		

Queue

		3	4						



Breadth first search



L : Level

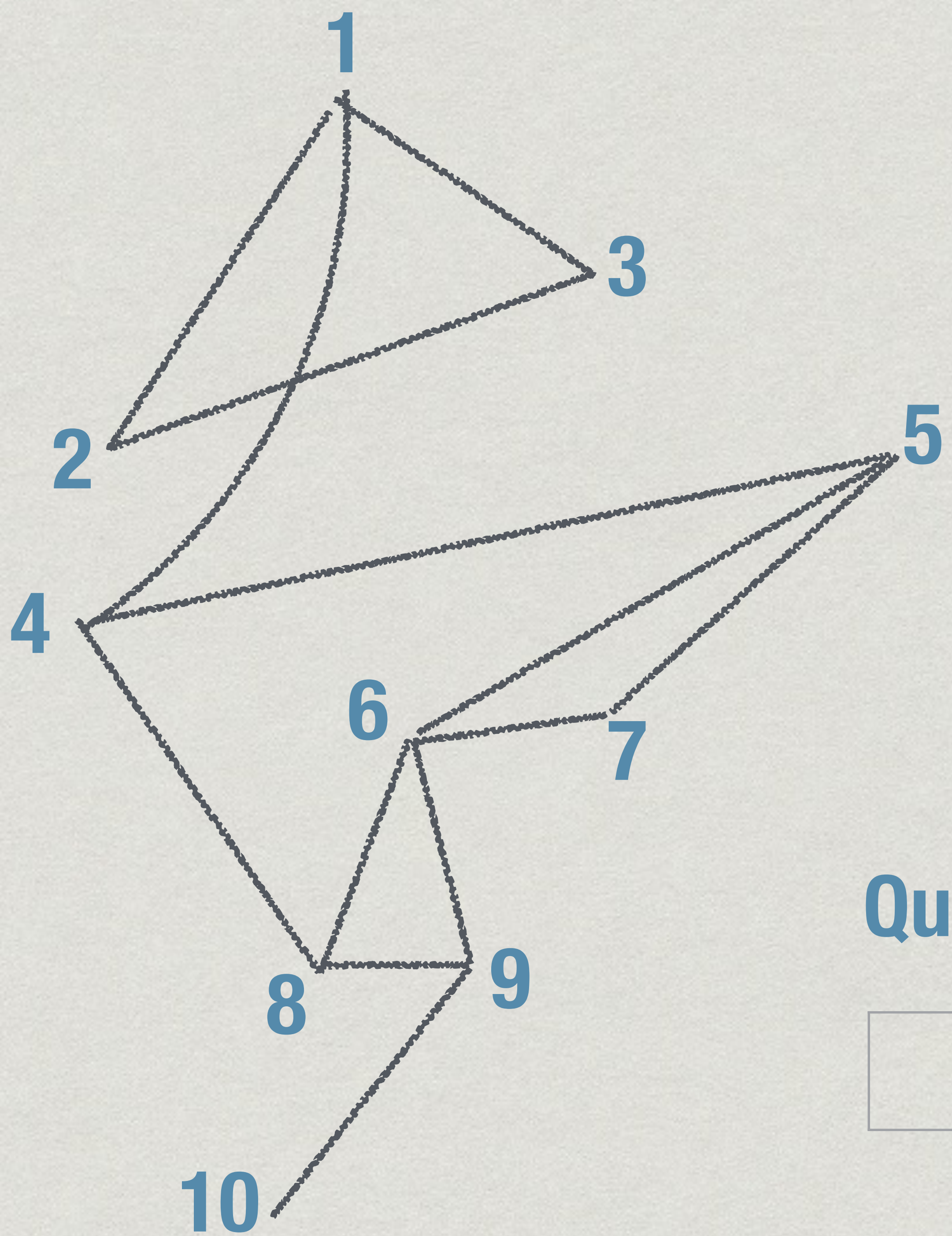
P : Parent

	L	P
1	0	-
2	1	1
3	1	1
4	1	1
5		
6		
7		
8		
9		
10		

Queue

			4						

Breadth first search

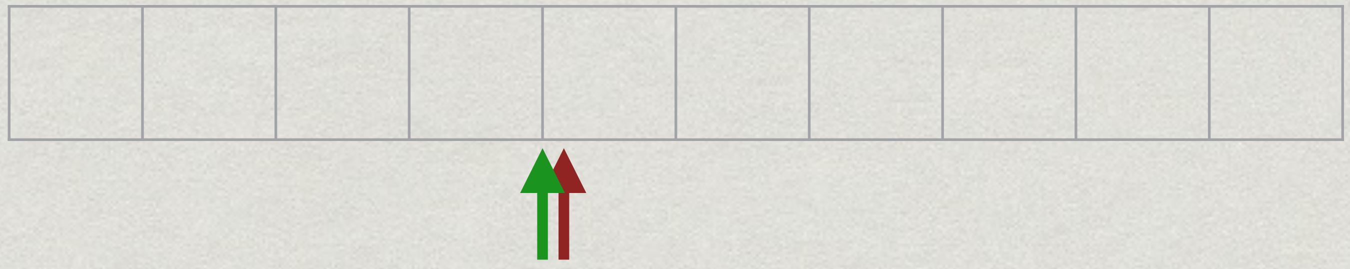


L : Level

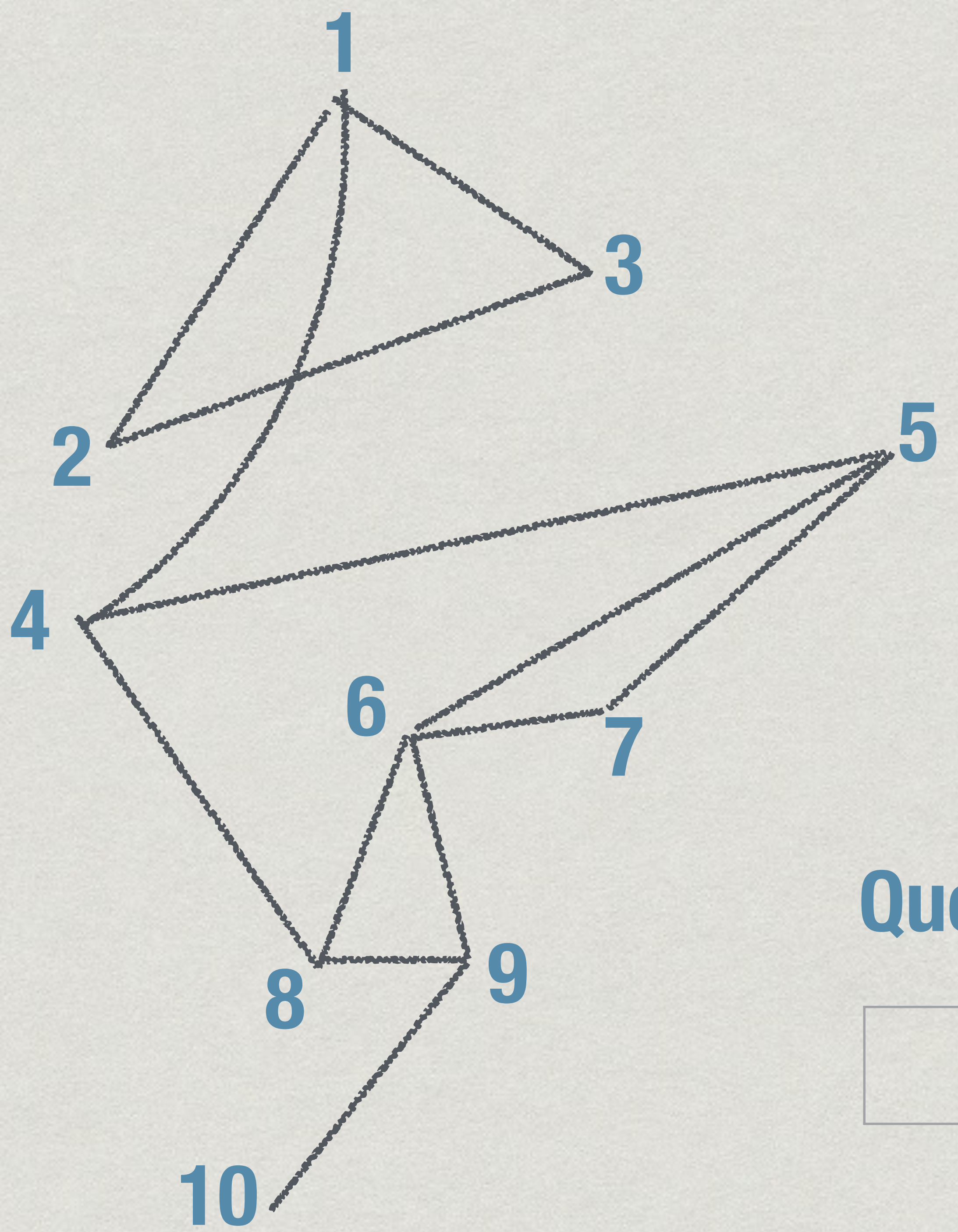
P : Parent

	L	P
1	0	-
2	1	1
3	1	1
