Automata for Real-time Systems

B. Srivathsan

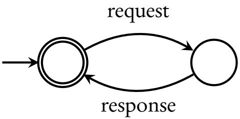
Chennai Mathematical Institute

Overview

Automata (*Finite State Machines*) are **good abstractions** of many real systems

hardware circuits, communication protocols, biological processes, . . .

Automata can model many **properties** of systems



every request is followed by a response

System $\longrightarrow$ Automaton $\mathcal{A}$

Property $\longrightarrow$ Automaton $\mathcal{B}$

$$\begin{array}{ccc} \text{System} & & \text{Property} \\ \downarrow & & \downarrow \\ \text{Automaton } \mathcal{A} & & \text{Automaton } \mathcal{B} \end{array}$$

Does system **satisfy** property?

System $\downarrow$ Automaton $\mathcal{A}$

Property $\downarrow$ Automaton $\mathcal{B}$

$$\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})?$$

Does system **satisfy** property?

Model-checking

System $\downarrow$ Automaton $\mathcal{A}$

Property $\downarrow$ Automaton $\mathcal{B}$

$$\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})?$$

Does system **satisfy** property?

In practice...

Huge system

Property

In practice...

Huge system	Property
$\downarrow$	$\downarrow$
Higher-level description	Higher-level description

In practice...

Huge system → Higher-level description $\xrightarrow{\text{translation}}$ Automaton $\mathcal{A}$

Property → Higher-level description $\xrightarrow{\text{translation}}$ Automaton $\mathcal{B}$

Model-Checker

$$\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})?$$

In practice...

Huge system → Higher-level description → (translation) → Automaton $\mathcal{A}$

Property → Higher-level description → (translation) → Automaton $\mathcal{B}$

Model-Checker

$$\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})?$$

Some model-checkers: SMV, NuSMV, SPIN, . . .

In practice...

$$\begin{array}{ccc}
\text{Huge system} & & \text{Property} \\
\downarrow & & \downarrow \\
\text{Higher-level description} & & \text{Higher-level description} \\
\Big\downarrow \text{translation} & & \Big\downarrow \text{translation} \\
\text{Automaton } \mathcal{A} & & \text{Automaton } \mathcal{B}
\end{array}$$

Model-Checker

$$\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})?$$

Some model-checkers: SMV, NuSMV, SPIN, ...

Turing Awards: Clarke, Emerson, Sifakis and Pnueli

Automata are **good abstractions** of many real systems

Automata are **good abstractions** of many real systems

Our course: Automata for **real-time** systems

Picture credits: F. Herbreteau

pacemaker, vehicle control systems, air traffic controllers, ...

Timed Automata

R. Alur and D. Dill in early 90s

Timed Automata

R. Alur and D. Dill in early 90s

Some model-checkers: UPPAAL, KRONOS, RED, …

Goals of our course

Study **language theoretic** and **algorithmic** properties of timed automata

Lecture 7:

Timed languages and timed automata

Σ : alphabet $\{a, b\}$

Σ^* : words $\{\varepsilon, a, b, aa, ab, ba, bb, aab, \ldots\}$

$L \subseteq \Sigma^*$: language $\longrightarrow$ *property over words*

$L_1 :=$ {set of words starting with an " a "}

$\{a, aa, ab, aaa, aab, \ldots\}$

$L_2 :=$ {set of words with a non-zero even length }

$\{aa, bb, ab, ba, abab, aaaa, \ldots\}$

Σ : alphabet $\{a, b\}$

Σ^* : words $\{\varepsilon, a, b, aa, ab, ba, bb, aab, \ldots\}$

$L \subseteq \Sigma^*$: language $\longrightarrow$ *property over words*

$L_1 :=$ {set of words starting with an " a "}

$\{a, aa, ab, aaa, aab, \ldots\}$

$L_2 :=$ {set of words with a non-zero even length }

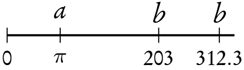
$\{aa, bb, ab, ba, abab, aaaa, \ldots\}$

Finite automata, pushdown automata, Turing machines, ...

