Lecture 24, 10 November 2026

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Programming and Data Structures with Python

Lecture 24, 10 Nov 2026

Dynamic sorted data

- Sorting is useful for efficient searching
- What if the data is changing dynamically?
  - Items are periodically inserted and deleted
- Insert/delete in a sorted list takes time $O(n)$

Priority Queue

Dynamic collection

insert()

delete_max()

2-D structure

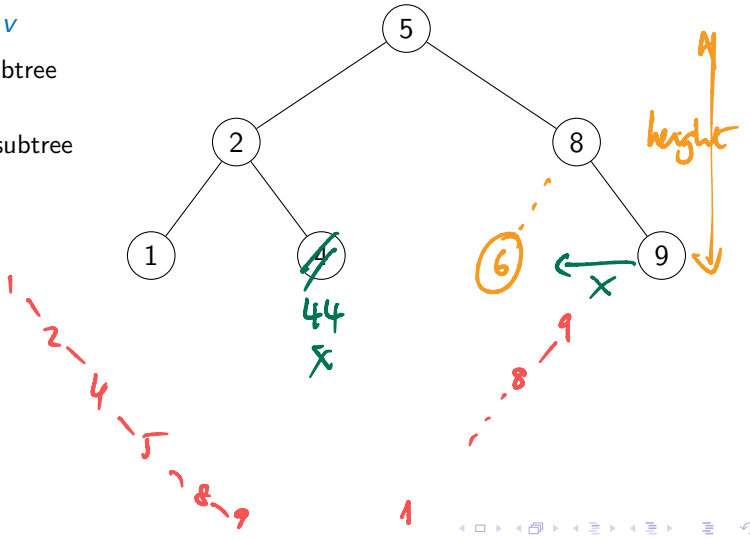
Heap

Dynamic sorted data

- Sorting is useful for efficient searching
- What if the data is changing dynamically?
  - Items are periodically inserted and deleted
- Insert/delete in a sorted list takes time $O(n)$
- Move to a tree structure, like heaps for priority queues

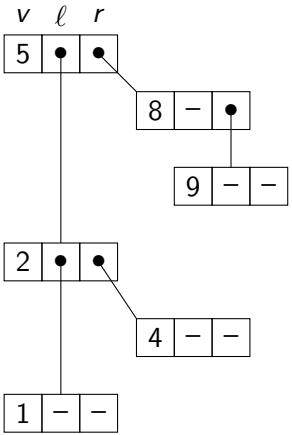
Binary search tree

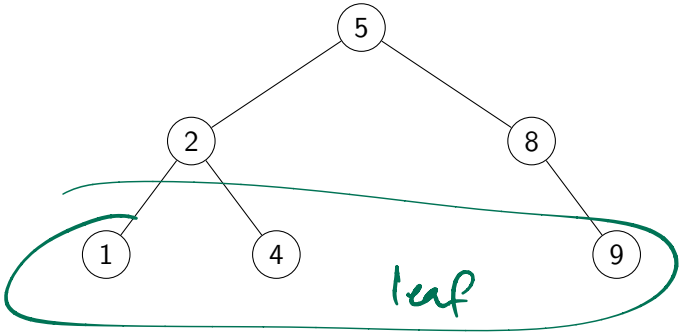
- For each node with value v
  - All values in the left subtree are $< v$
  - All values in the right subtree are $> v$
- No duplicate values



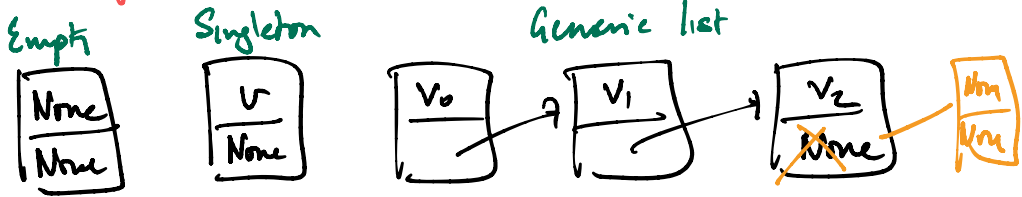
Implementing a binary search tree

- Each node has a value and pointers to its children





Inductive defn → inductive/recursive exploration



append (v)

- if empty --
- if singleton --

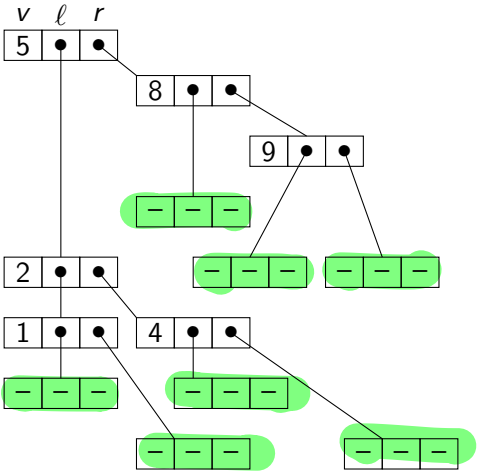
] 2 base case

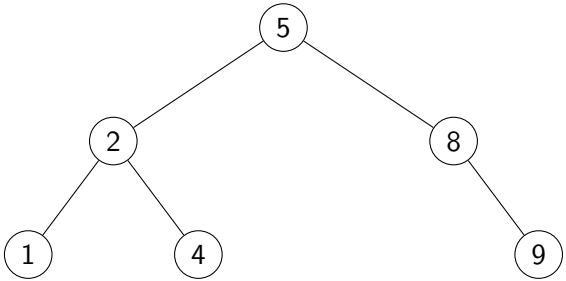
else

next. append

Implementing a binary search tree

- Each node has a value and pointers to its children

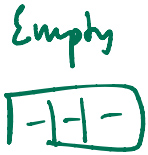


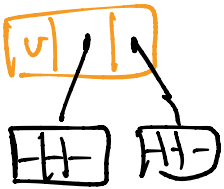


- Add a frontier with empty nodes, all fields –
  - Empty tree is single empty node
  - Leaf node points to empty nodes
- Easier to implement operations recursively