4	1	1
5		
6		
7		
8		
9		
10		

Queue



Breadth first search



L : Level

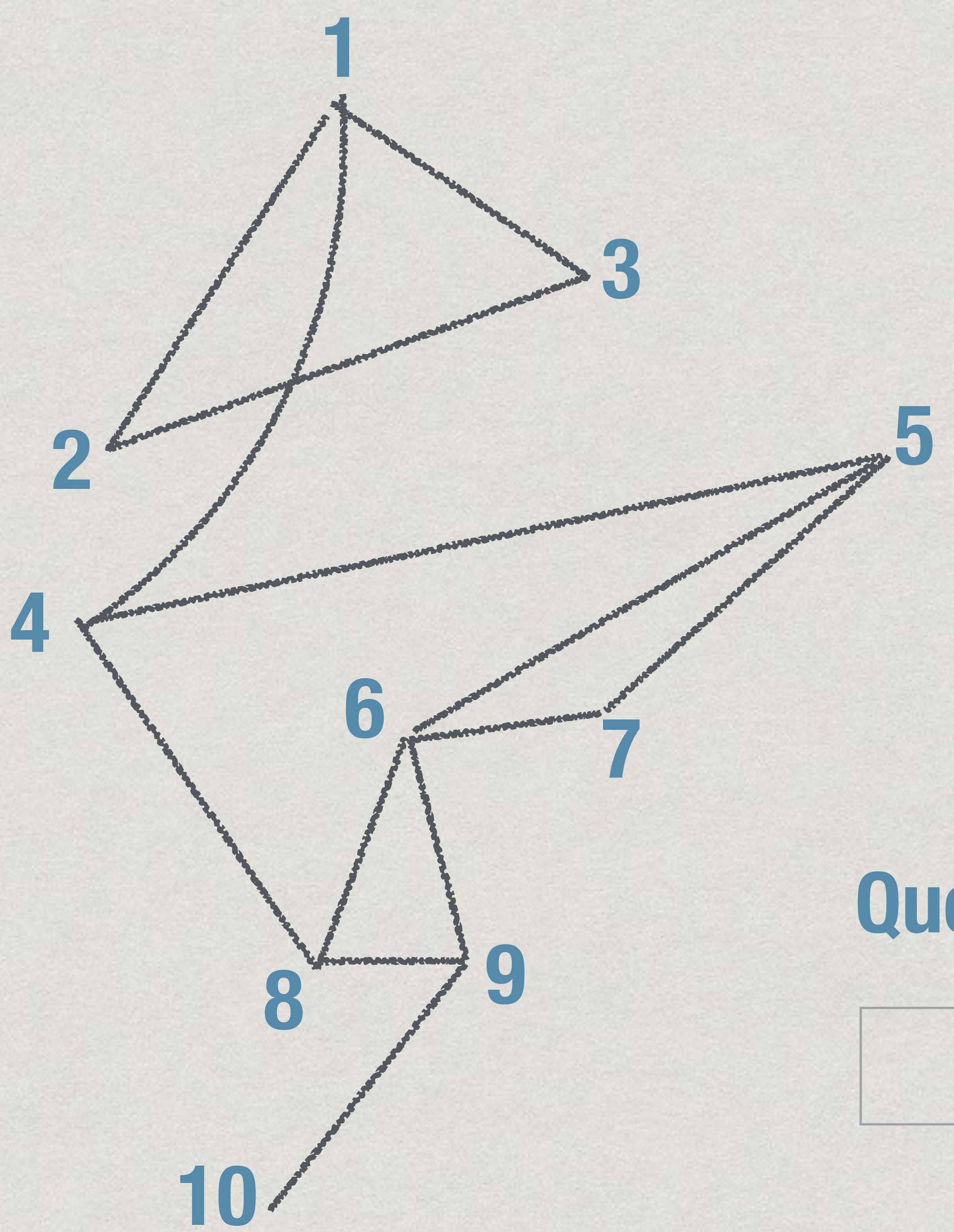
P : Parent

	L	P
1	0	-
2	1	1
3	1	1
4	1	1
5	2	4
6		
7		
8		
9		
10		

Queue

				5					

Breadth first search



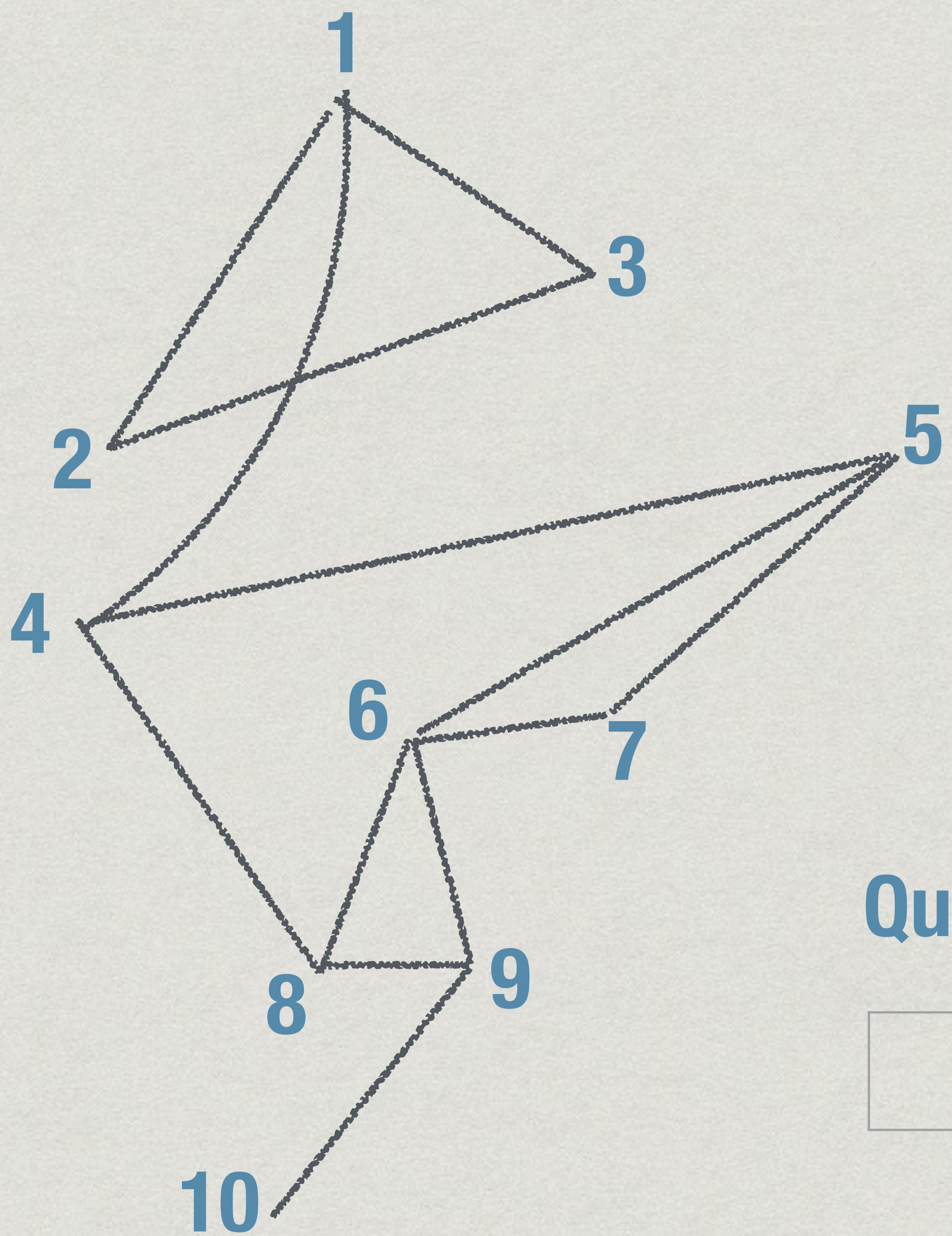
L : Level
P : Parent

	L	P
1	0	-
2	1	1
3	1	1
4	1	1
5	2	4
6		
7		
8	2	4
