Lecture 26, 20 November 2025

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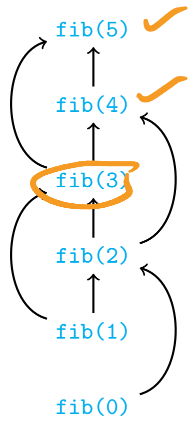
Programming and Data Structures with Python

Lecture 26, 20 Nov 2025

Dynamic programming

- Anticipate the structure of subproblems
  - Derive from inductive definition
  - Dependencies are acyclic
- Solve subproblems in appropriate order
  - Start with base cases — no dependencies
  - Evaluate a value after all its dependencies are available
  - Fill table iteratively
  - Never need to make a recursive call

Evaluating `fib(5)`



Longest common subsequence

- Subsequence — can drop some letters in between

- Given two strings, find the (length of the) longest common subsequence
  - `"secret"`, `"secretary"` — `"secret"`, length 6
  - `"bisect"`, `"trisect"` — `"isect"`, length 5
  - `"bisect"`, `"secret"` — `"sect"`, length 4
  - `"director"`, `"secretary"` — `"ectr"`, `"retr"`, length 4

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  - "bisect", "secret" — "sect", length 4
  - "director", "secretary" — "ectr", "retr", length 4
- LCS is the longest path connecting non-zero LCW entries, moving right/down

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	0	0	0	0	0	0	0
1	i	0	0	0	0	0	0	0
2	s	3	0	0	0	0	0	0
3	e	0	2	0	0	1	0	0
4	c	0	0	1	0	0	0	0
5	t	0	0	0	0	0	1	0
6	•	0	0	0	0	0	0	0

Applications

- Analyzing genes
  - DNA is a long string over `A`, `T`, `G`, `C`
  - Two species are similar if their DNA has long common subsequences
- `diff` command in Unix/Linux
  - Compares text files
  - Find the longest matching subsequence of lines
  - Each line of text is a "character"

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1	i	**0**	**0**	**0**	**0**	**0**	**0**	**0**
2	s	**3**	**0**	**0**	**0**	**0**	**0**	**0**
3	e	**0**	**2**	**0**	**0**	**1**	**0**	**0**
4	c	**0**	**0**	**1**	**0**	**0**	**0**	**0**
5	t	**0**	**0**	**0**	**0**	**0**	**1**	**0**
6	•	**0**	**0**	**0**	**0**	**0**	**0**	**0**

Inductive structure

- $u = a_0 a_1 \ldots a_{m-1}$, $v = b_0 b_1 \ldots b_{n-1}$
- $LCS(i,j)$ — length of longest common subsequence in $a_i a_{i+1} \ldots a_{m-1}$, $b_j b_{j+1} \ldots b_{n-1}$

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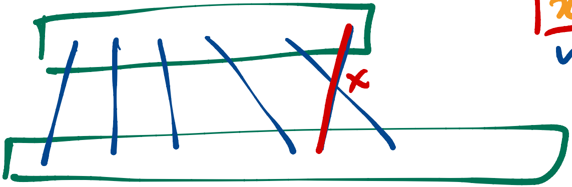

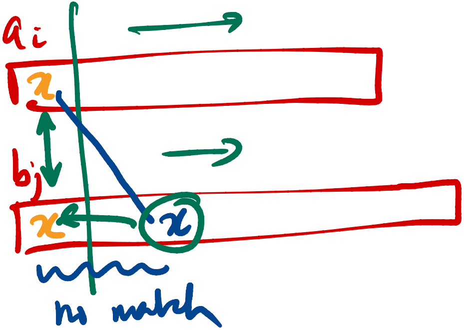

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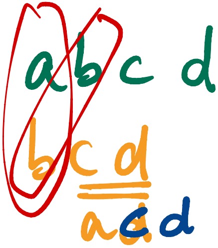
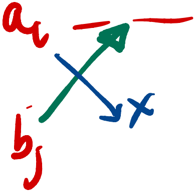
  - Which one should we drop?





