Lecture 25, 18 November 2025

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Programming and Data Structures with Python

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Inductive definitions, recursive programs, subproblems

- Factorial
  - $fact(0) = 1$
  - $fact(n) = n \times fact(n-1)$

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def fact(n):
  if n <= 0:
    return(1)
  else:
    return(n * fact(n-1))
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- Insertion sort
  - $isort([\,]) = [\,]$
  - $isort([x_0, x_1, \ldots, x_n]) =$
    $insert(isort([x_0, x_1, \ldots, x_{n-1}]), x_n)$

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- Insertion sort
  - $isort([\,]) = [\,]$
  - $isort([x_0, x_1, \ldots, x_n]) = insert(isort([x_0, x_1, \ldots, x_{n-1}]), x_n)$

- $fact(n{-}1)$ is a subproblem of $fact(n)$
  - So are $fact(n{-}2)$, $fact(n{-}3)$, $\ldots$, $fact(0)$
- $isort([x_0, x_1, \ldots, x_{n-1}])$ is a subproblem of $isort([x_0, x_1, \ldots, x_n])$
  - So is $isort([x_0, \ldots, x_j])$ for any $0 < j < n$
- Solution to original problem can be derived by combining solutions to subproblems

Tradeoff – cost of recursive calls

Evaluating subproblems

- Fibonacci numbers
  - $fib(0) = 0$
  - $fib(1) = 1$
  - $fib(n) = fib(n-1) + fib(n-2)$

0 1 2 3 4 5

0 1 1 2 3 5

fib(5)

```
def fib(n):
  if n <= 1:
    value = n
  else:
    value = fib(n-1) + fib(n-2)
  return(value)
```

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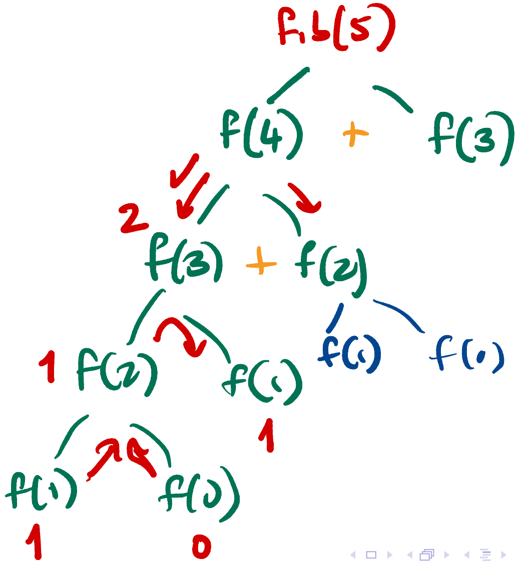

Evaluating subproblems

- Build a table of values already computed
  - Memory table

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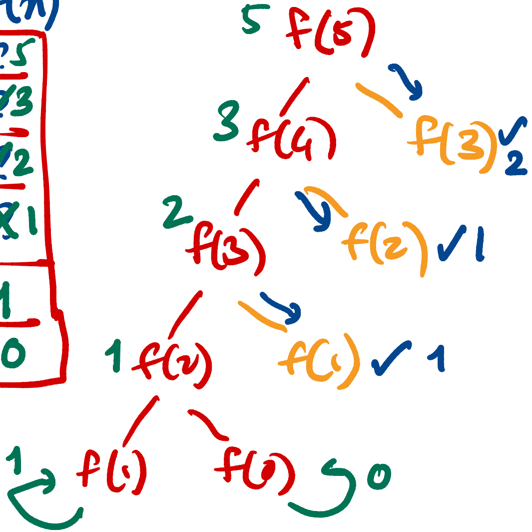
  - Memory table
- Memoization
  - Check if the value to be computed was already seen before

Evaluating subproblems

- Build a table of values already computed
  - Memory table
- Memoization
  - Check if the value to be computed was already seen before
- Store each newly computed value in a table
- Look up the table before making a recursive call
- Computation tree becomes linear

	n	f(n)
	5	~~?~~5
	4	~~?~~3
	3	~~?~~2
3	2	~~X~~1
1	1	1
2	0	0



Memoizing recursive implmentations

```
def fib(n):
  if n <= 1:
    value = n
  else:
    value = fib(n-1) + fib(n-2)
  return(value)
```