9		
10		

Queue

				5	8				

Breadth first search



L : Level

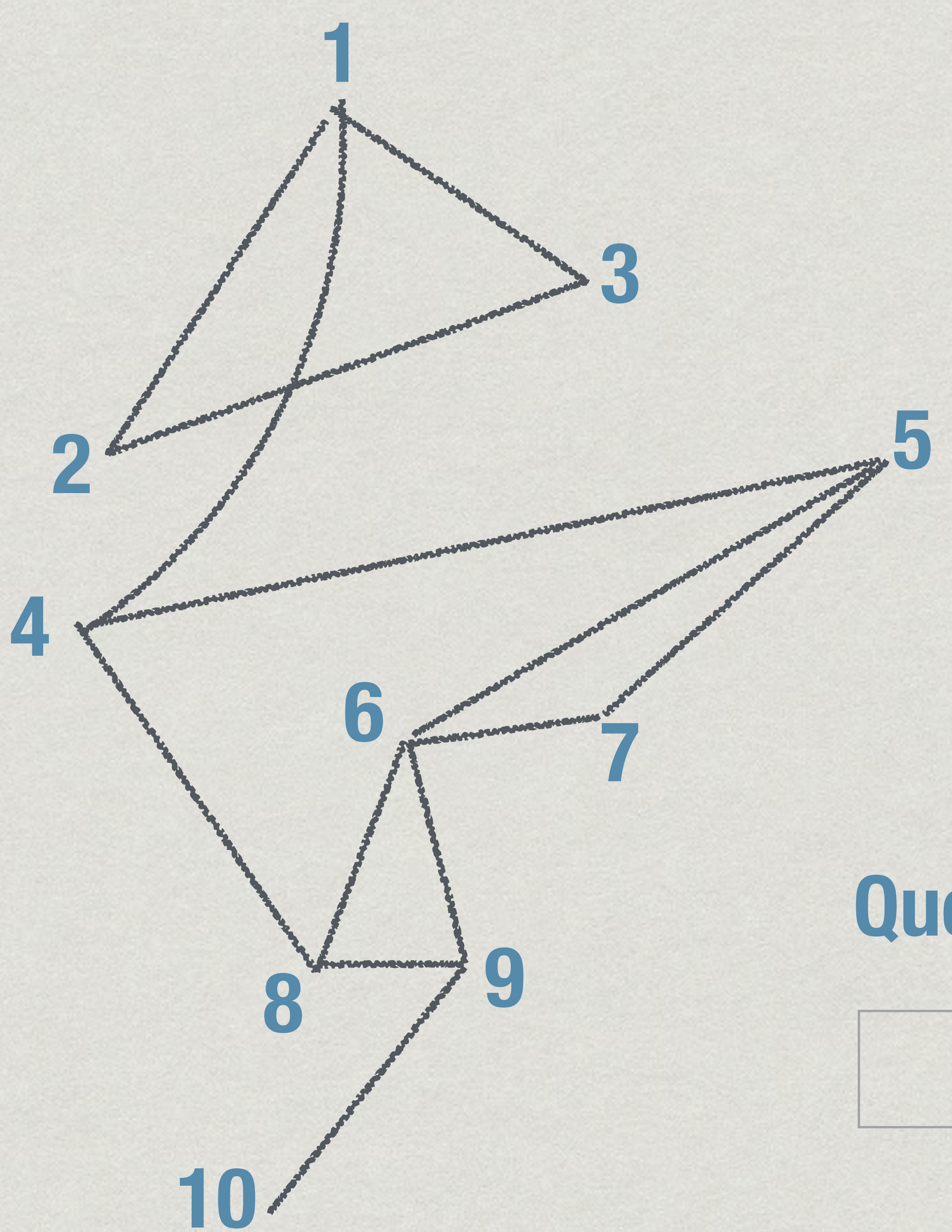
P : Parent

	L	P
1	0	-
2	1	1
3	1	1
4	1	1
5	2	4
6		
7		
8	2	4
9		
10		

Queue

					8				

Breadth first search



L : Level

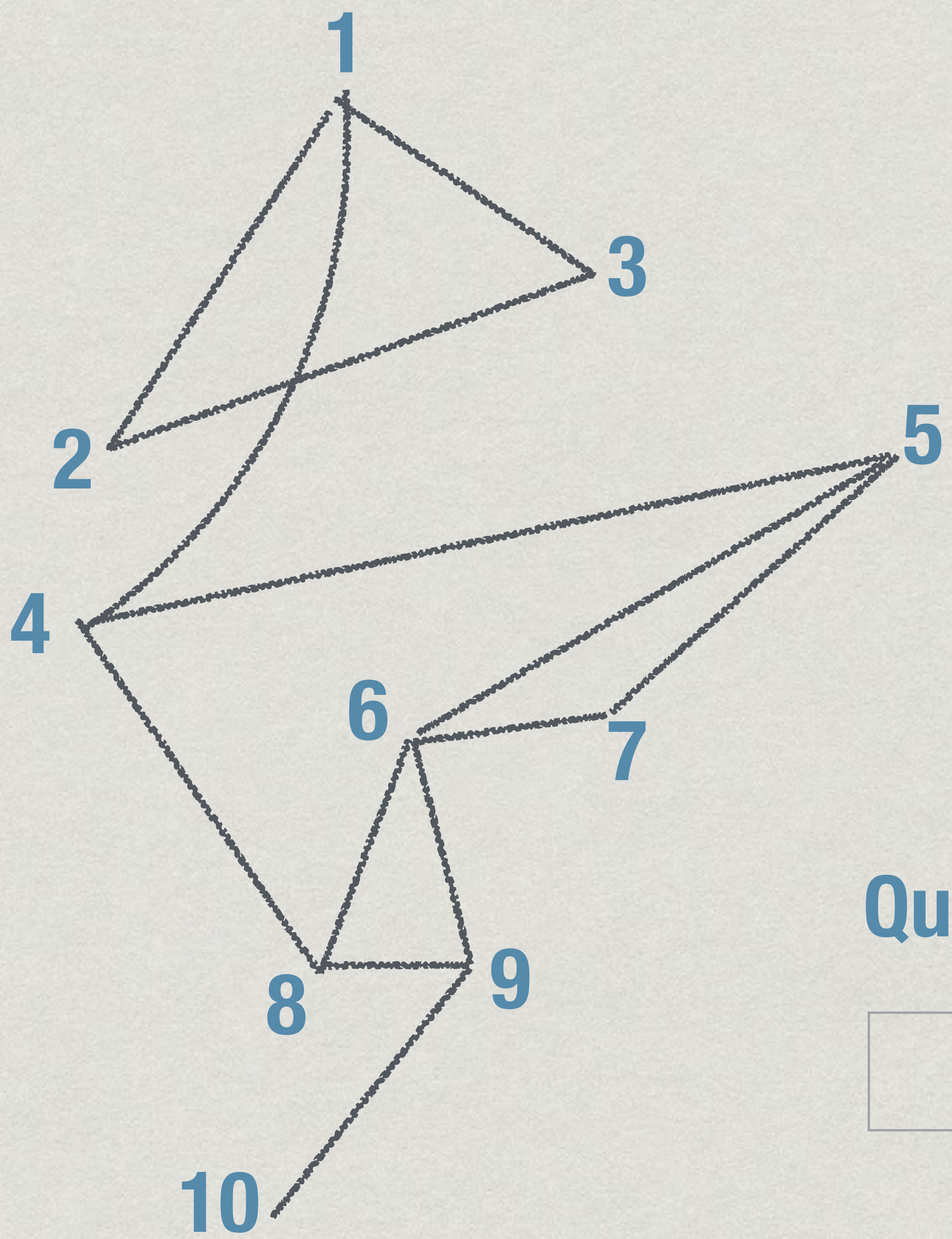
P : Parent

	L	P
1	0	-
2	1	1
3	1	1
4	1	1
5	2	4
6	3	5
7		
8	2	4
9		