The class Tree

- Three local fields, `value`, `left`, `right`
- Value `None` for empty value –
- Empty tree has all fields `None`
- Leaf has a nonempty `value` and empty `left` and `right`






```
class Tree:

    # Constructor:
    def __init__(self,initval=None):
        self.value = initval
        if self.value != None:
            self.left = Tree()
            self.right = Tree()
        else:
            self.left = None
            self.right = None
        return

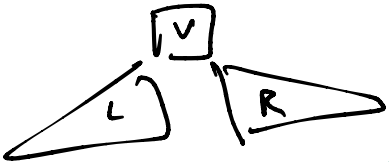
    # Only empty node has value None
    def isempty(self):
        return (self.value == None)

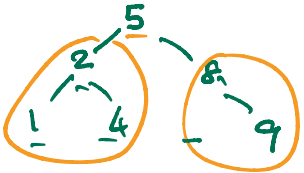
    # Leaf nodes have both children empty
    def isleaf(self):
        return (self.value != None and
                self.left.isempty() and
                self.right.isempty())
```

Create frontier

Inorder traversal

- List the left subtree, then the current node, then the right subtree
- Lists values in sorted order
- Use to print the tree





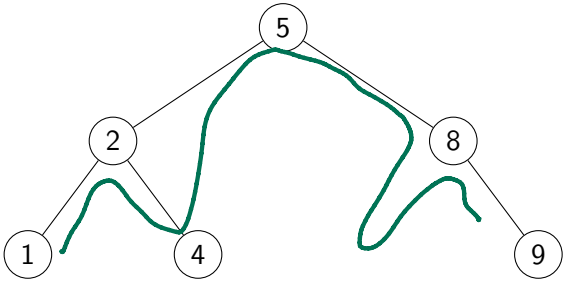

```
class Tree:
    ...
    # Inorder traversal
    def inorder(self):
        if self.isempty():
            return([])
        else:
            return(self.left.inorder()+
                   [self.value]+
                   self.right.inorder())

    # Display Tree as a string
    def __str__(self):
        return(str(self.inorder()))
```

1 - 2 - 4 - 5 - 8 - 9

Inorder traversal

- List the left subtree, then the current node, then the right subtree
- Lists values in sorted order
- Use to print the tree




```
class Tree:
    ...
    # Inorder traversal
    def inorder(self):
        if self.isempty():
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        else:
            return(self.left.inorder()+
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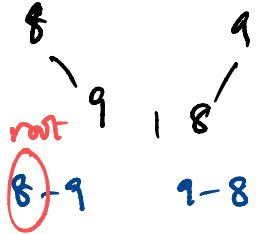
    # Display Tree as a string
    def __str__(self):
        return(str(self.inorder()))
```

Preorder

[v] - left - right

Postorder

left - right - [v]



Find a value v

- Check value at current node
- If v smaller than current node, go left
- If v smaller than current node, go right
- Natural generalization of binary search

```
class Tree:
    ...
    # Check if value v occurs in tree
    def find(self,v):
        if self.isempty():
            return(False)

        if self.value == v:
            return(True)

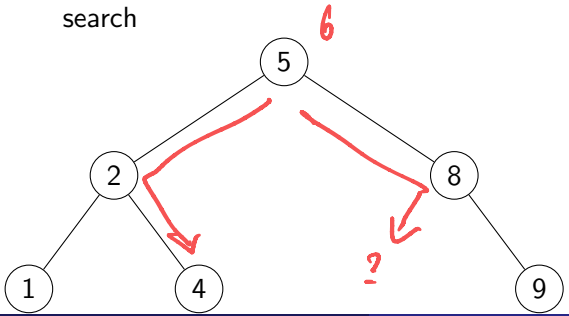
        if v < self.value:
            return(self.left.find(v))

        if v > self.value:
            return(self.right.find(v))
```

No check for leaf node

Find a value v

- Check value at current node
- If v smaller than current node, go left
- If v smaller than current node, go right
- Natural generalization of binary search




```
class Tree:
    ...
    # Check if value v occurs in tree
    def find(self,v):
        if self.isempty():
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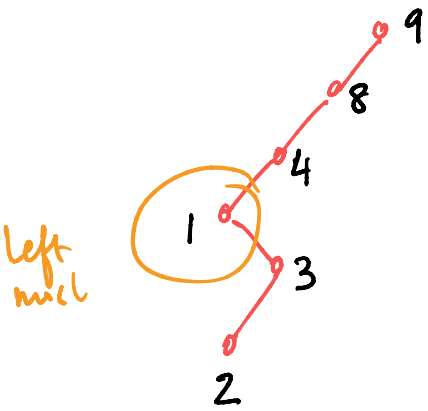
        if self.value == v:
            return(True)

        if v < self.value:
            return(self.left.find(v))

        if v > self.value:
            return(self.right.find(v))
```