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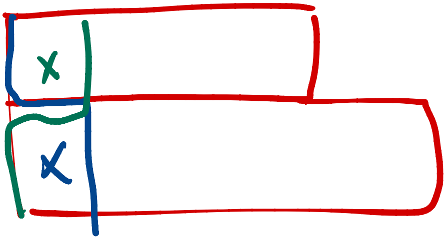
skip b_j skip a_i



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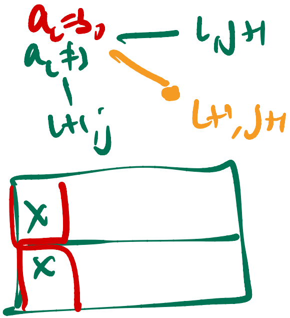
- If $a_i \neq b_j$, a_i and b_j cannot both belong to LCS
  - Which one should we drop?
  - $LCS(i,j) = \max(LCS(i,j+1), LCS(i+1,j))$
- Base cases as with LCW
  - $LCS(i,n) = 0$ for all $0 \leq i \leq m$
  - $LCS(m,j) = 0$ for all $0 \leq j \leq n$



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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							
1	i							
2	s							
3	e							
4	c							
5	t							
6	•							

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							**0**
1	i							**0**
2	s							**0**
3	e							**0**
4	c							**0**
5	t							**0**
6	•							**0**

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b						1	0
1	i						1	0
2	s						1	0
3	e						1	0
4	c		$c \neq t$				1	0
5	t						1	0
6	•						0	0

max

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		s	e	c	r	e	t	•
0	b					**2**	**1**	**0**
1	i					**2**	**1**	**0**
2	s					**2**	**1**	**0**
3	e					**2**	**1**	**0**
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0	b			**2**	**2**	**2**	**1**	**0**
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1	i	4	3	2	2	2	1	0
2	s	4	3	2	2	2	1	0
3	e	3	3	2	2	2	1	0
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Reading off the solution

- Trace back the path by which each entry was filled

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	4	3	2	2	2	1	0
1	i	4	3	2	2	2	1	0
2	s	4	3	2	2	2	1	0
3	e	3	3	2	2	2	1	0
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Reading off the solution

- Trace back the path by which each entry was filled
- Each diagonal step is an element of LCS

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	4	3	2	2	2	1	0
1	i	4	3	2	2	2	1	0
2	s	4	3	2	2	2	1	0
3	e	3	3	2	2	2	1	0
4	c	2	2	2	1	1	1	0
5	t	1	1	1	1	1	1	0
6	•	0	0	0	0	0	0	0