10		

Queue

					8	6			

Breadth first search



L : Level

P : Parent

	L	P
1	0	-
2	1	1
3	1	1
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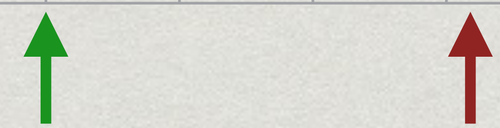
5	2	4
6	3	5
7	3	5
8	2	4
9		
10		

Queue

					8	6	7		



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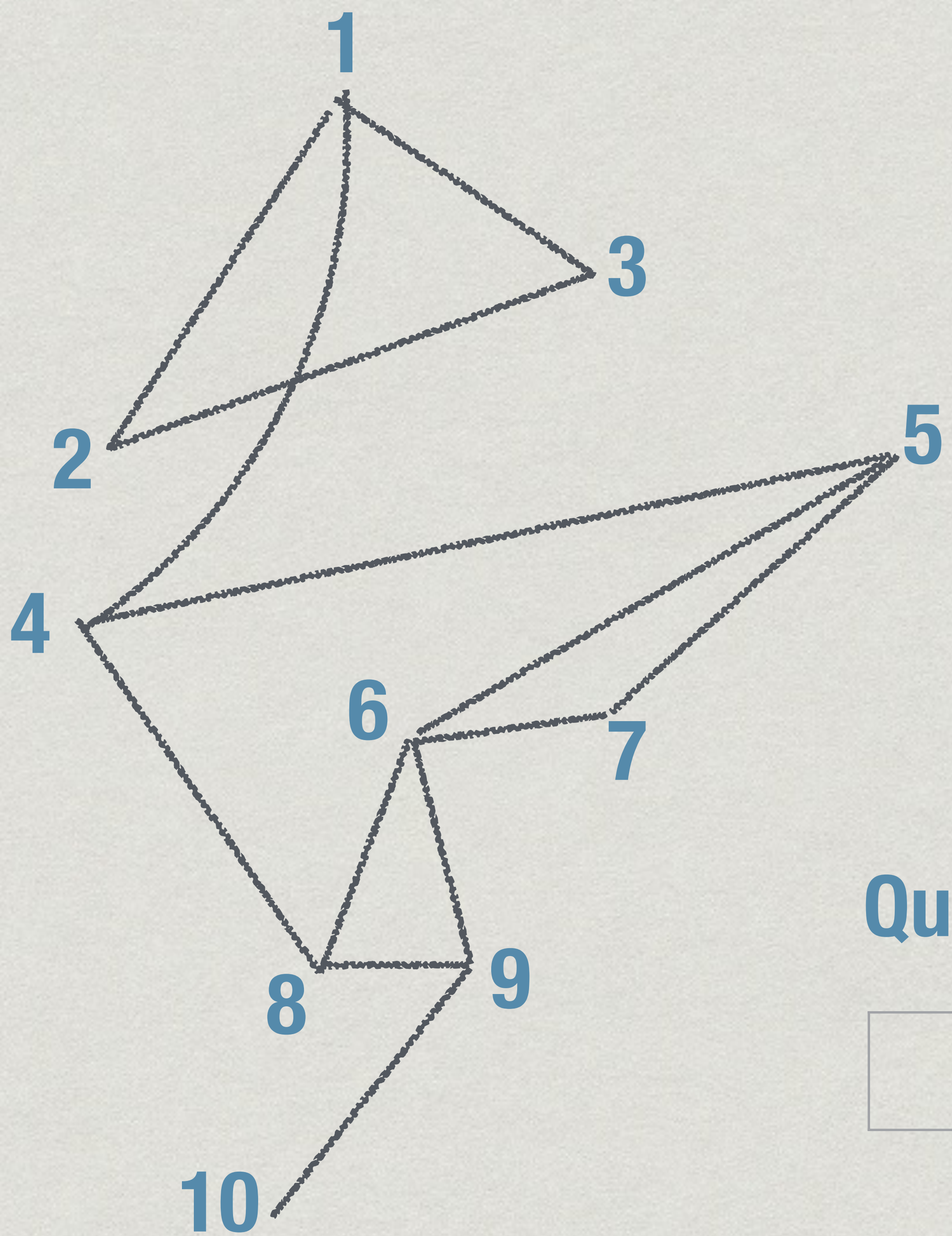