Memoizing recursive implmentations

← fibtable = {}

```
def fib(n):
  if n in fibtable.keys():
    return(fibtable[n])
  if n <= 1:
    value = n
  else:
    value = fib(n-1) + fib(n-2)
  fibtable[n] = value ✓
  return(value)
```

Memoizing recursive implmentations

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def fib(n):
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  else:
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  return(value)
```

In general

```
def f(x,y,z):
  if (x,y,z) in ftable.keys():
    return(ftable[(x,y,z)])
  recursively compute value
    from subproblems
  ftable[(x,y,z)] = value
  return(value)
```

Dynamic programming

- Anticipate the structure of subproblems
  - Derive from inductive definition
  - Dependencies are acyclic

Recursion

1 – Wasteful recomputation

2 – Cost of recursive calls

1 Memoization ✓

2 Translate to iteration

Dynamic programming

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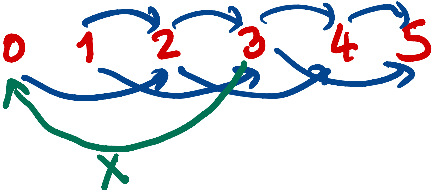


Evaluating `fib(5)`

`fib(5)`

`fib(4)`

`fib(3)`

`fib(2)`

`fib(1)`

`fib(0)`

Dynamic programming

- Anticipate the structure of subproblems
  - Derive from inductive definition
  - Dependencies are acyclic
- Solve subproblems in appropriate order
  - Start with base cases — no dependencies
  - Evaluate a value after all its dependencies are available
  - Fill table iteratively
  - Never need to make a recursive call

0 1 1 2 3 5 8 13 21

Evaluating `fib(5)`

`fib(5)`

`fib(4)`

`fib(3)`

`fib(2)`

`fib(1)`

`fib(0)`

Grid paths

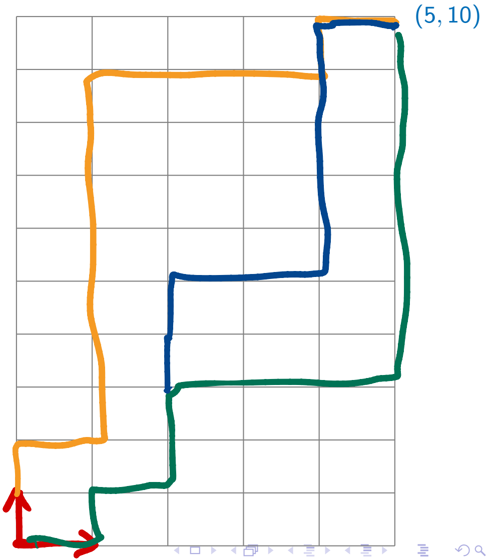
- Rectangular grid of one-way roads
- Can only go up and right
- How many paths from $(0,0)$ to (m,n)?

$m+n$ steps, each R, U

5 R
10 U

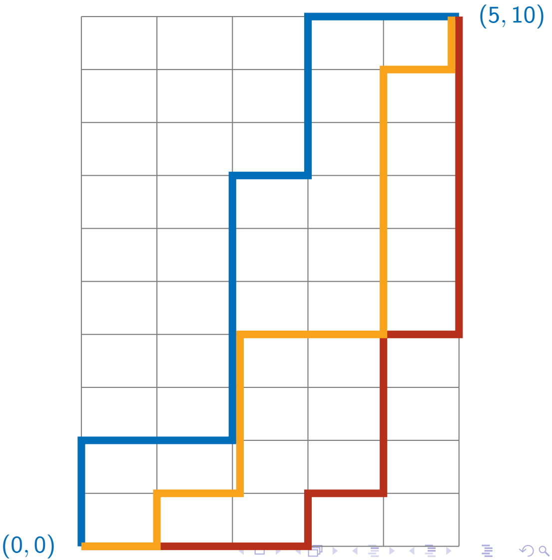
RURRUU - - R
1 2 3 - - - 15

$\binom{15}{5}$ $\binom{m+n}{n}$ $\binom{m+n}{m}$ $\binom{15}{10}$



Combinatorial solution

- Every path from $(0,0)$ to $(5,10)$ has 15 segments
  - Out of 15, exactly 5 are right moves, 10 are up moves