Σ : alphabet $\{a, b\}$

$T\Sigma^*$: timed words

$(aa;\ 0.8, 2.5)$



$(abb;\ \pi, 203, 312.3)$

Σ : alphabet $\{a, b\}$

$T\Sigma^*$: timed words

$(aa;\ 0.8, 2.5)$ $(abb;\ \pi, 203, 312.3)$

(w, τ)

Word Time sequence

$w = a_1 \dots a_n$

$a_i \in \Sigma$

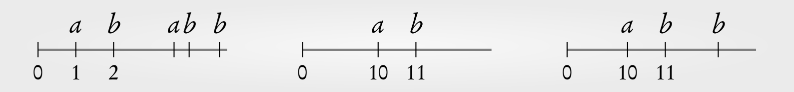
$\tau = \tau_1 \dots \tau_n$

$\tau_i \in \mathbb{R}_{\geq 0}$

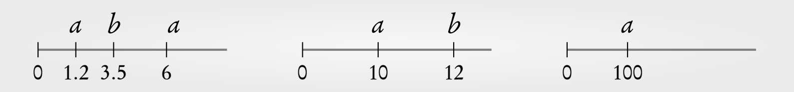
$\tau_1 \leq \dots \leq \tau_n$

$L \subseteq T\Sigma^*$: Timed language $\longrightarrow$ *property over timed words*

$$L_1 := \{(\ ab(a+b)^*, \tau\ ) \mid \tau_2 - \tau_1 = 1\}$$

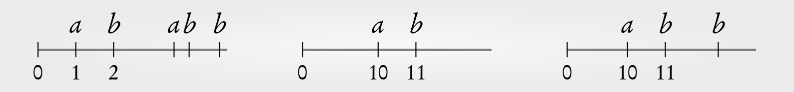


$$L_2 := \{\ (w, \tau) \mid \tau_{i+1} - \tau_i \geq 2 \text{ for all } i < |w|\}$$

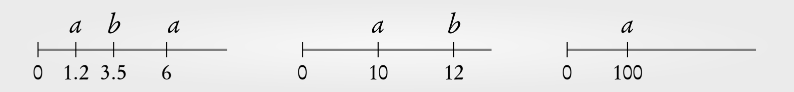


$L \subseteq T\Sigma^*$: Timed language $\longrightarrow$ *property over timed words*

$$L_1 := \{(\, ab(a+b)^*, \tau\ ) \mid \tau_2 - \tau_1 = 1\}$$

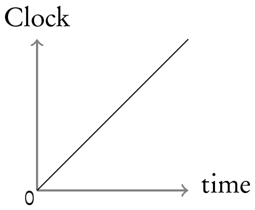


$$L_2 := \{\ (w, \tau) \mid \tau_{i+1} - \tau_i \geq 2 \text{ for all } i < |w|\}$$

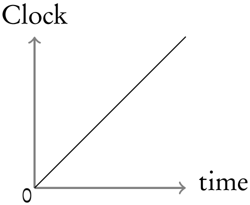


Timed automata

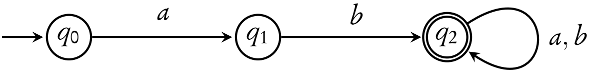
Timed automaton: Finite automaton + Finite no. of *Clocks*



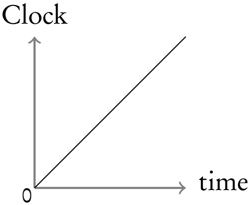
Timed automaton: Finite automaton + Finite no. of *Clocks*



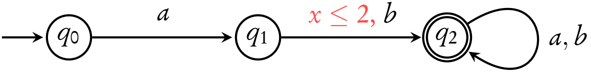
$$\{(\, ab(a+b)^*, \tau) \mid \tau_2 \leq 2\}$$



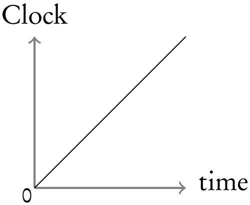
Timed automaton: Finite automaton + Finite no. of *Clocks*