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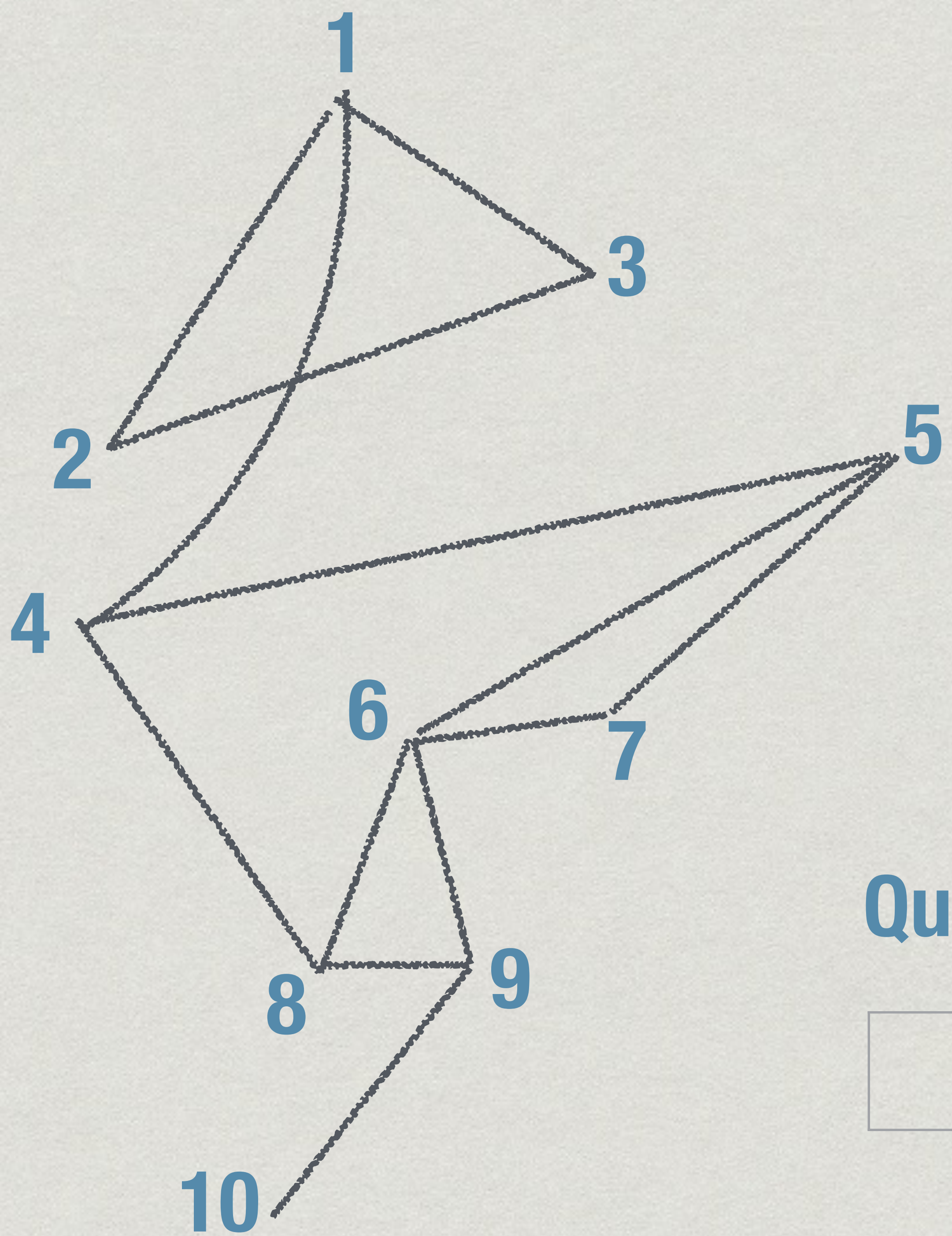
	L	P
1	0	-
2	1	1
3	1	1
4	1	1
5	2	4
6	3	5
7	3	5
8	2	4
9		
10		