Implementation

```
def LCS(u,v):
  import numpy as np
  (m,n) = (len(u),len(v))
  lcs = np.zeros((m+1,n+1))

  for j in range(n-1,-1,-1):
    for i in range(m-1,-1,-1):
      if u[i] == v[j]:
        lcs[i,j] = 1 + lcs[i+1,j+1]
      else:
        lcs[i,j] = max(lcs[i+1,j],
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  return(lcs[0,0])
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Complexity

- Again $O(mn)$, using dynamic programming or memoization
  - Fill a table of size $O(mn)$
  - Each table entry takes constant time to compute

Document similarity

- "The students were able to appreciate the concept optimal substructure property and its use in designing algorithms"
- "The lecture taught the students to appreciate how the concept of optimal substructures can be used in designing algorithms"

Document similarity

- ~~"The students were able to appreciate the concept optimal substructure property and its use in designing algorithms"~~ Delete
- "The lecture taught the students to appreciate how the concept of optimal substructures can be used in designing algorithms" Insert
- Edit operations to transform documents
  - Insert a character
  - Delete a character
  - Substitute one character by another Delete + Insert

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- insert, ~~delete~~, substitute

Edit distance

- Minimum number of edit operations needed
- In our example, 24 characters inserted, 18 ~~deleted~~, 2 substituted
- Edit distance is at most 44

Edit distance

- Minimum number of editing operations needed to transform one document to the other
  - Insert a character
  - Delete a character
  - Substitute one character by another
- Also called Levenshtein distance
  - Vladimir Levenshtein, 1965
- Applications
  - Suggestions for spelling correction
  - Genetic similarity of species

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  - `bisect`, `secret` — LCS is `sect`

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  - ~~bi~~sect, sec~~re~~t — LCS is sect
  - Delete b, i in bisect and r, e in secret

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- Deleting a letter from u is equivalent to inserting it in v
  re
  - `~~bi~~sect`, `secret` — LCS is `sect`
  - Delete `b`, `i` in `bisect` and `r`, `e` in `secret`
  - Delete `b`, `i` and then insert `r`, `e` in `bisect`
- From LCS, we can compute edit distance without substitution

Inductive structure for edit distance

- $u = a_0 a_1 \ldots a_{m-1}$, $v = b_0 b_1 \ldots b_{n-1}$

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- Recall LCS

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- Edit distance — aim is to transform u to v

b_j

$\cancel{a_i}\, a_{i+1} \ldots\ a_{m-1}$

$\cancel{b_j}\, b_{j+1} \ldots\ b_{n-1}$

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- Edit distance — aim is to transform u to v
- If $a_i = b_j$, nothing to be done

Inductive structure for edit distance

- $u = a_0 a_1 \ldots a_{m-1}$, $v = b_0 b_1 \ldots b_{n-1}$

- Recall LCS
- $LCS(i,j)$ — length of longest common subsequence in $a_i a_{i+1} \ldots a_{m-1}$, $b_j b_{j+1} \ldots b_{n-1}$
- If $a_i = b_j$,
  $LCS(i,j) = 1 + LCS(i{+}1, j{+}1)$
- If $a_i \neq b_j$,
  $LCS(i,j) = \max[\ LCS(i, j{+}1),$
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- Edit distance — aim is to transform u to v
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- If $a_i = b_j$, nothing to be done
- If $a_i \neq b_j$, best among
  - Substitute a_i by b_j
  - Delete a_i
  - Insert b_j before a_i

all changes only on u

Inductive structure for edit distance

- $u = a_0 a_1 \ldots a_{m-1}$, $v = b_0 b_1 \ldots b_{n-1}$
- $ED(i,j)$ — edit distance for $a_i a_{i+1} \ldots a_{m-1}$, $b_j b_{j+1} \ldots b_{n-1}$
- If $a_i = b_j$, nothing to be done
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- If $a_i = b_j$,
  $ED(i,j) = ED(i+1, j+1)$
- If $a_i \neq b_j$,
  $ED(i,j) = 1 + \min[\ ED(i+1, j+1),$ subst
  $ED(i+1, j),$ del
  $ED(i, j+1)\ ]$ insert

Inductive structure for edit distance

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$\overbrace{a_0 - a_{i-1}}^{i}\ \underline{\cancel{a_i\, a_{i+1} \ldots a_{m-1}}}$

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  $ED(i,j) = ED(i+1, j+1)$
- If $a_i \neq b_j$,
  $ED(i,j) = 1 + \min[\ ED(i+1,j+1),\ ED(i+1,j),\ ED(i,j+1)\ ]$

  min
- Base cases
  - $ED(m,n) = 0$
  - $ED(i,n) = \underline{m-i}$ for all $0 \leq i \leq m$
    Delete $a_i a_{i+1} \ldots a_{m-1}$ from u
  - $ED(m,j) = n-j$ for all $0 \leq j \leq n$
    Insert $b_j b_{j+1} \ldots b_{n-1}$ into u

Subproblem dependency

- Subproblems are $ED(i,j)$, for $0 \leq i \leq m$, $0 \leq j \leq n$

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							
1	i							
2	s							
3	e							
4	c							
5	t							
6	•							

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		s	e	c	r	e	t	•
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2	s							
3	e							
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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							**6**
1	i							**5**
2	s							**4**
3	e							**3**
4	c							**2**
5	t							**1**
6	•							**0**

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b						**5**	**6**
1	i						**4**	**5**
2	s						**3**	**4**
3	e						**2**	**3**
4	c						**1**	**2**
5	t						**0**	**1**
6	•						**1**	**0**

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b					**4**	**5**	**6**
1	i					**3**	**4**	**5**
2	s					**2**	**3**	**4**
3	e					**1**	**2**	**3**
4	c					**1**	**1**	**2**
5	t					**1**	**0**	**1**
6	•					**2**	**1**	**0**

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		s	e	c	r	e	t	•
0	b				4	4	5	6
1	i				3	3	4	5
2	s				2	2	3	4
3	e				2	1	2	3
4	c				2	1	1	2
5	t				2	1	0	1
6	•				3	2	1	0

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		s	e	c	r	e	t	•
0	b			4	4	4	5	6
1	i			3	3	3	4	5
2	s			3	2	2	3	4
3	e			3	2	1	2	3
4	c			2	2	1	1	2
5	t			3	2	1	0	1
6	•			4	3	2	1	0

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		s	e	c	r	e	t	•
0	b		4	4	4	4	5	6
1	i		4	3	3	3	4	5
2	s		3	3	2	2	3	4
3	e		2	3	2	1	2	3
4	c		3	2	2	1	1	2
5	t		4	3	2	1	0	1
6	•		5	4	3	2	1	0

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	4	4	4	4	4	5	6
1	i	3	4	3	3	3	4	5
2	s	2	3	3	2	2	3	4
3	e	3	2	3	2	1	2	3
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Reading off the solution

- Transform `bisect` to `secret`

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	4	4	4	4	4	5	6
1	i	3	4	3	3	3	4	5
2	s	2	3	3	2	2	3	4
3	e	3	2	3	2	1	2	3
4	c	4	3	2	2	1	1	2
5	t	5	4	3	2	1	0	1
6	•	6	5	4	3	2	1	0

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Reading off the solution

- Transform `bisect` to `secret`
- Delete `b`

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	4	4	4	4	4	5	6
1	i	3	4	3	3	3	4	5
2	s	2	3	3	2	2	3	4
3	e	3	2	3	2	1	2	3
4	c	4	3	2	2	1	1	2
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Reading off the solution

- Transform `bisect` to `secret`
- Delete `b` , Delete `i`

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	4	4	4	4	4	5	6
1	i	3	4	3	3	3	4	5
2	s	2	3	3	2	2	3	4
3	e	3	2	3	2	1	2	3
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Reading off the solution

- Transform `bisect` to `secret`
- Delete `b` , Delete `i` , Insert `r`

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	4	4	4	4	4	5	6
1	i	3	4	3	3	3	4	5
2	s	2	3	3	2	2	3	4
3	e	3	2	3	2	1	2	3
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Reading off the solution

- Transform `bisect` to `secret`
- Delete `b` , Delete `i` , Insert `r` , Insert `e`

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	4	4	4	4	4	5	6
1	i	3	4	3	3	3	4	5
2	s	2	3	3	2	2	3	4
3	e	3	2	3	2	1	2	3
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5	t	5	4	3	2	1	0	1
6	•	6	5	4	3	2	1	0

Implementation