  - Fix the positions of the 5 right moves among the 15 positions overall



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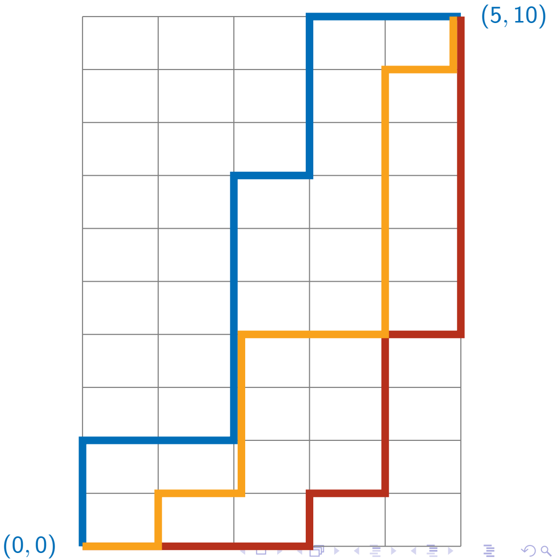
  - Same as $\binom{15}{10}$ — fix the 10 up moves



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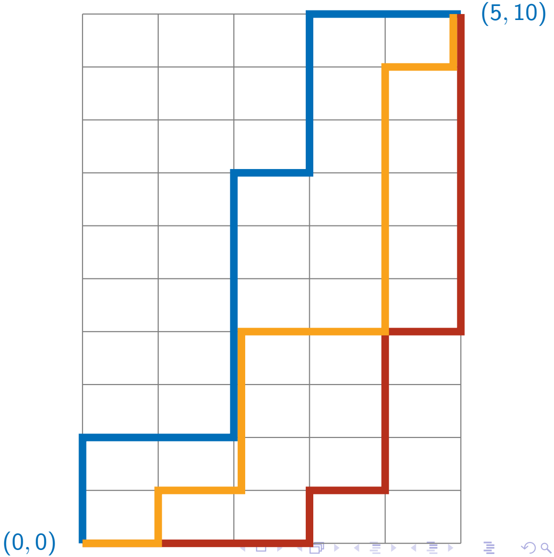
  - $\binom{15}{5} = \frac{15!}{10! \cdot 5!} = 3003$
  - Same as $\binom{15}{10}$ — fix the 10 up moves
- In general $m+n$ segments from $(0,0)$ to (m,n)



Holes

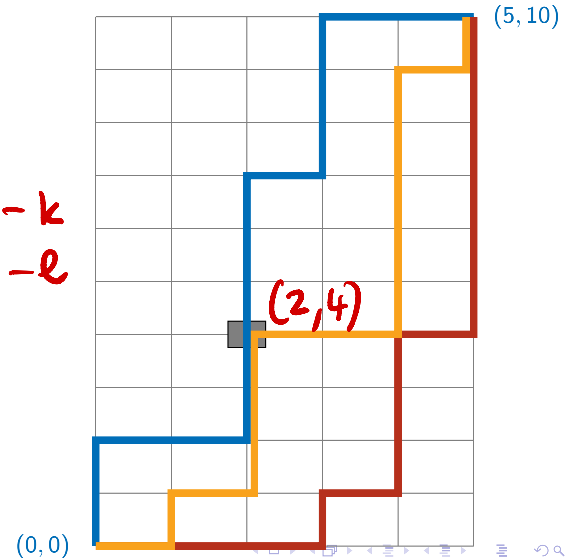
- What if an intersection is blocked?
  - For instance, $(2, 4)$

Count paths $(0,0) - (2,4) - k$

$(2,4) - (5,10) - \ell$

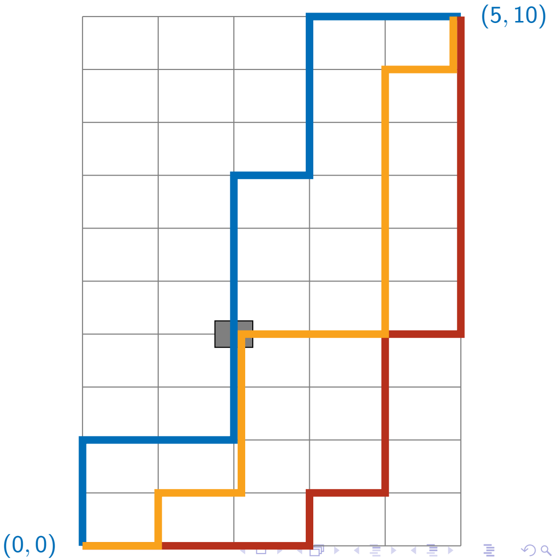
$k \cdot \ell$ bad paths

$\binom{m+n}{n} - k \cdot \ell$



Combinatorial solution for holes

- Discard paths passing through $(2, 4)$



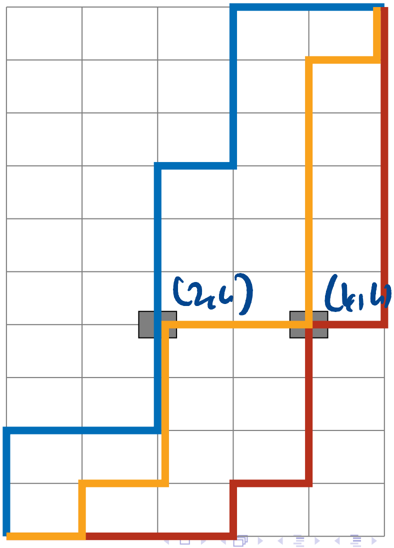
More holes

- What if two intersections are blocked?
- Discard paths via $(2,4)$, $(4,4)$
  - Some paths are counted twice