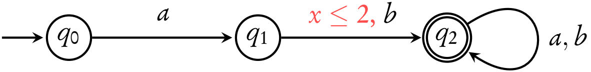
$$\{(\, ab(a+b)^*, \tau) \mid \tau_2 \leq 2\}$$

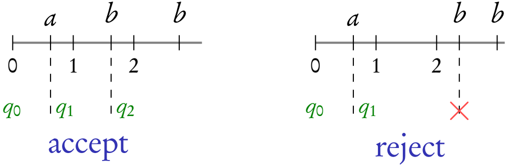


Timed automaton: Finite automaton + Finite no. of *Clocks*

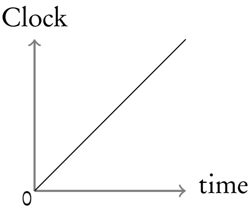


$$\{(\,ab(a+b)^*, \tau) \mid \tau_2 \leq 2\}$$





Timed automaton: Finite automaton + Finite no. of *Clocks*

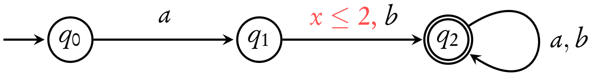


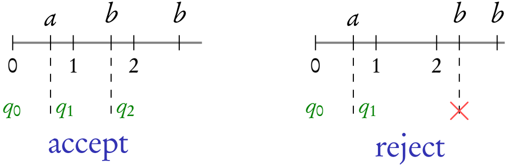
Guards

$$\phi := x \leq c \mid x \geq c \mid \neg\phi \mid \phi \wedge \phi$$

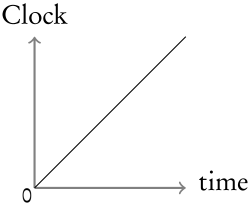
$$x \in \mathit{Clocks}\ ,\quad c \in \mathbb{Q}_{\geq 0}$$

$$\{(\, ab(a+b)^*, \tau) \mid \tau_2 \leq 2\}$$





Timed automaton: Finite automaton + Finite no. of *Clocks*

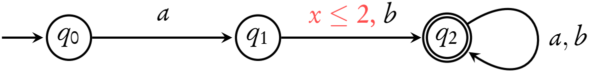


Guards

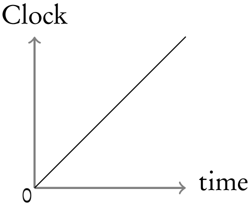
$$\phi := x \le c \mid x \ge c \mid \neg\phi \mid \phi \wedge \phi$$

$$x \in \mathit{Clocks}\ ,\ \ c \in \mathbb{Q}_{\ge 0}$$

$$\{(\, ab(a+b)^*, \tau) \mid \tau_2 - \tau_1 \le 2\}$$



Timed automaton: Finite automaton + Finite no. of *Clocks*



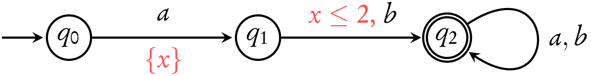
Guards

$$\phi := x \leq c \mid x \geq c \mid \neg\phi \mid \phi \wedge \phi$$

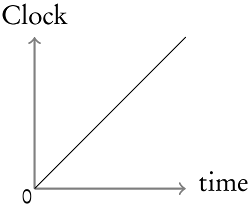
$$x \in \mathit{Clocks}\ ,\ \ c \in \mathbb{Q}_{\geq 0}$$

Resets

$$\{(\, ab(a+b)^*, \tau) \mid \tau_2 - \tau_1 \leq 2\}$$



Timed automaton: Finite automaton + Finite no. of *Clocks*



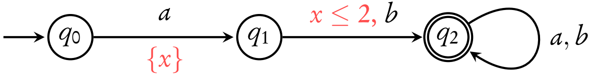
Guards

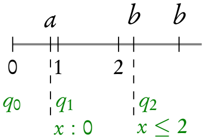
$$\phi := x \le c \mid x \ge c \mid \neg\phi \mid \phi \wedge \phi$$