```
def ED(u,v):
  import numpy as np
  (m,n) = (len(u),len(v))
  ed = np.zeros((m+1,n+1))

  for i in range(m-1,-1,-1):
    ed[i,n] = m-i
  for j in range(n-1,-1,-1):
    ed[m,j] = n-j

  for j in range(n-1,-1,-1):
    for i in range(m-1,-1,-1):
      if u[i] == v[j]:
        ed[i,j] = ed[i+1,j+1]
      else:
        ed[i,j] = 1 + min(ed[i+1,j+1],
                          ed[i,j+1],
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  return(ed[0,0])
```

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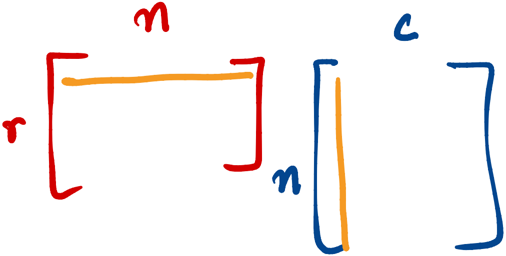
                          ed[i+1,j])
  return(ed[0,0])
```

Complexity

- Again $O(mn)$, using dynamic programming or memoization
  - Fill a table of size $O(mn)$
  - Each table entry takes constant time to compute

- Multiply matrices A, B
  - $AB[i,j] = \sum_{k=0}^{n-1} A[i,k]B[k,j]$

$(r_1 \bowtie (r_2) \bowtie r_3)$



r · c entries

k time to compute

Multiplying matrices

- Multiply matrices A, B
  - $AB[i,j] = \sum_{k=0}^{n-1} A[i,k]B[k,j]$
- Dimensions must be compatible
  - $A : m \times n$, $B : n \times p$
  - $AB : m \times p$

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- Computing each entry in AB is $O(n)$
- Overall, computing AB is $O(mnp)$

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- Let $A : 1 \times 100$, $B : 100 \times 1$, $C : 1 \times 100$

Multiplying matrices

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- Dimensions must be compatible
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  - $AB : m \times p$
- Computing each entry in AB is $O(n)$
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- Let $A : 1 \times 100$, $B : 100 \times 1$, $C : 1 \times 100$
- Computing $A(BC)$
  - $BC : 100 \times 100$, takes $100 \cdot 1 \cdot 100 = 10000$ steps to compute
  - $A(BC) : 1 \times 100$, takes $1 \cdot 100 \cdot 100 = 10000$ steps to compute

Multiplying matrices

- Multiply matrices A, B
  - $AB[i,j] = \sum_{k=0}^{n-1} A[i,k]B[k,j]$
- Dimensions must be compatible
  - $A : m \times n$, $B : n \times p$
  - $AB : m \times p$
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- Computing $(AB)C$
  - $AB : 1 \times 1$, takes $1 \cdot 100 \cdot 1 = 100$ steps to compute
  - $(AB)C) : 1 \times 100$, takes $1 \cdot 1 \cdot 100 = 100$ steps to compute

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  - $AB : 1 \times 1$, takes $1 \cdot 100 \cdot 1 = 100$ steps to compute
  - $(AB)C) : 1 \times 100$, takes $1 \cdot 1 \cdot 100 = 100$ steps to compute
- 20000 steps vs 200 steps!