- Add back the paths that pass through both holes
- Inclusion-exclusion — counting is messy

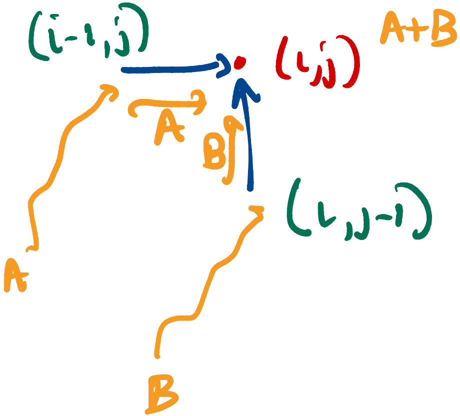
$(2,4)$ $k_1 \cdot l_1 = A$

$(4,4)$ $k_2 \cdot l_2 = B$

Yellow path



- How can a path reach (i, j)



Inductive formulation

- How can a path reach (i, j)
  - Move up from $(i, j - 1)$

Inductive formulation

- How can a path reach (i,j)
  - Move up from $(i,j-1)$
  - Move right from $(i-1,j)$

$$\begin{tikzcd}
(\boldsymbol{i-1,j}) \arrow[r] & (\boldsymbol{i,j}) \\
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Inductive formulation

- How can a path reach (i, j)
    - Move up from $(i, j-1)$
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- Each path to these neighbours extends to a unique path to (i, j)

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- Recurrence for $P(i, j)$, number of paths from $(0, 0)$ to (i, j)
  - $P(i, j) = P(i-1, j) + P(i, j-1)$

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$P(i-2,j)$ $P(i-1,j)$ $P(i,j)$