Queue

						6	7		

Breadth first search



L : Level

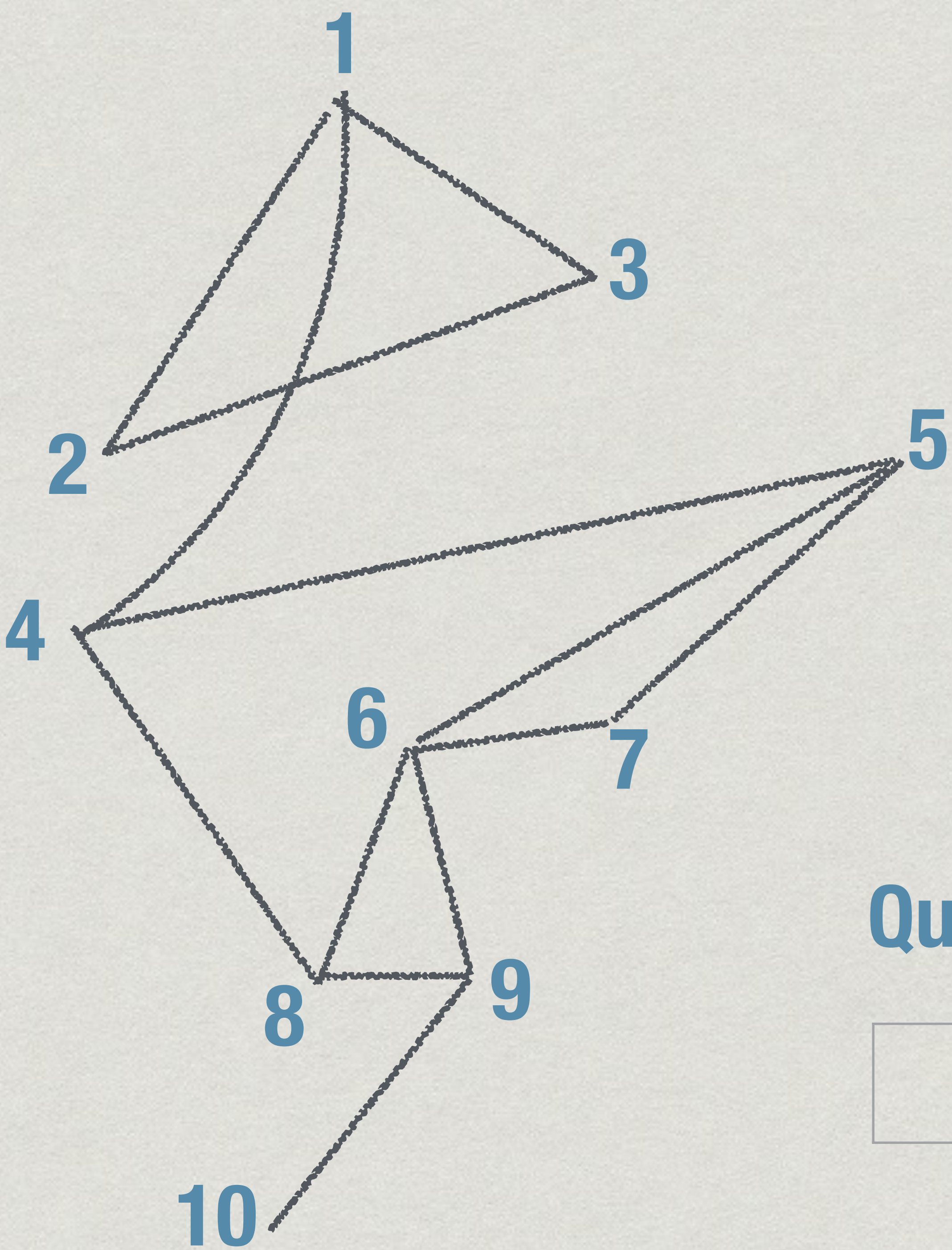
P : Parent

	L	P
1	0	-
2	1	1
3	1	1
4	1	1
5	2	4
6	3	5
7	3	5
8	2	4
9	3	8
10		

Queue

						6	7	9	

Breadth first search



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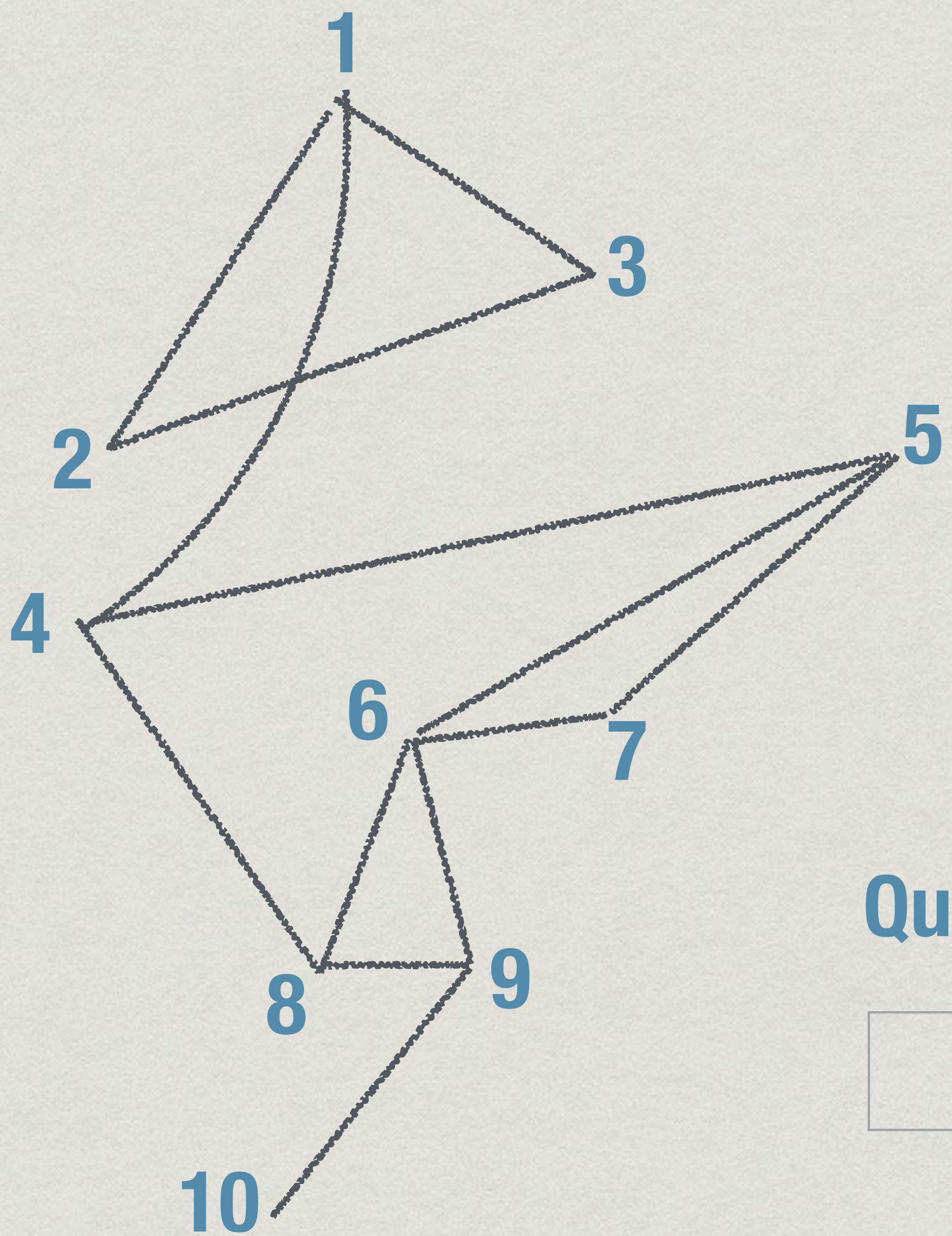
P : Parent