Multiplying matrices

- Multiply matrices A, B
  - $AB[i,j] = \sum_{k=0}^{n-1} A[i,k]B[k,j]$
- Dimensions must be compatible
  - $A : m \times n$, $B : n \times p$
  - $AB : m \times p$
- Computing each entry in AB is $O(n)$
- Overall, computing AB is $O(mnp)$
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  - . . . but can affect the complexity!

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  - Dimensions match: $r_j = c_{j-1}$, $0 < j < n$
  - Product $M_0 \cdot M_1 \cdots M_{n-1}$ can be computed

$(M_0 M_1)\, M_2\, (M_3 M_4)$

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  - Product $M_0 \cdot M_1 \cdots M_{n-1}$ can be computed
- Find an optimal order to compute the product
  - Multiply two matrices at a time
  - Bracket the expression optimally

Inductive structure

- Final step combines two subproducts
  $(M_0 \cdot M_1 \cdots M_{k-1}) \cdot (M_k \cdot M_{k+1} \cdots M_{n-1})$
  for some $0 < k < n$

$M_0 \mid M_1 \mid M_2 \mid \;--\; M_{n-2} \mid M_{n-1}$

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- First factor is $r_0 \times c_{k-1}$, second is $r_k \times c_{n-1}$, where $r_k = c_{k-1}$

$r_0 \times c_{k-1} = r_k \quad \times c_{n-1}$

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  - Generic subproblem is $M_j \cdot M_{j+1} \cdots M_k$

$M_0 - M_{k-1} \quad M_k - M_{n-1}$

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- $C(j, k) =$
  $\min_{j<\ell\leq k} [C(j, \ell{-}1) + C(\ell, k) + \underline{r_j r_\ell c_k}]$

last step

Inductive structure

- Final step combines two subproducts $(M_0 \cdot M_1 \cdots M_{k-1}) \cdot (M_k \cdot M_{k+1} \cdots M_{n-1})$ for some $0 < k < n$
- First factor is $r_0 \times c_{k-1}$, second is $r_k \times c_{n-1}$, where $r_k = c_{k-1}$
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  $\min_{j<\ell\leq k} [C(j, \ell-1) + C(\ell, k) + r_j r_\ell c_k]$
- Base case: $C(j, j) = 0$ for $0 \leq j < n$

$$M_0 \cdot \left[M_1 - - \quad M_{n-1} \right]$$

$$C(0,0) \quad \checkmark \quad C(1,n)$$

$$0$$

Subproblem dependency

- Compute $C(i,j)$, $0 \leq i,j < n$

	0	$\cdots$	i	$\cdots$	$\cdots$	j	$\cdots$	$n-1$
0								
$\cdots$								
i								
$\cdots$								
$\cdots$								
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Subproblem dependency

- Compute $C(i,j)$, $0 \leq i,j < n$
  - Only for $i \leq j$
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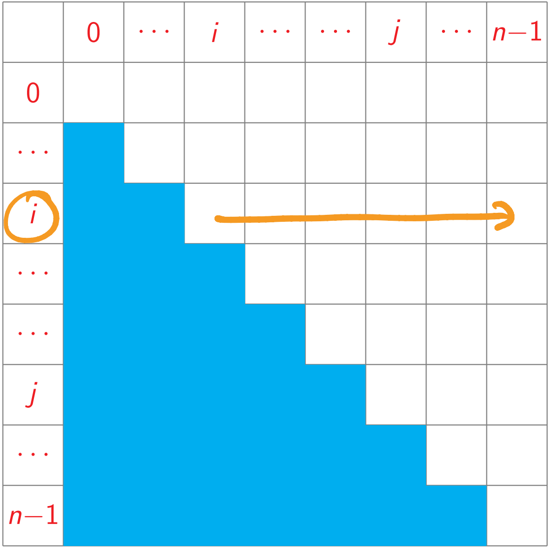

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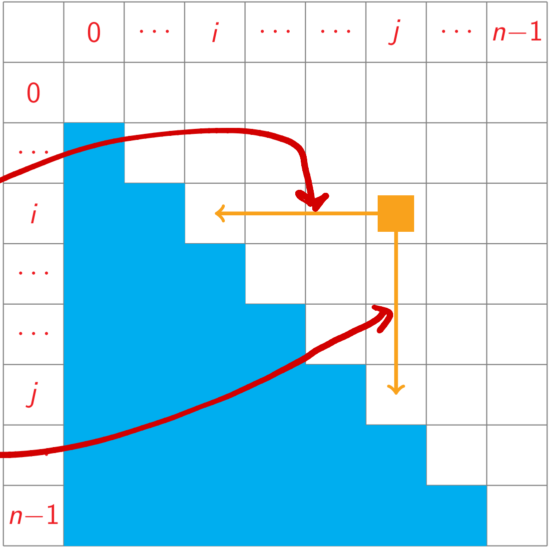