$$\begin{tikzcd}
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& (i,j-1) \arrow[u]
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$P(i-1,j-1)$ $P(i,j-1)$

$P(i-1,j-1)$ $P(i,j-2)$

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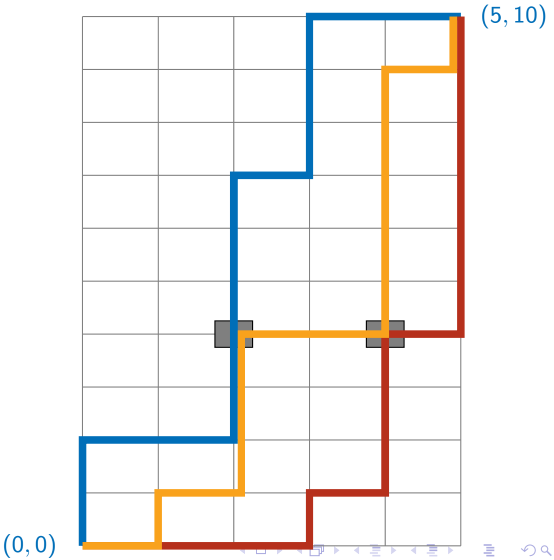
  - $P(0, j) = P(0, j-1)$ — left column
- $P(i, j) = 0$ if there is a hole at (i, j)

$$
\begin{tikzcd}
(\boldsymbol{i}-\mathbf{1},\boldsymbol{j}) \arrow[r] & (\boldsymbol{i},\boldsymbol{j}) \\
 & (\boldsymbol{i},\boldsymbol{j}-\mathbf{1}) \arrow[u]
\end{tikzcd}
$$

Computing $P(i, j)$

- Naive recursion recomputes same subproblem repeatedly



Computing $P(i, j)$

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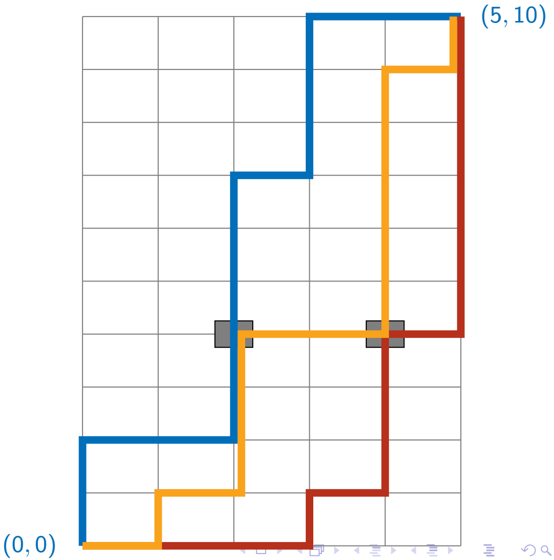
  - $P(5, 10)$ requires $P(4, 10)$, $P(5, 9)$



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- Naive recursion recomputes same subproblem repeatedly
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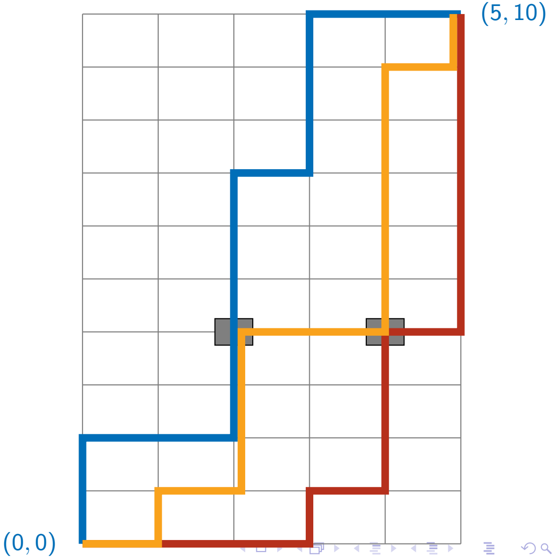