$$x \in \mathit{Clocks}\ , \quad c \in \mathbb{Q}_{\ge 0}$$

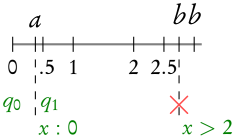
Resets

$$\{(\, ab(a+b)^*, \tau) \mid \tau_2 - \tau_1 \le 2\}$$





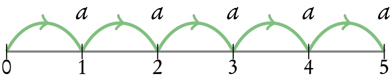
accept



reject

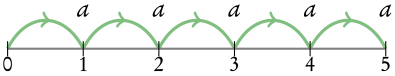
$$L_3 := \{\ (\ a^k, \tau\ ) \mid k > 0,\ \tau_i = i \text{ for all } i \leq k\}$$

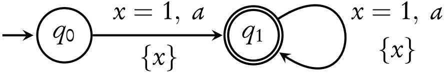
An "a" occurs in every integer from $1, \ldots, k$



$$L_3 := \{\ (\ a^k, \tau\ ) \mid k > 0,\ \tau_i = i \text{ for all } i \leq k\}$$

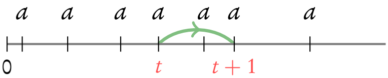
An "a" occurs in every integer from $1, \ldots, k$





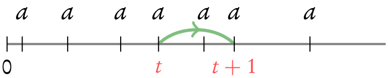
$$L_4 := \{\ (\ a^k, \tau\ ) \mid \text{exist } i, j \text{ s.t. } \tau_j - \tau_i = 1\}$$

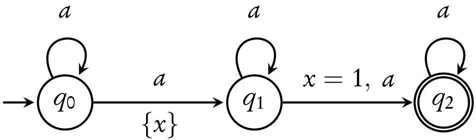
There are 2 "a"s which are at distance 1 apart



$$L_4 := \{ \, (\, a^k, \tau \,) \mid \text{exist } i, j \text{ s.t. } \tau_j - \tau_i = 1 \}$$

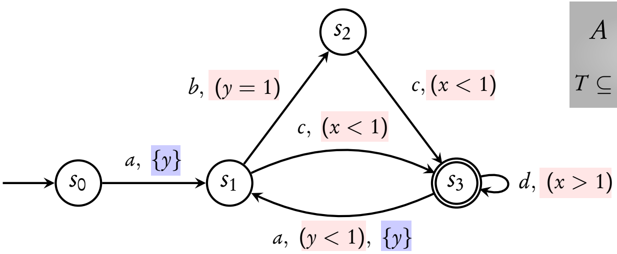
There are 2 "a"s which are at distance 1 apart





Three **mechanisms** to exploit:

- Reset: to **start** measuring time
- Guard: to **impose** time constraint on action
- Non-determinism: for **existential** time constraints



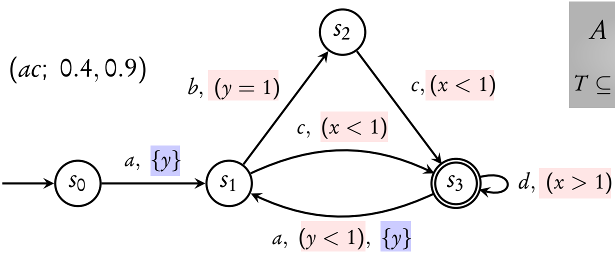
$$A = (Q, \Sigma, X, T, Q_0, F)$$

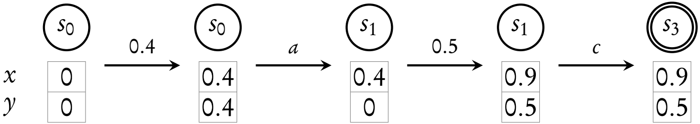
$$T \subseteq Q \times \Sigma \times \text{guard} \times \text{reset} \times Q$$

$$A = (Q, \Sigma, X, T, Q_0, F)$$

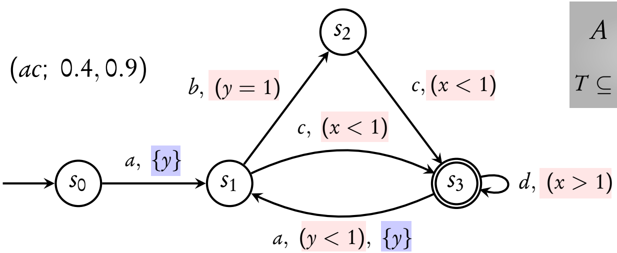
$$T \subseteq Q \times \Sigma \times \text{guard} \times \text{reset} \times Q$$