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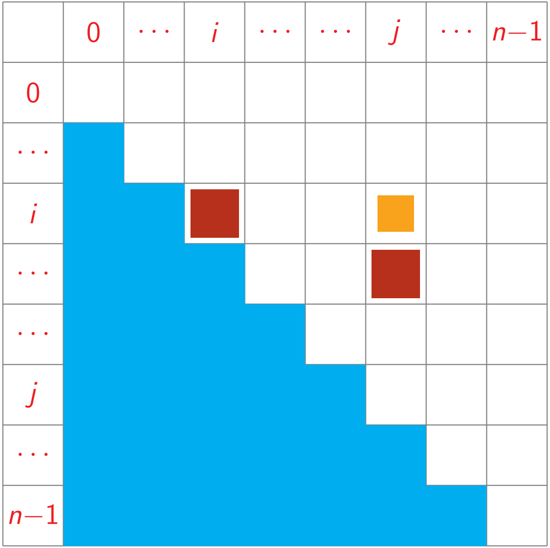

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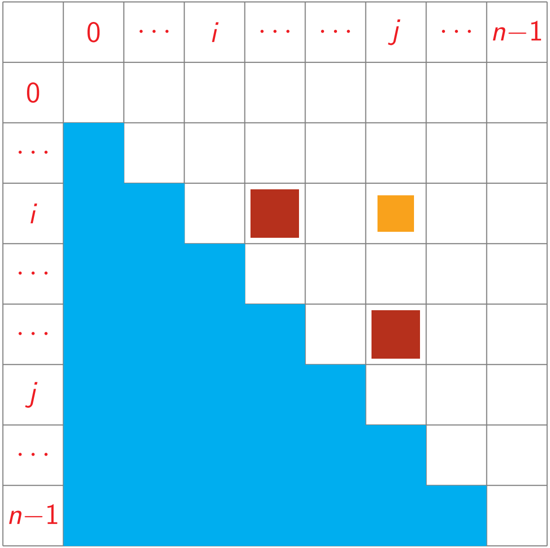

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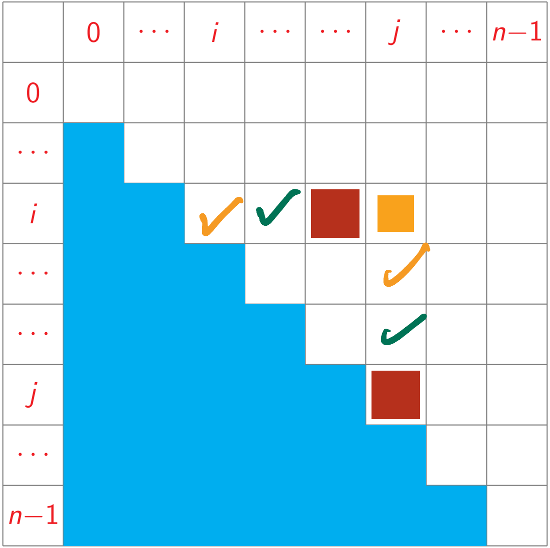

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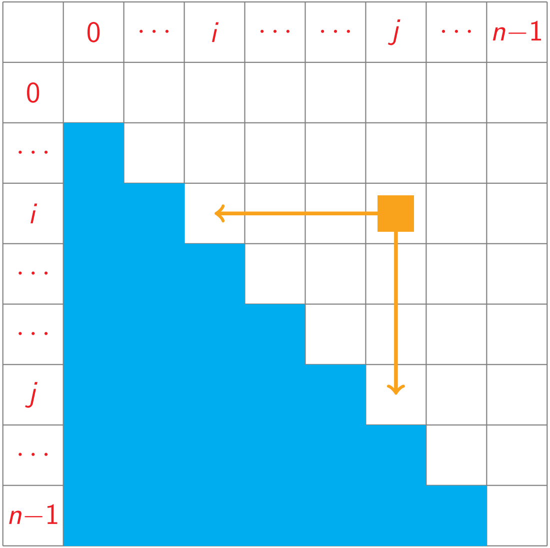

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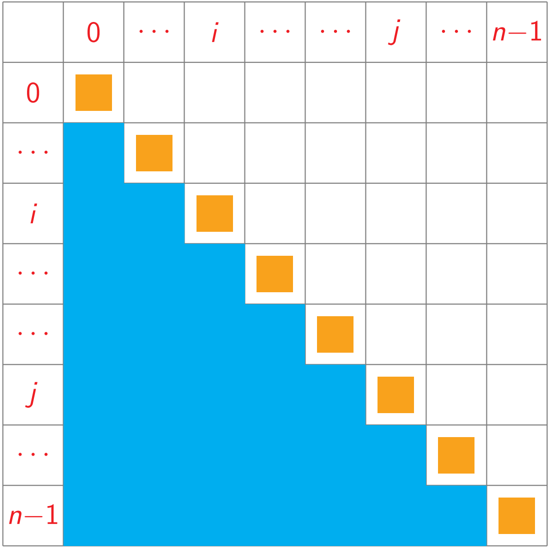