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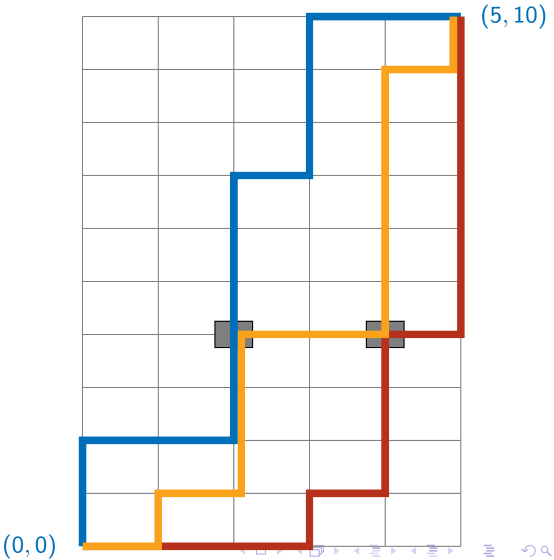
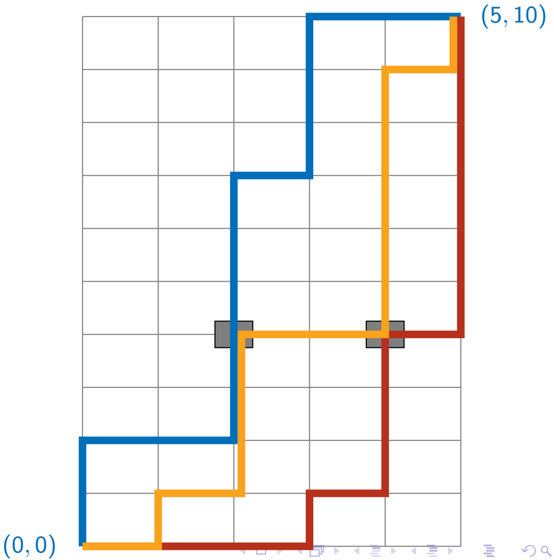
- Use memoization . . .



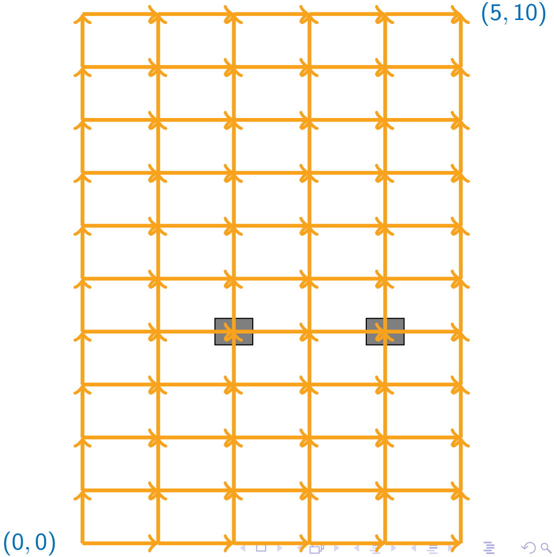
Computing $P(i, j)$

- Naive recursion recomputes same subproblem repeatedly
  - $P(5, 10)$ requires $P(4, 10)$, $P(5, 9)$
  - Both $P(4, 10)$, $P(5, 9)$ require $P(4, 9)$
- Use memoization ...
- ...or find a suitable order to compute the subproblems



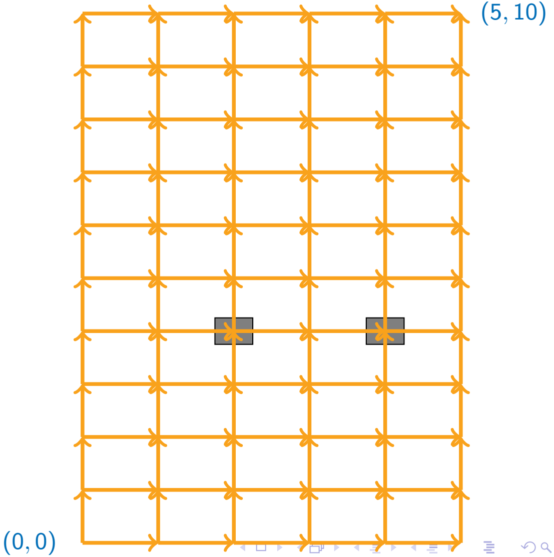
Dynamic programming

- Identify subproblem structure



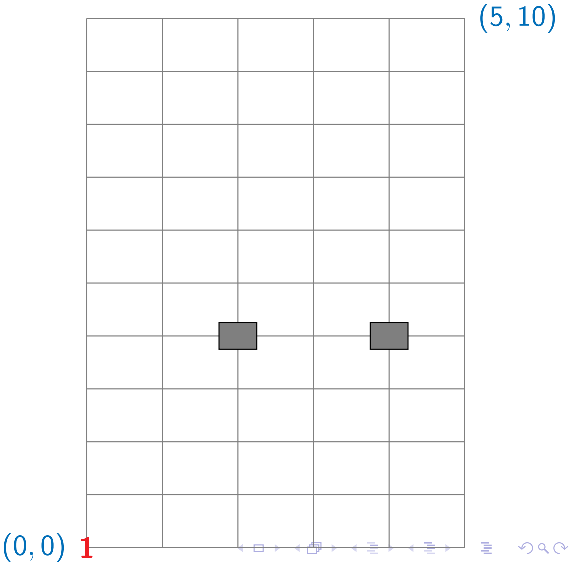
Dynamic programming

- Identify subproblem structure
- $P(0,0)$ has no dependencies



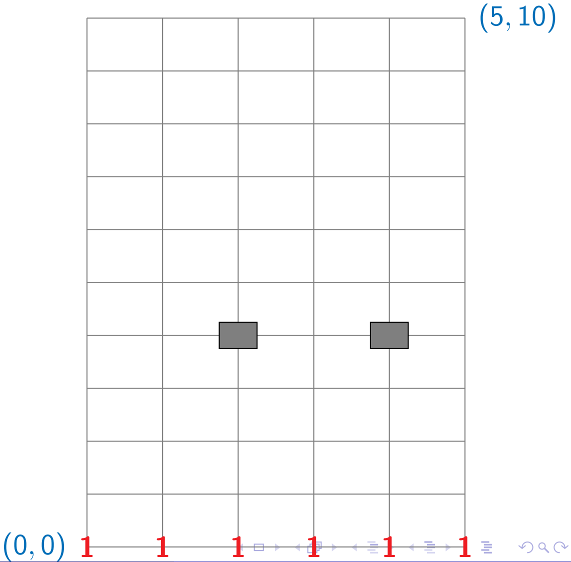
Dynamic programming

- Identify subproblem structure
- $P(0,0)$ has no dependencies
- Start at $(0,0)$



Dynamic programming

- Identify subproblem structure
- $P(0,0)$ has no dependencies
- Start at $(0,0)$
- Fill row by row



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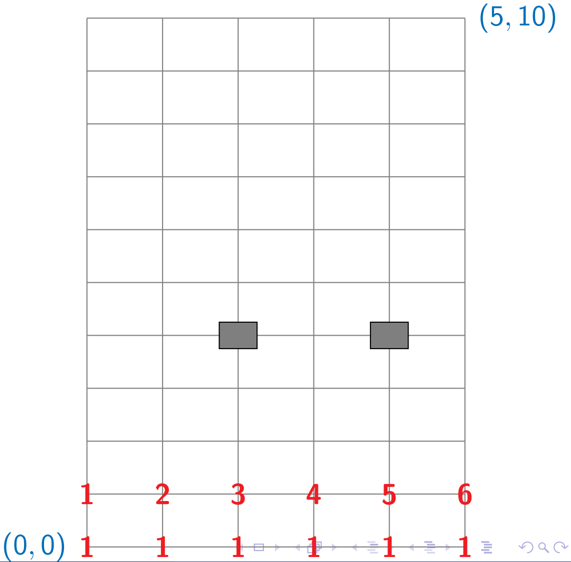