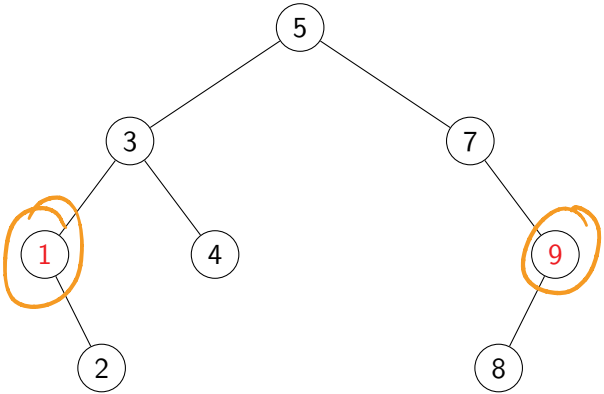
Minimum and maximum

- Minimum is left most node in the tree
- Maximum is right most node in the tree



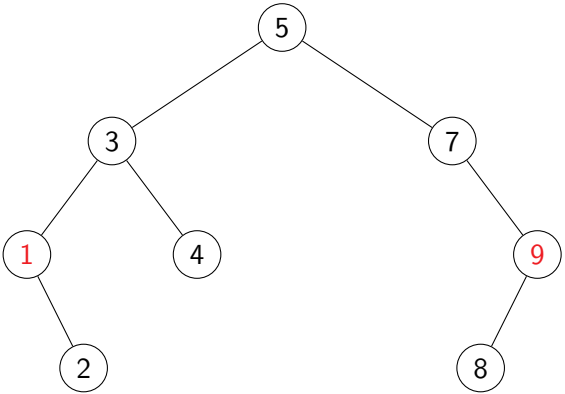
Minimum and maximum

- Minimum is left most node in the tree
- Maximum is right most node in the tree



Minimum and maximum

- Minimum is left most node in the tree
- Maximum is right most node in the tree




```
class Tree:
    ...
    def minval(self):
        if self.left.isempty():
            return(self.value)
        else:
            return(self.left.minval())


    def maxval(self):
        if self.right.isempty():
            return(self.value)
        else:
            return(self.right.maxval())
```

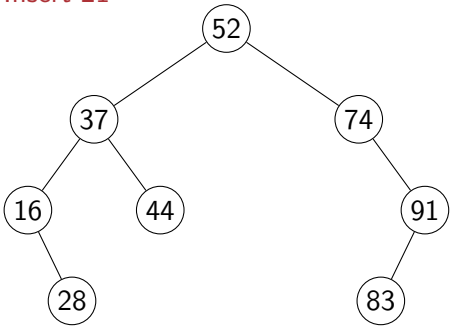
Insert a value v

- Try to find `v`
- Insert at the position where `find` fails

Insert a value v

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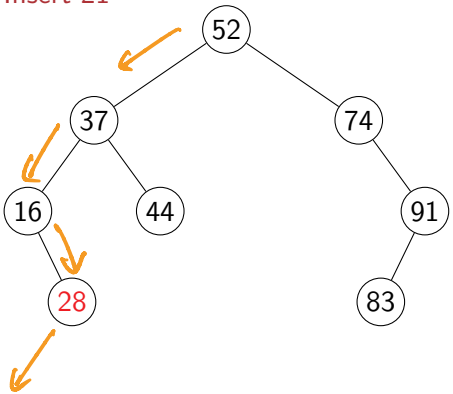
Insert 21