	L	P
1	0	-
2	1	1
3	1	1
4	1	1
5	2	4
6	3	5
7	3	5
8	2	4
9	3	8
10		

Queue

							7	9	

Breadth first search



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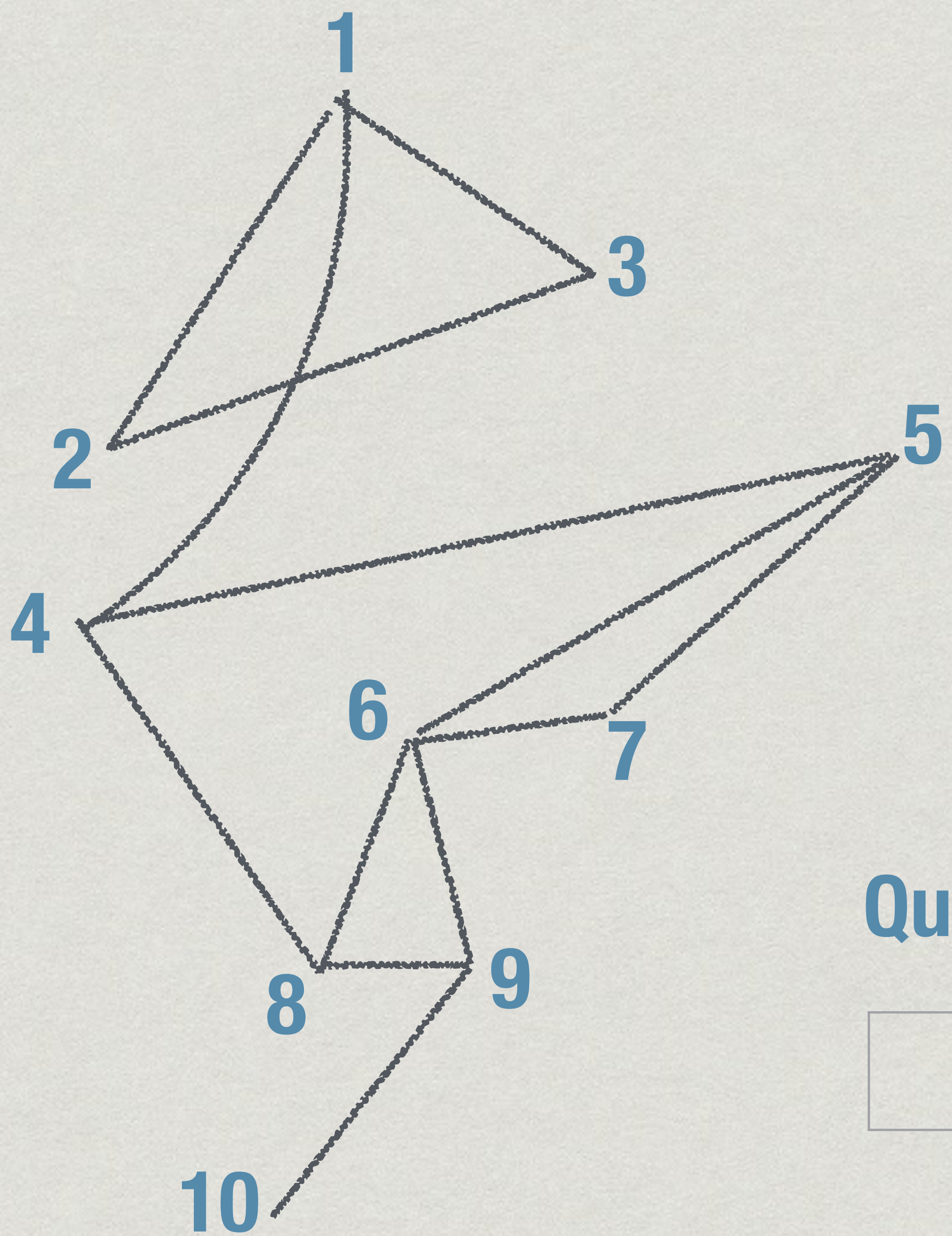
5	2	4
6	3	5
7	3	5
8	2	4
9	3	8
10		

Queue

								9	

Breadth first search



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P : Parent

	L	P
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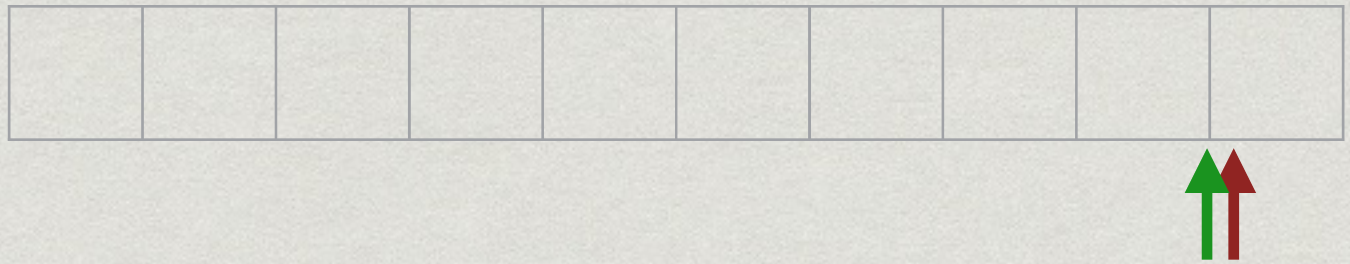
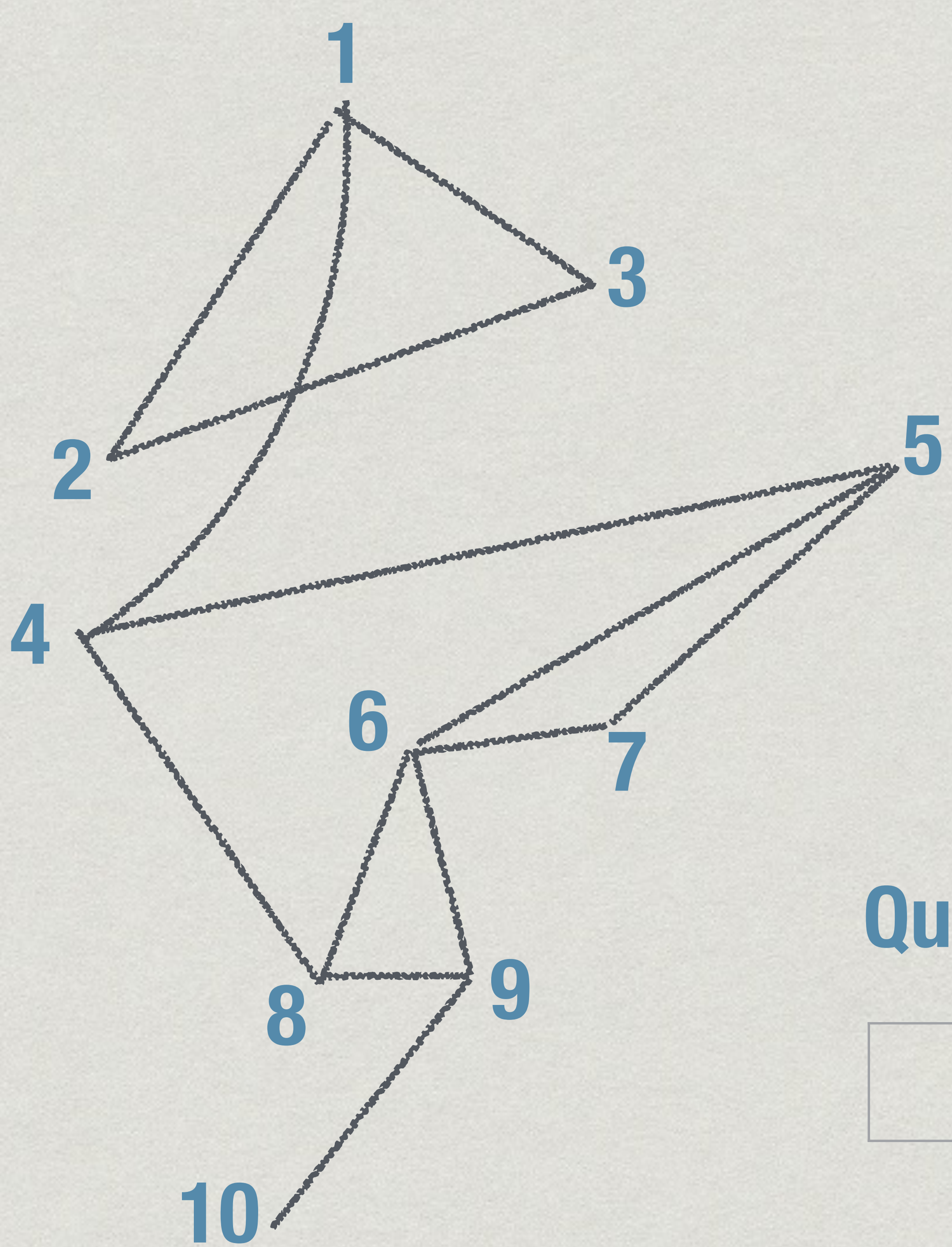
2	1	1
3	1	1
4	1	1
5	2	4
6	3	5
7	3	5
8	2	4
9	3	8
10		