$(ac;\ 0.4, 0.9)$



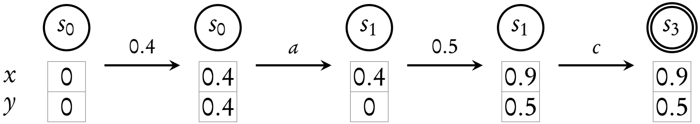


$(ac;\ 0.4, 0.9)$



$$A = (Q, \Sigma, X, T, Q_0, F)$$

$$T \subseteq Q \times \Sigma \times \text{guard} \times \text{reset} \times Q$$



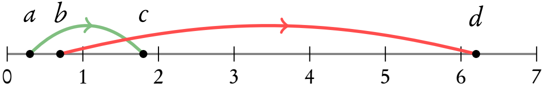
Run of A over $(a_1 a_2 \ldots a_k;\ \tau_1 \tau_2 \ldots \tau_k)$ $\qquad \delta_i := \tau_i - \tau_{i-1};\ \tau_0 := 0$

$$(q_0, v_0) \xrightarrow{\delta_1} (q_0, v_0 + \delta_1) \xrightarrow{a_1} (q_1, v_1) \xrightarrow{\delta_2} (q_1, v_1 + \delta_2) \ \cdots \ \xrightarrow{a_k} (q_k, v_k)$$

$(w, \tau) \in \mathcal{L}(A)$ if A has an **accepting** run over (w, τ)

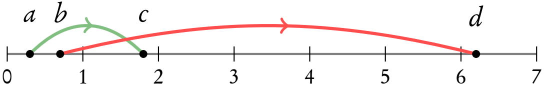
$$L_5 := \{\ (\ abcd.\Sigma^*, \tau\ )\mid \tau_3 - \tau_1 \leq 2 \text{ and } \tau_4 - \tau_2 \geq 5\}$$

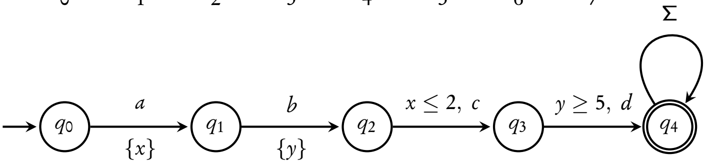
Interleaving distances



$$L_5 := \{\ (\ abcd.\Sigma^*, \tau\ )\ |\ \tau_3 - \tau_1 \leq 2 \text{ and } \tau_4 - \tau_2 \geq 5\}$$

Interleaving distances





n interleavings $\Rightarrow$ need n clocks

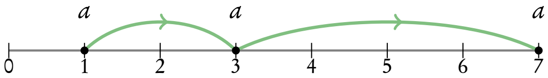
$n + 1$ clocks more expressive than n clocks