Insert a value v

- Try to find `v`
- Insert at the position where `find` fails

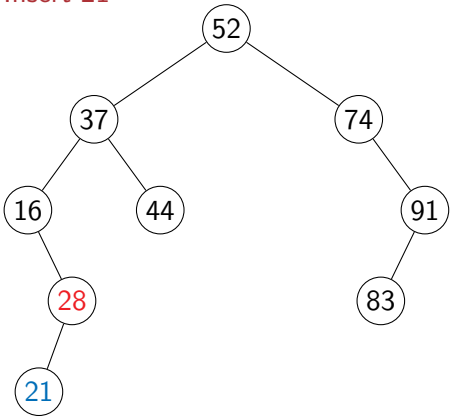
Insert 21



Insert a value v

- Try to find `v`
- Insert at the position where `find` fails

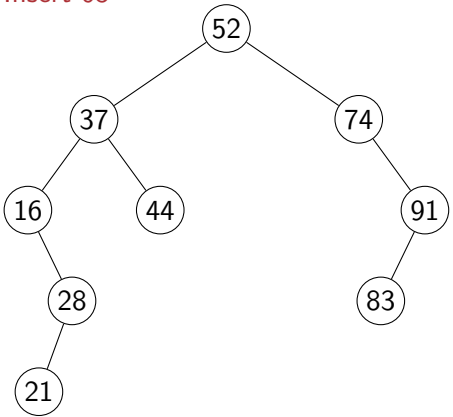
Insert 21



Insert a value v

- Try to find `v`
- Insert at the position where `find` fails

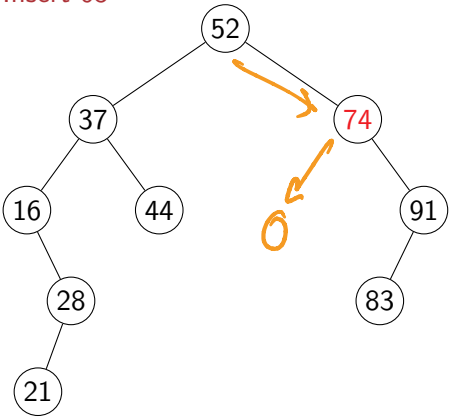
Insert 65



Insert a value v

- Try to find `v`
- Insert at the position where `find` fails

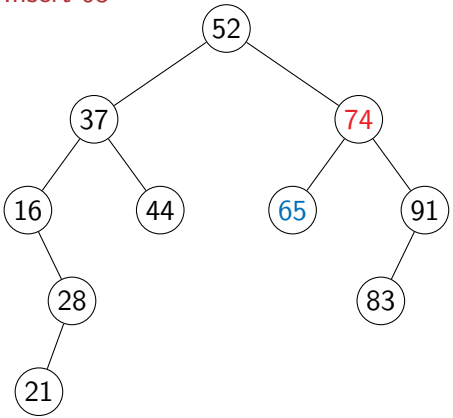
Insert 65



Insert a value v

- Try to find `v`
- Insert at the position where `find` fails

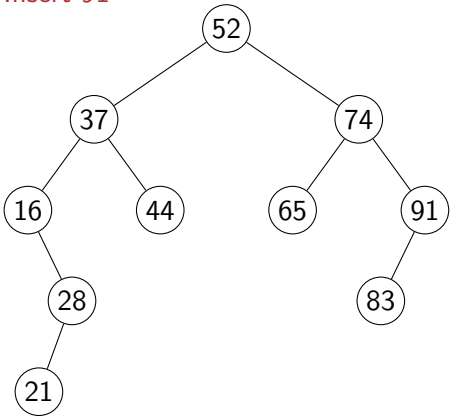
Insert 65



Insert a value v

- Try to find `v`
- Insert at the position where `find` fails

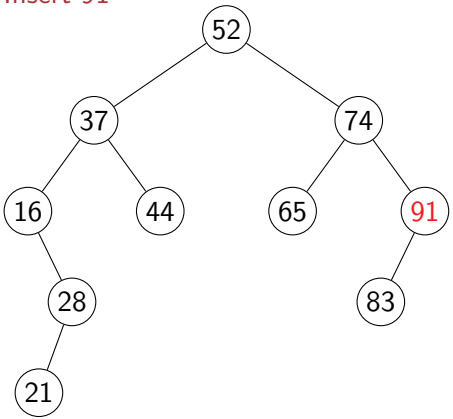
Insert 91



Insert a value v

- Try to find `v`
- Insert at the position where `find` fails

Insert 91




```
class Tree:
    ...
    def insert(self,v):
        if self.isempty():
            self.value = v
            self.left = Tree()
            self.right = Tree()

        if self.value == v:
            return

        if v < self.value:
            self.left.insert(v)
            return

        if v > self.value:
            self.right.insert(v)
            return
```

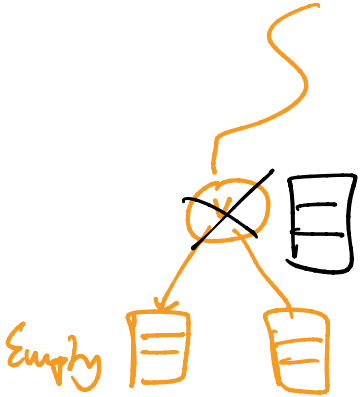
restore frontier

Delete a value v

- If v is present, delete

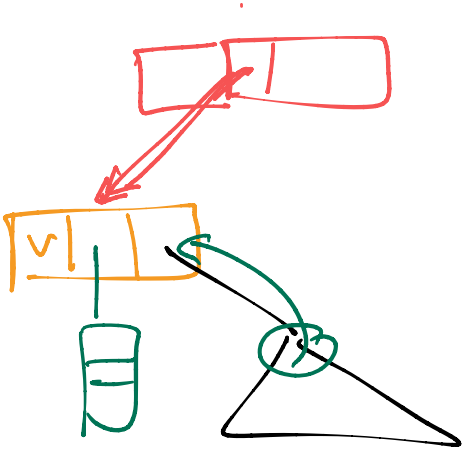
Delete a value v

- If v is present, delete
- Leaf node? No problem

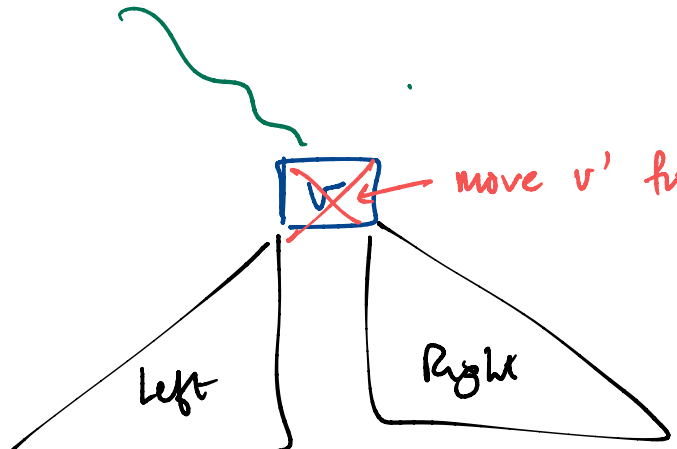


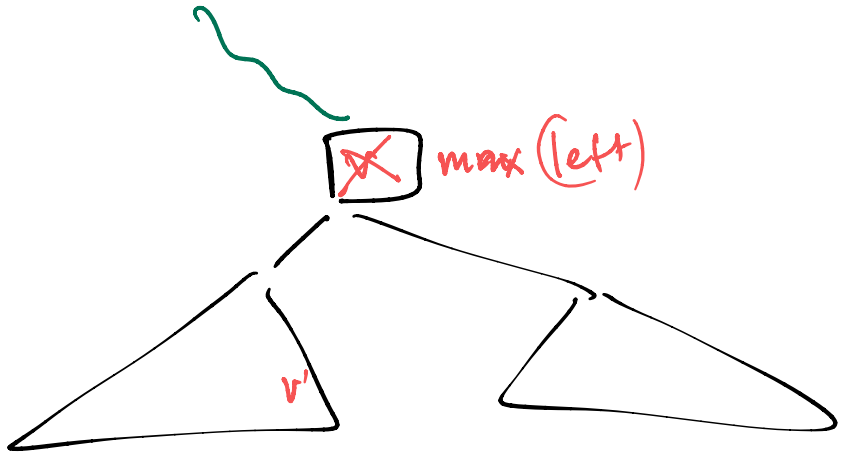
Delete a value v

- If v is present, delete
- Leaf node? No problem
- If only one child, promote that subtree