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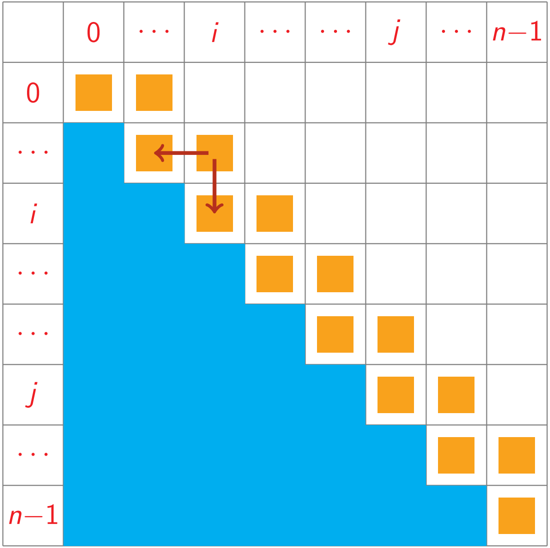

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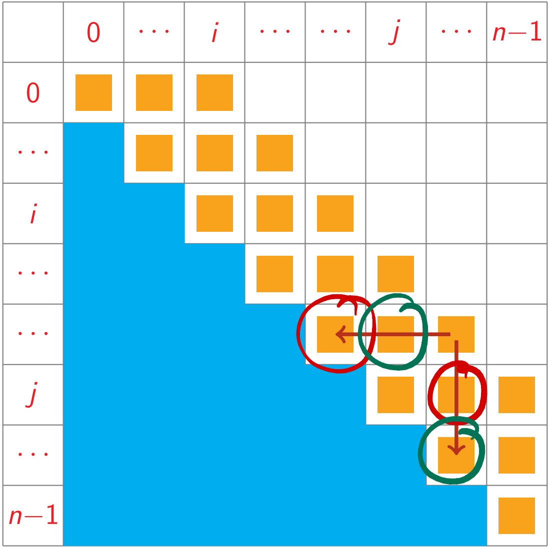

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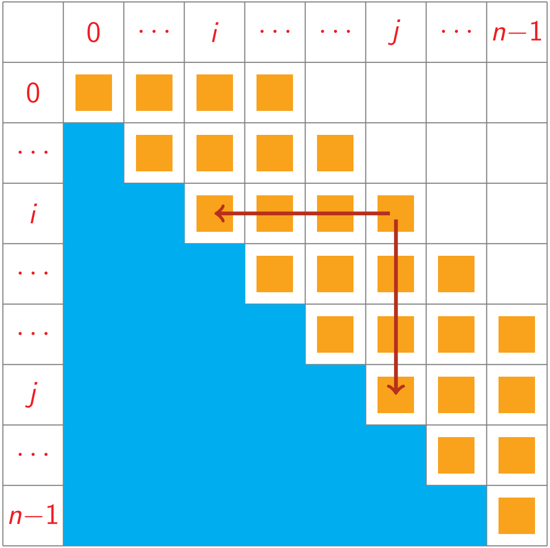

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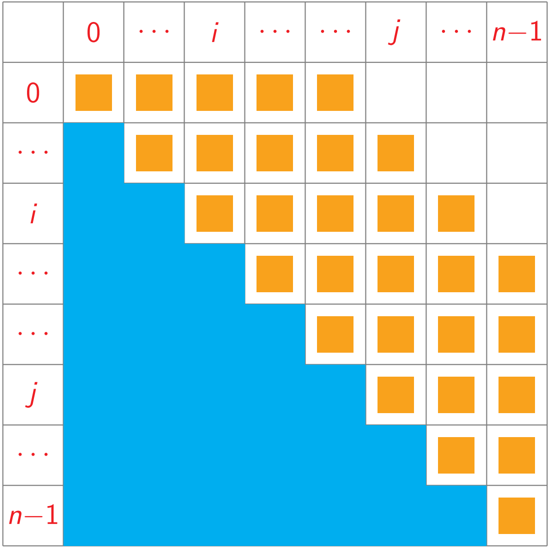