Timed automata

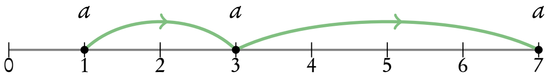
Runs

1 clock $<$ 2 clocks $< \ldots$

$$L_6 := \{\ (\ a^k, \tau\ ) \mid \tau_i \text{ is some integer for each } i\}$$



$L_6 := \{ (a^k, \tau) \mid \tau_i \text{ is some integer for each } i \}$



Claim: **No timed automaton** can accept L_6

Step 1: *Suppose* $L_6 = \mathcal{L}(A)$

Let c_{max} be the maximum constant appearing in a guard of A

Step 1: *Suppose* $L_6 = \mathcal{L}(A)$

Let c_{max} be the maximum constant appearing in a guard of A

Step 2: For a clock x,

$$x = \lceil c_{max} \rceil + 1 \text{ and } x = \lceil c_{max} \rceil + 1.1$$

satisfy the same guards

Step 1: *Suppose* $L_6 = \mathcal{L}(A)$

Let c_{max} be the maximum constant appearing in a guard of A

Step 2: For a clock x,

$$x = \lceil c_{max} \rceil + 1 \text{ and } x = \lceil c_{max} \rceil + 1.1$$

satisfy the same guards

Step 3: $(a; \lceil c_{max} \rceil + 1) \in L_6$ and so A has an accepting run

$$(q_0, v_0) \xrightarrow{\delta = \lceil c_{max} \rceil + 1} (q_0, v_0 + \delta) \xrightarrow{a} (q_F, v_F)$$

Step 1: *Suppose* $L_6 = \mathcal{L}(A)$

Let c_{max} be the maximum constant appearing in a guard of A

Step 2: For a clock x,

$$x = \lceil c_{max} \rceil + 1 \text{ and } x = \lceil c_{max} \rceil + 1.1$$

satisfy the same guards

Step 3: $(a; \lceil c_{max} \rceil + 1) \in L_6$ and so A has an accepting run

$$(q_0, v_0) \xrightarrow{\delta = \lceil c_{max} \rceil + 1} (q_0, v_0 + \delta) \xrightarrow{a} (q_F, v_F)$$

Step 4: By Step 2, the following is an accepting run

$$(q_0, v_0) \xrightarrow{\delta' = \lceil c_{max} \rceil + 1.1} (q_0, v_0 + \delta') \xrightarrow{a} (q_F, v'_F)$$

Step 1: *Suppose* $L_6 = \mathcal{L}(A)$

Let c_{max} be the maximum constant appearing in a guard of A

Step 2: For a clock x,

$$x = \lceil c_{max} \rceil + 1 \text{ and } x = \lceil c_{max} \rceil + 1.1$$

satisfy the same guards

Step 3: $(a;\ \lceil c_{max} \rceil + 1) \in L_6$ and so A has an accepting run

$$(q_0, v_0) \xrightarrow{\delta \,=\, \lceil c_{max} \rceil + 1} (q_0, v_0 + \delta) \xrightarrow{a} (q_F, v_F)$$

Step 4: By Step 2, the following is an accepting run

$$(q_0, v_0) \xrightarrow{\delta' \,=\, \lceil c_{max} \rceil + 1.1} (q_0, v_0 + \delta') \xrightarrow{a} (q_F, v'_F)$$

Hence $(a;\ \lceil c_{max} \rceil + 1.1) \in \mathcal{L}(A) \neq L_6$

Therefore **no timed automaton** can accept L_6 $\square$

Timed automata

Runs

1 clock $<$ 2 clocks $< \ldots$

Role of max constant

Timed automata

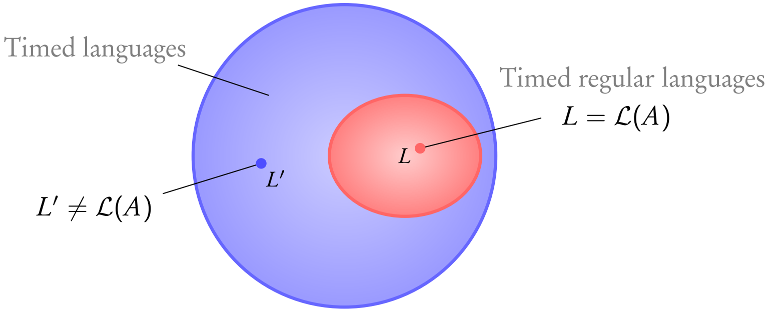
Runs

1 clock $<$ 2 clocks $< \ldots$

Role of max constant

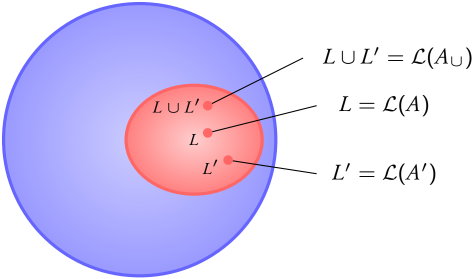
Timed regular lngs.