v has 2 children




max (left)
v'
v' has no right child

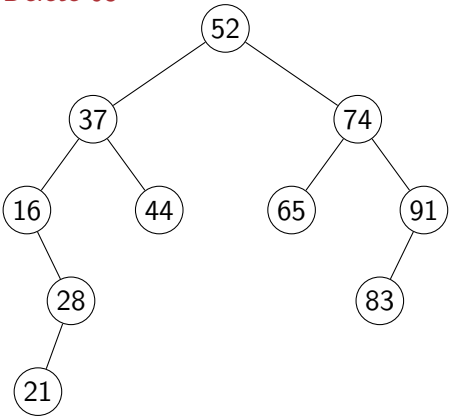
Delete a value v

- If `v` is present, delete
- Leaf node? No problem
- If only one child, promote that subtree
- Otherwise, replace `v` with `self.left.maxval()` and delete `self.left.maxval()`
  - `self.left.maxval()` has no right child

```
class Tree:
    ...
    def delete(self,v):
        if self.isempty():
            return
        if v < self.value:
            self.left.delete(v)
            return
        if v > self.value:
            self.right.delete(v)
            return
        if v == self.value:
            if self.isleaf():
                self.makeempty()
            elif self.left.isempty():
                self.copyright()
            elif self.right.isempty():
                self.copyleft()
            else:
                self.value = self.left.maxval()
                self.left.delete(self.left.maxval())
            return
```

Delete a value v

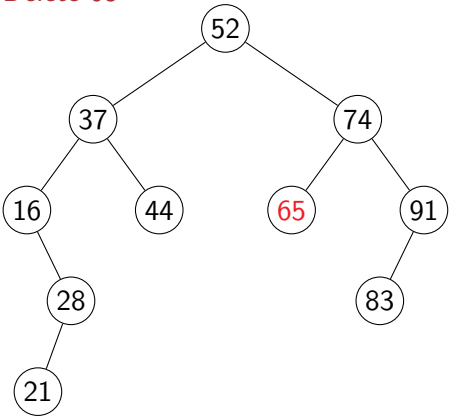
Delete 65




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            return
        if v > self.value:
            self.right.delete(v)
            return
        if v == self.value:
            if self.isleaf():
                self.makeempty()
            elif self.left.isempty():
                self.copyright()
            elif self.right.isempty():
                self.copyleft()
            else:
                self.value = self.left.maxval()
                self.left.delete(self.left.maxval())
            return
```

Delete a value v

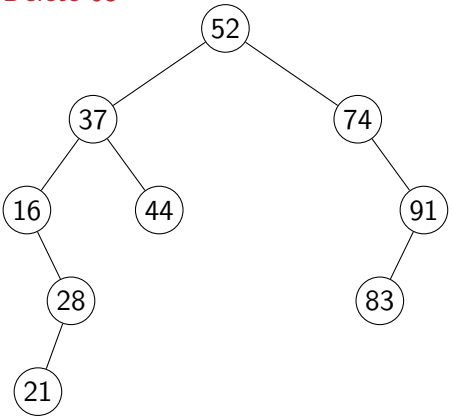
Delete 65




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    ...
    def delete(self,v):
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        if v < self.value:
            self.left.delete(v)
            return
        if v > self.value:
            self.right.delete(v)
            return
        if v == self.value:
            if self.isleaf():
                self.makeempty()
            elif self.left.isempty():
                self.copyright()
            elif self.right.isempty():
                self.copyleft()
            else:
                self.value = self.left.maxval()
                self.left.delete(self.left.maxval())
            return
```

Delete a value v

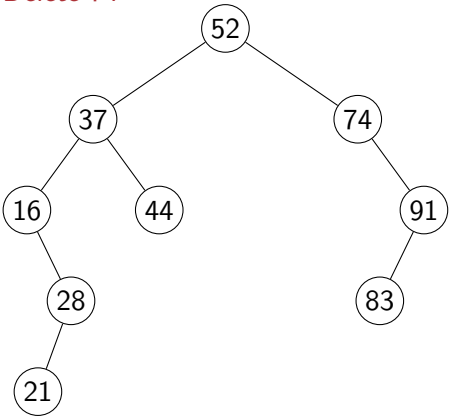
Delete 65




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class Tree:
    ...
    def delete(self,v):
        if self.isempty():
            return
        if v < self.value:
            self.left.delete(v)
            return
        if v > self.value:
            self.right.delete(v)
            return
        if v == self.value:
            if self.isleaf():
                self.makeempty()
            elif self.left.isempty():
                self.copyright()
            elif self.right.isempty():
                self.copyleft()
            else:
                self.value = self.left.maxval()
                self.left.delete(self.left.maxval())
            return
```

Delete a value v

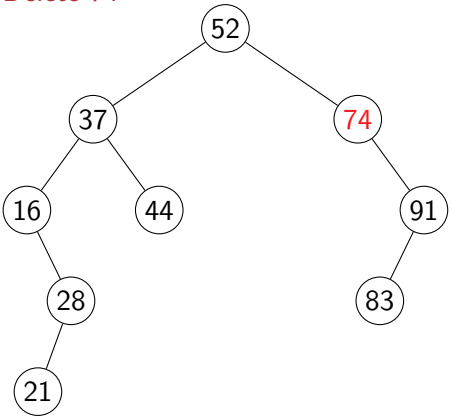
Delete 74