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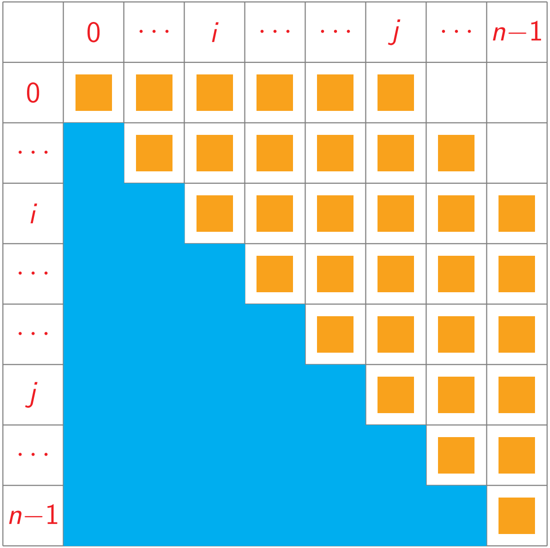

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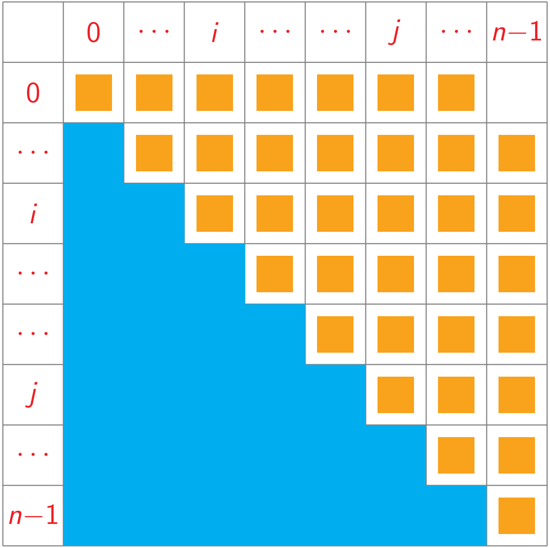

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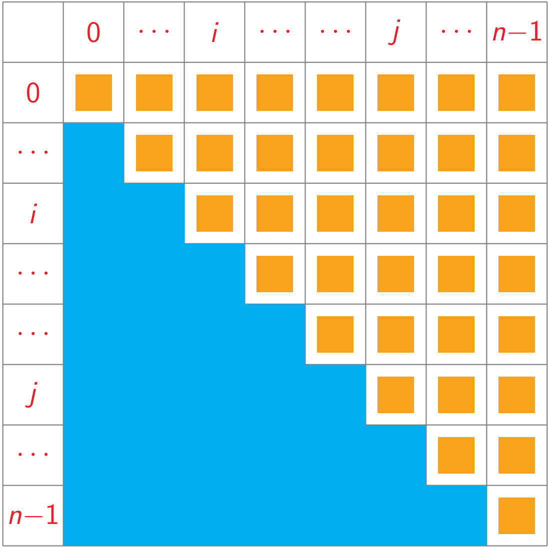
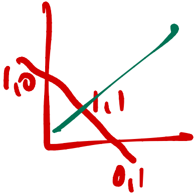


Implementation

```
def C(dim):
  # dim: dimension matrix,
  #      entries are pairs (r_i,c_i)
  import numpy as np
  n = dim.shape[0]
  C = np.zeros((n,n))
  for i in range(n):
    C[i,i] = 0
  for diff in range(1,n):
    for i in range(0,n-diff):
      j = i + diff
      C[i,j] = C[i,i] +
               C[i+1,j] +
               dim[i][0]*dim[i+1][0]*dim[j][1]
      for k in range(i+1,j+1):
        C[i,j] = min(C[i,j],
                     C[i,k-1] + C[k,j] +
                     dim[i][0]*dim[k][0]*dim[j][1])
  return(C[0,n-1])
```



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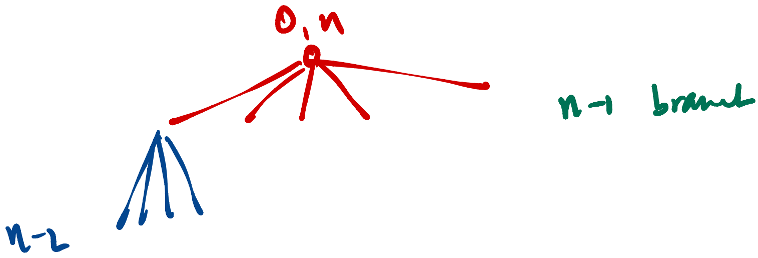
```

Complexity

- We have to fill a table of size $O(n^2)$
- Filling each entry takes $O(n)$
- Overall, $O(n^3)$

Brute force is exponential


0, n
n-1 branch
n-2