Queue



Breadth first search



L : Level

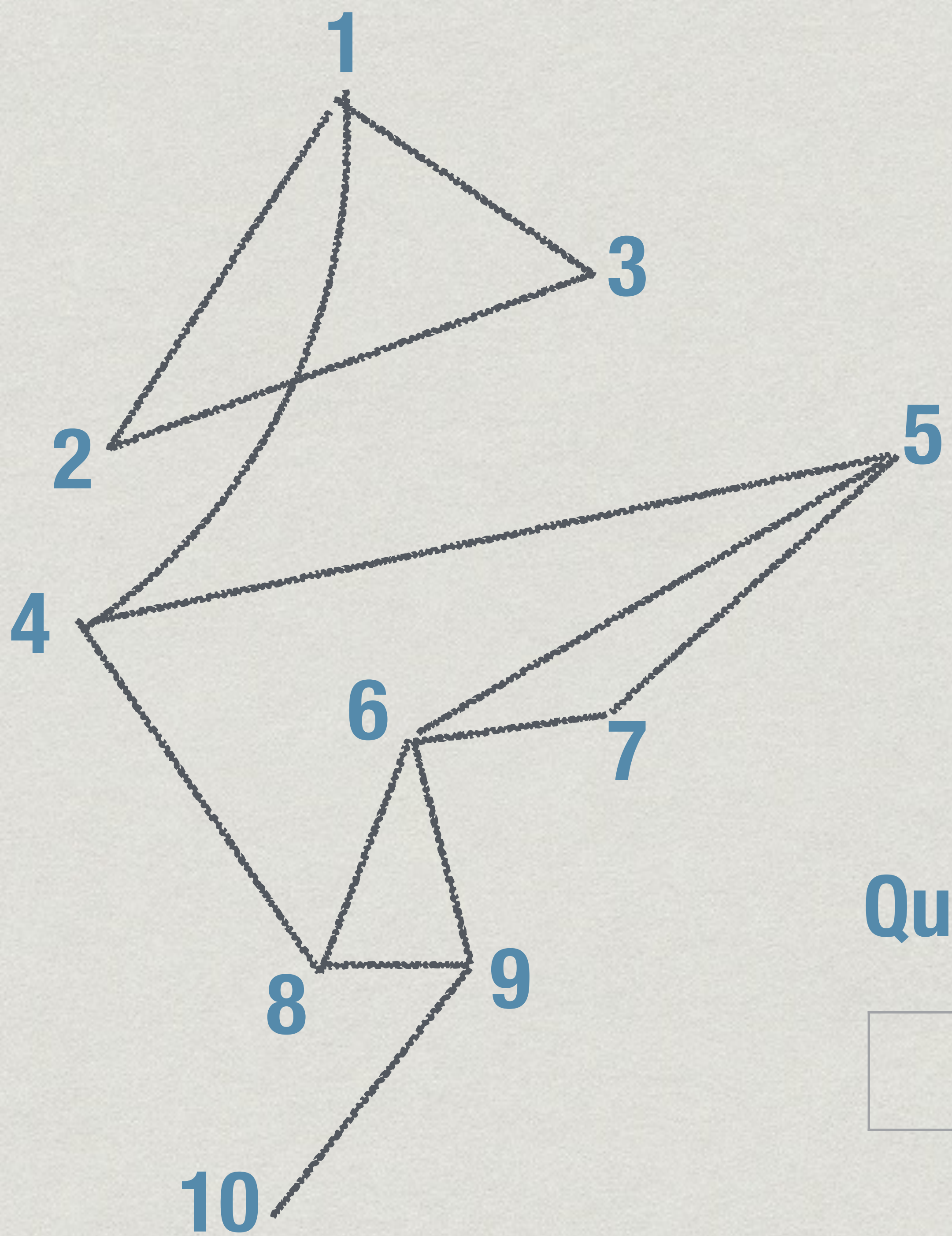
P : Parent

	L	P
1	0	-
2	1	1
3	1	1
4	1	1
5	2	4
6	3	5
7	3	5
8	2	4
9	3	8
10	4	9

Queue

									10

Breadth first search



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	L	P
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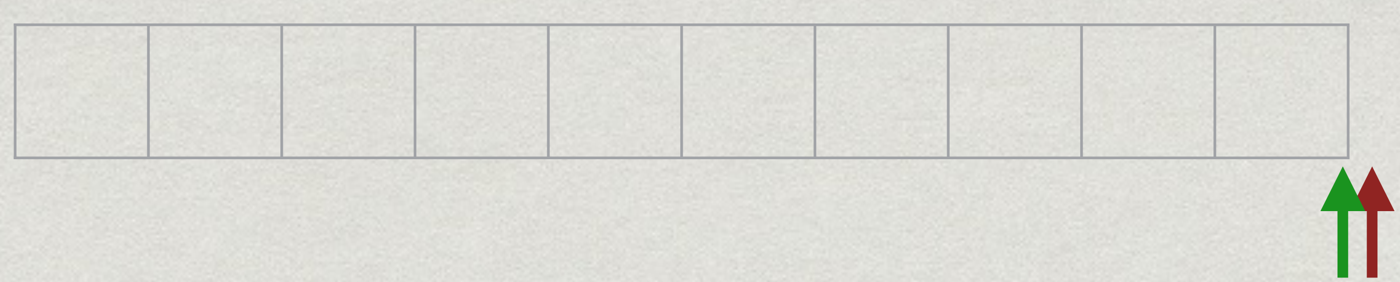
5	2	4
6	3	5
7	3	5
8	2	4
9	3	8
10	4	9

Queue



Recording distances

- BFS with level[ ] gives us the shortest path to each node in terms of number of edges
- In general, edges are labelled by a cost (money, time, distance …)
  - Min cost path not same as fewest edges
- Will look at shortest paths in **weighted** graphs later
  - BFS computes shortest paths if all costs are 1