```
class Tree:
    ...
    def delete(self,v):
        if self.isempty():
            return
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            self.left.delete(v)
            return
        if v > self.value:
            self.right.delete(v)
            return
        if v == self.value:
            if self.isleaf():
                self.makeempty()
            elif self.left.isempty():
                self.copyright()
            elif self.right.isempty():
                self.copyleft()
            else:
                self.value = self.left.maxval()
                self.left.delete(self.left.maxval())
            return
```

Delete a value v

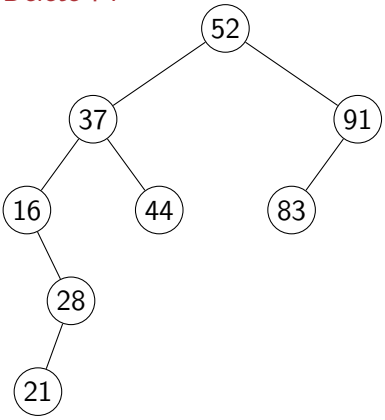
Delete 74




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            self.right.delete(v)
            return
        if v == self.value:
            if self.isleaf():
                self.makeempty()
            elif self.left.isempty():
                self.copyright()
            elif self.right.isempty():
                self.copyleft()
            else:
                self.value = self.left.maxval()
                self.left.delete(self.left.maxval())
            return
```

Delete a value v

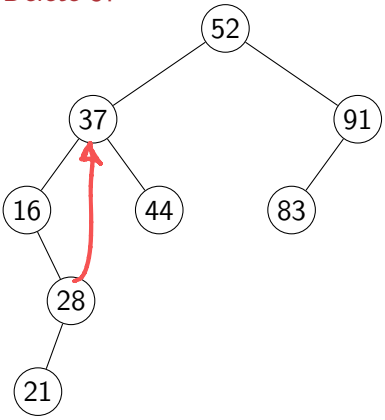
Delete 74




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class Tree:
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            self.right.delete(v)
            return
        if v == self.value:
            if self.isleaf():
                self.makeempty()
            elif self.left.isempty():
                self.copyright()
            elif self.right.isempty():
                self.copyleft()
            else:
                self.value = self.left.maxval()
                self.left.delete(self.left.maxval())
            return
```

Delete a value v

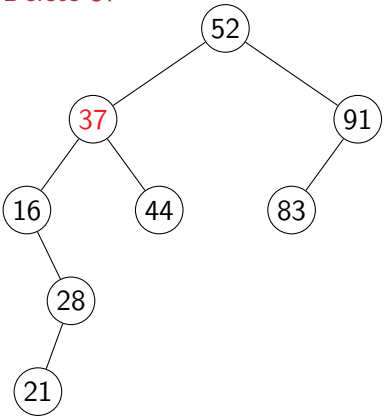
Delete 37




```
class Tree:
    ...
    def delete(self,v):
        if self.isempty():
            return
        if v < self.value:
            self.left.delete(v)
            return
        if v > self.value:
            self.right.delete(v)
            return
        if v == self.value:
            if self.isleaf():
                self.makeempty()
            elif self.left.isempty():
                self.copyright()
            elif self.right.isempty():
                self.copyleft()
            else:
                self.value = self.left.maxval()
                self.left.delete(self.left.maxval())
            return
```

Delete a value v

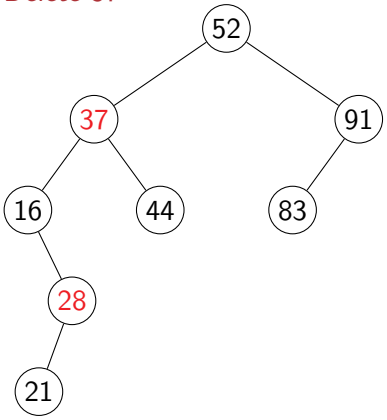
Delete 37




```
class Tree:
    ...
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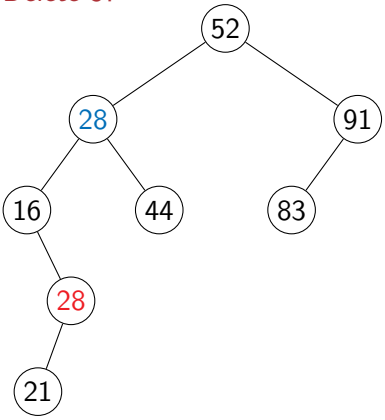
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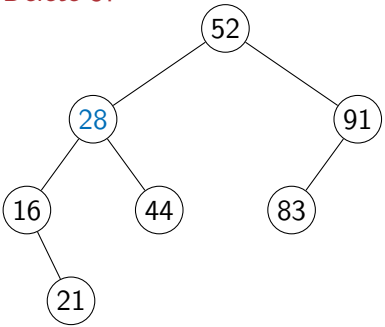
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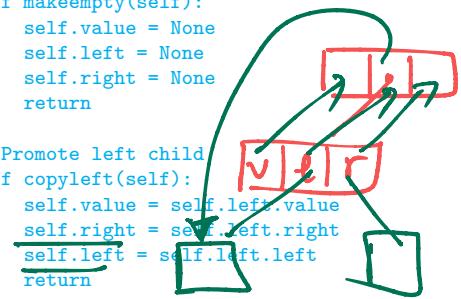
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            return
```

```
# Convert leaf node to empty node
def makeempty(self):
    self.value = None
    self.left = None
    self.right = None
    return

# Promote left child
def copyleft(self):
    self.value = self.left.value
    self.right = self.left.right
    self.left = self.left.left
    return