Timed regular languages



Definition

A timed language is called **timed regular** if it can be **accepted** by a timed automaton

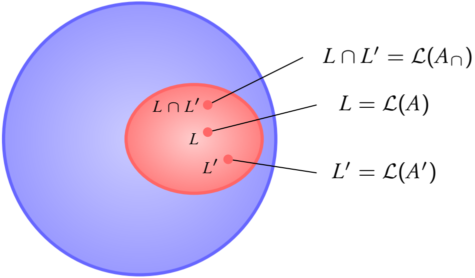


$$A = (Q, \Sigma, X, T, Q_0, F) \qquad A' = (Q', \Sigma, X', T', Q_0', F')$$

$$A_\cup = (\ Q \cup Q'\ ,\ \Sigma\ ,\ X \cup X'\ ,\ T \cup T'\ ,\ Q_0 \cup Q_0'\ ,\ F \cup F'\ )$$

$$\mathcal{L}(A) \cup \mathcal{L}(A') = \mathcal{L}(A_\cup)$$

Timed regular languages are **closed** under **union**



$$A = (Q, \Sigma, X, T, Q_0, F) \qquad\qquad A' = (Q', \Sigma, X', T', Q_0', F')$$

$$A_\cap = (\ Q \times Q'\ ,\ \Sigma\ ,\ X \cup X'\ ,\ T_\cap\ ,\ Q_0 \times Q_0'\ ,\ F \times F'\ )$$

$$T_\cap : \quad (q_1, q_1') \xrightarrow[R \cup R']{a,\ g \wedge g'} (q_2, q_2') \text{ if}$$

$$q_1 \xrightarrow[R]{a,\ g} q_2 \in T \quad \text{and} \quad q_1' \xrightarrow[R']{a,\ g'} q_2' \in T'$$

Timed regular languages are **closed** under **intersection**

L : a timed language over Σ

$$\text{Untime}(L) \equiv \{w \in \Sigma^* \mid \exists \tau.\ (w, \tau) \in L\}$$

Untiming construction

For **every timed** automaton A there is a **finite automaton** A_u s.t.

$$\text{Untime}(\ \mathcal{L}(A)\ ) = \mathcal{L}(A_u)$$

more about this later . . .

Complementation

$$\Sigma : \{a, b\}$$

$$\begin{aligned} L &= \{\, (w, \tau) \mid \text{there is an } a \text{ at some time } t \text{ and} \\ &\qquad \text{no action occurs at time } t+1 \,\} \end{aligned}$$

$$\begin{aligned} \overline{L} &= \{\, (w, \tau) \mid \text{every } a \text{ has an action at} \\ &\qquad \text{a distance 1 from it} \,\} \end{aligned}$$

Complementation

$$\Sigma : \{a, b\}$$

$$\begin{aligned} L &= \{\, (w, \tau) \mid \text{there is an } a \text{ at some time } t \text{ and} \\ &\qquad \text{no action occurs at time } t+1 \,\} \end{aligned}$$

$$\begin{aligned} \overline{L} &= \{\, (w, \tau) \mid \text{every } a \text{ has an action at} \\ &\qquad \text{a distance 1 from it} \,\} \end{aligned}$$

Claim: **No timed automaton** can accept $\overline{L}$

Decision problems for timed automata: A survey

Alur, Madhusudhan. *SFM'04: RT*

Step 1: $\overline{L} \;=\; \{\, (w, \tau) \mid$ every a has an action at a distance 1 from it $\}$

Suppose $\overline{L}$ is timed regular

Step 1: $\overline{L} \; = \; \{ \, (w, \tau) \mid$ every a has an action at a distance 1 from it $\}$

Suppose $\overline{L}$ is timed regular

Step 2: Let $L' = \{ \, (a^* b^*, \tau) \mid$ all a's occur before time 1 and no two a's happen at same time $\}$

Clearly L' is timed regular

Step 1: $\overline{L} = \{ (w, \tau) \mid$ every a has an action at a distance 1 from it $\}$