# Promote right child
def copyright(self):
    self.value = self.right.value
    self.left = self.right.left
    self.right = self.right.right
    return
```



Complexity

- `find()`, `insert()` and `delete()` all walk down a single path
- Worst-case: height of the tree
- An unbalanced tree with n nodes may have height $O(n)$
- Balanced trees have height $O(\log n)$
- How can we maintain balance as tree grows and shrinks?



Operations on search trees

Defining balance

- Left and right subtrees should be "equal"
  - Two possible measures: `size` and `height`

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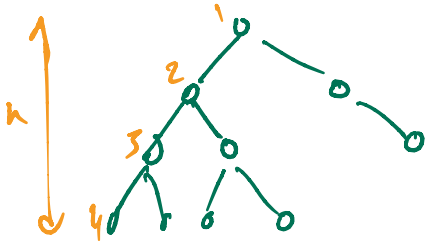
Operations on search trees

Defining balance

- Left and right subtrees should be "equal"
  - Two possible measures: `size` and `height`
- `self.left.size()` and `self.right.size()` are equal?
  - Only possible for complete binary trees
- `self.left.size()` and `self.right.size()` differ by at most 1?
  - Plausible, but difficult to maintain

Height balanced trees

- `self.height()` — number of nodes on longest path from root to leaf
  - 0 for empty tree
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  - 1 + max of heights of left and right subtrees, in general



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  - AVL trees — Adelson-Velskii, Landis

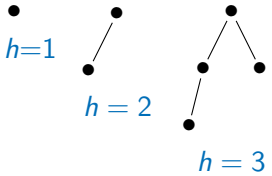
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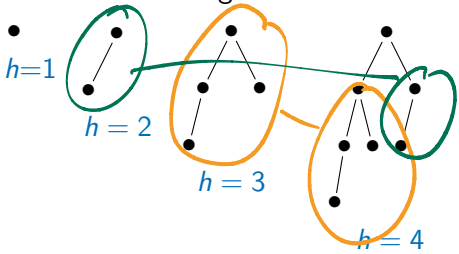
- Minimum size height-balanced trees



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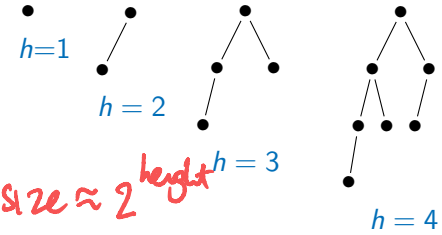
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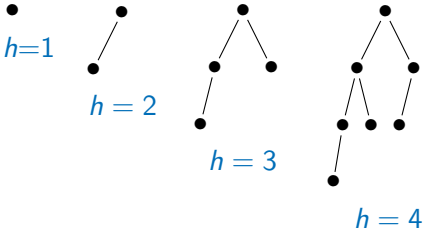


Want size $\approx 2^{\text{height}}$

- General strategy to build a small balanced tree of height h
  - Smallest balanced tree of height $h-1$ as left subtree
  - Smallest balanced tree of height $h-2$ as right subtree

Height balanced trees

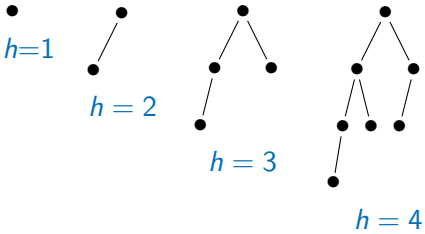
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Height balanced trees

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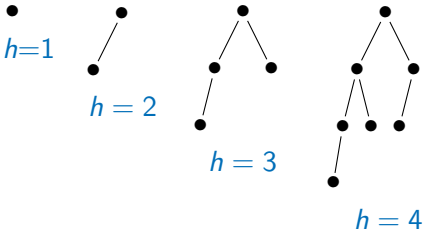


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- $S(h)$, size of smallest height-balanced tree of height h

Height balanced trees

- Minimum size height-balanced trees

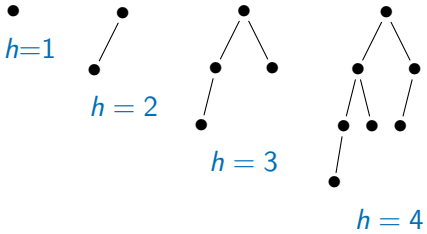


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- $S(h)$, size of smallest height-balanced tree of height h
- Recurrence
  - $S(0) = 0$, $S(1) = 1$
  - $S(h) = 1 + S(h-1) + S(h-2)$

Height balanced trees

- Minimum size height-balanced trees



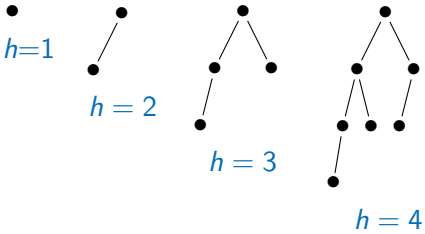
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- Recurrence
  - $S(0) = 0$, $S(1) = 1$
  - $S(h) = 1 + S(h-1) + S(h-2)$
- Compare to Fibonacci sequence
  - $F(0) = 0$, $F(1) = 1$
  - $F(n) = F(n-1) + F(n-2)$

$S(h) \geq F(h)$

Height balanced trees

- Minimum size height-balanced trees



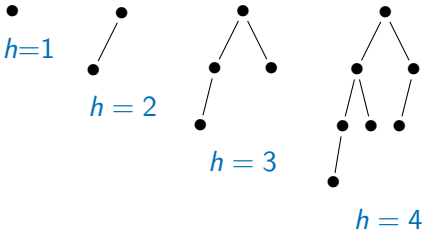
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  - $F(n) = F(n-1) + F(n-2)$
- $S(h)$ grows exponentially with h

because F(n) grows exponentially

Height balanced trees

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- $S(h)$ grows exponentially with h
- For size n, h is $O(\log n)$

Correcting imbalance

- Slope of a node : `self.left.height() - self.right.height()`

Correcting imbalance

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- Balanced tree — slope is $\{-1, 0, 1\}$



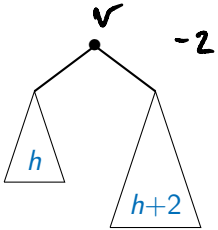
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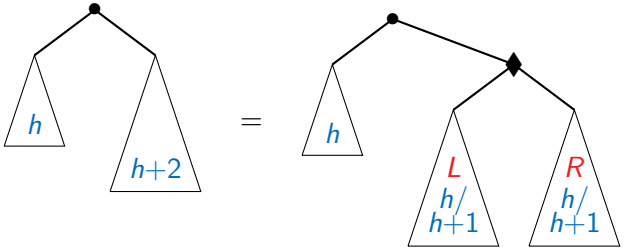
Left rotation



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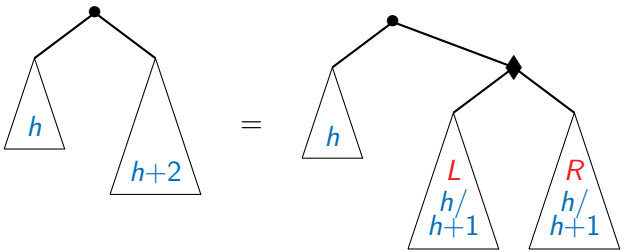
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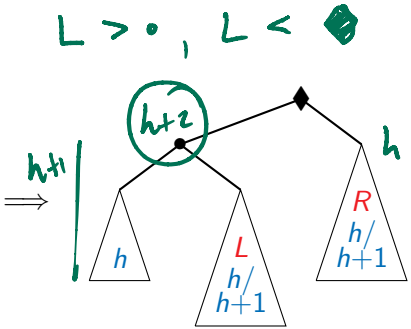


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Left rotation — converts slope -2 to $\{0, 1, 2\}$

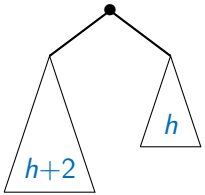




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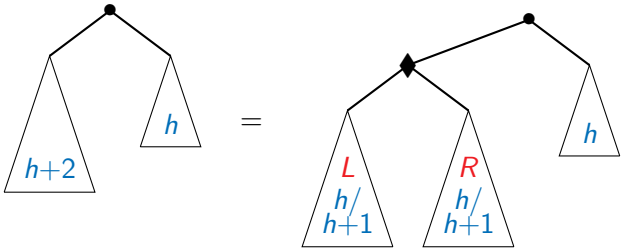
Right rotation



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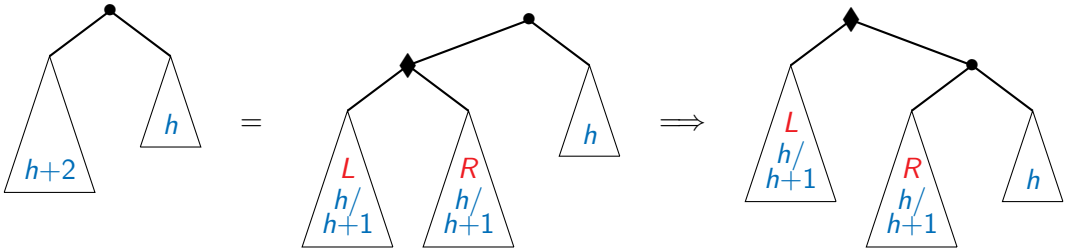
Right rotation



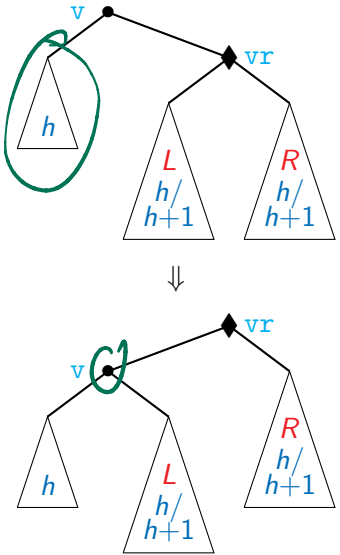
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- `t.insert(v)`, `t.delete(v)` can alter slope to -2 or $+2$

Right rotation — converts slope $+2$ to $\{-2, -1, 0\}$



Implementing rotations




```
class Tree:
  ...

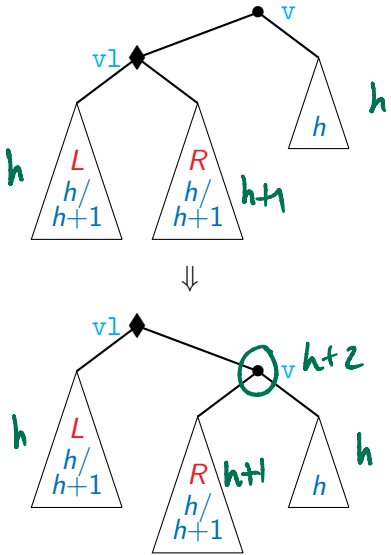
  def leftrotate(self):
    v = self.value
    vr = self.right.value
    tl = self.left
    trl = self.right.left
    trr = self.right.right

    newleft = Tree(v)
    newleft.left = tl
    newleft.right = trl

    self.value = vr
    self.left = newleft
    self.right = trr

    return
```

Implementing rotations




```
class Tree:
  ...

  def rightrotate(self):
    v = self.value
    vl = self.left.value
    tll = self.left.left
    tlr = self.left.right
    tr = self.right

    newright = Tree(v)
    newright.left = tlr
    newright.right = tr

    self.value = vl
    self.left = tll
    self.right = newright

    return
```