Suppose $\overline{L}$ is timed regular

Step 2: Let $L' = \{ (a^* b^*, \tau) \mid$ all a's occur before time 1 and no two a's happen at same time $\}$

Clearly L' is timed regular

Step 3: Untime($\overline{L} \cap L'$) should be a regular language

Step 1: $\overline{L} \;=\; \{\, (w, \tau) \mid$ every a has an action at a distance 1 from it $\}$

Suppose $\overline{L}$ is timed regular

Step 2: Let $L' = \{\, (a^* b^*, \tau) \mid$ all a's occur before time 1 and no two a's happen at same time $\}$

Clearly L' is timed regular

Step 3: Untime($\overline{L} \cap L'$) should be a regular language

Step 4: But, Untime($\overline{L} \cap L'$) $= \{a^n b^m \mid m \geq n\}$, ***not regular!***

Step 1: $\overline{L} \;=\; \{\, (w, \tau) \mid$ every a has an action at a distance 1 from it $\}$

Suppose $\overline{L}$ is timed regular

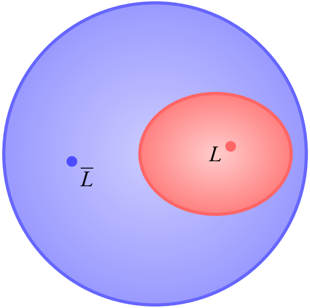
Step 2: Let $L' = \{\, (a^* b^*, \tau) \mid$ all a's occur before time 1 and no two a's happen at same time $\}$

Clearly L' is timed regular

Step 3: Untime($\overline{L} \cap L'$) should be a regular language

Step 4: But, Untime($\overline{L} \cap L'$) $= \{a^n b^m \mid m \geq n\}$, *not regular!*

Therefore $\overline{L}$ **cannot be timed regular** $\square$



Timed regular languages are **not closed** under **complementation**

Timed automata

Runs

1 clock $<$ 2 clocks $< \dots$

Role of max constant

Timed regular lngs.

Closure under $\cup$, $\cap$

Non-closure under complement

Timed automata

Runs

1 clock $<$ 2 clocks $< \ldots$

Role of max constant

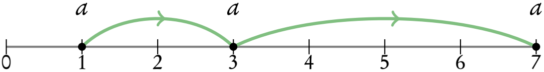
Timed regular lngs.

Closure under $\cup, \cap$

Non-closure under complement

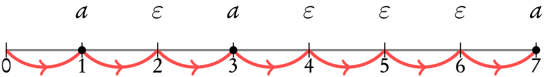
ε-transitions

$$L_6 := \{\ (\ a^k, \tau\ )\ |\ \tau_i \text{ is some integer for each } i\}$$

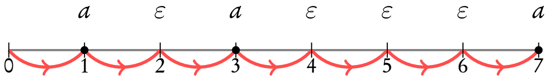


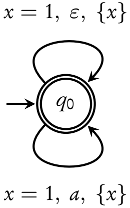
Claim: **No timed automaton** can accept L_6

$$L_6 := \{\ (\ a^k, \tau\ )\ |\ \tau_i \text{ is some integer for each } i\}$$



$$L_6 := \{\ (\ a^k, \tau\ )\ |\ \tau_i \text{ is some integer for each } i\}$$





ε-transitions

ε-transitions **add expressive power** to timed automata.

Characterization of the expressive power of silent transitions in timed automata

Bérard, Diekert, Gastin, Petit. *Fundamenta Informaticae'98*

ε-transitions

ε-transitions **add expressive power** to timed automata. However, they add power **only** when a clock is **reset** in an ε-transition.

Characterization of the expressive power of silent transitions in timed automata

Bérard, Diekert, Gastin, Petit. *Fundamenta Informaticae'98*

Timed automata

Runs

1 clock $<$ 2 clocks $< \dots$

Role of max constant

Timed regular lngs.

Closure under $\cup$, $\cap$

Non-closure under complement

ε-transitions

More expressive

$\xrightarrow{\varepsilon}$ without reset $\equiv$ TA