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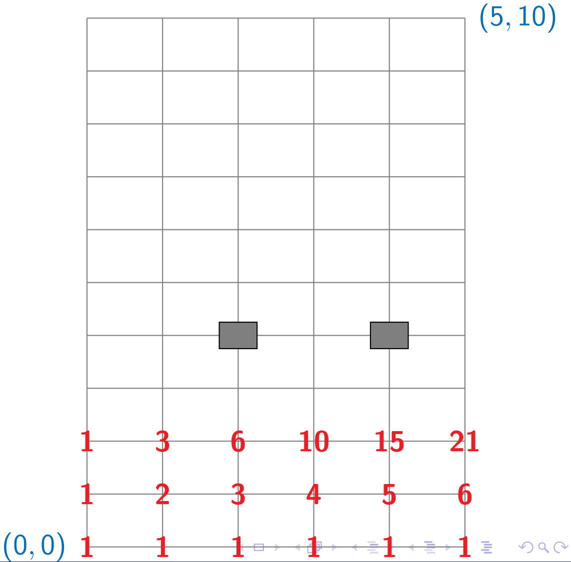

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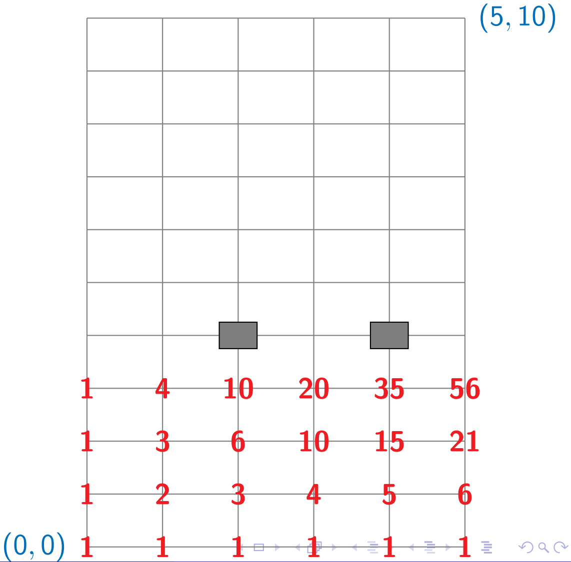

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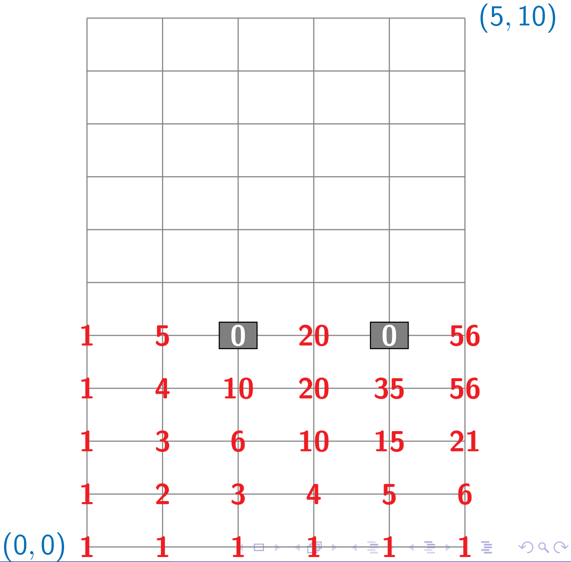

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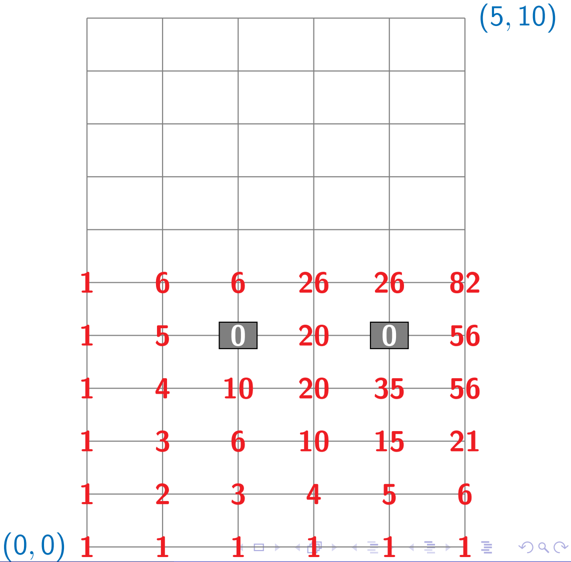

Dynamic programming

- Identify subproblem structure
- $P(0,0)$ has no dependencies
- Start at $(0,0)$
- Fill row by row

$(5,10)$

1	11	51	181	526	1358
1	10	40	130	345	832
1	9	30	90	215	487
1	8	21	60	125	272
1	7	13	39	65	147
1	6	6	26	26	82
1	5	0	20	0	56
1	4	10	20	35	56
1	3	6	10	15	21
1	2	3	4	5	6
1	1	1	1	1	1

$(0,0)$

Dynamic programming

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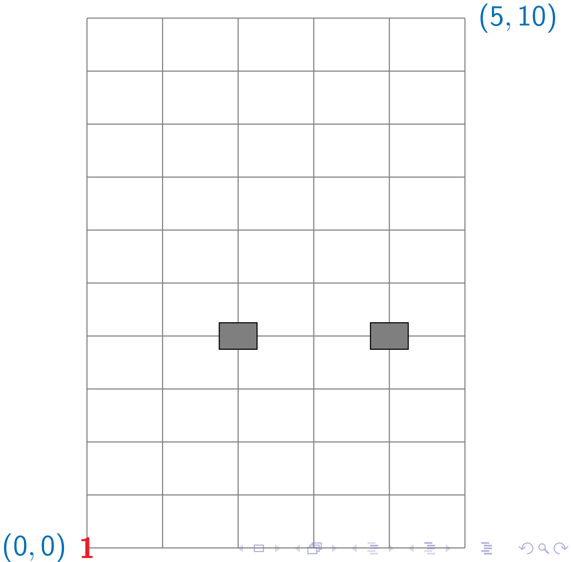

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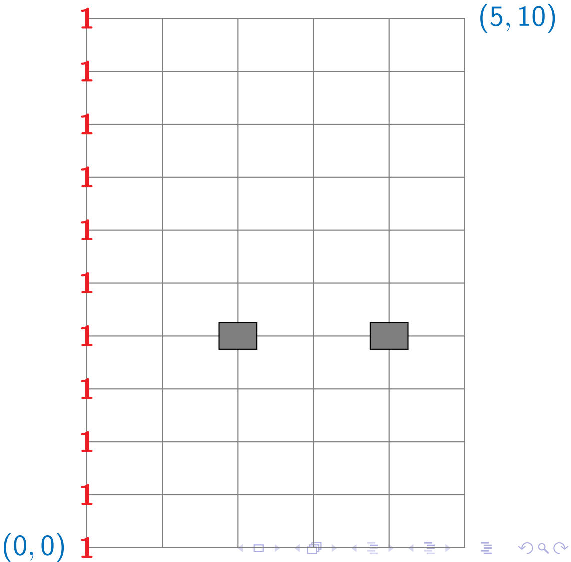

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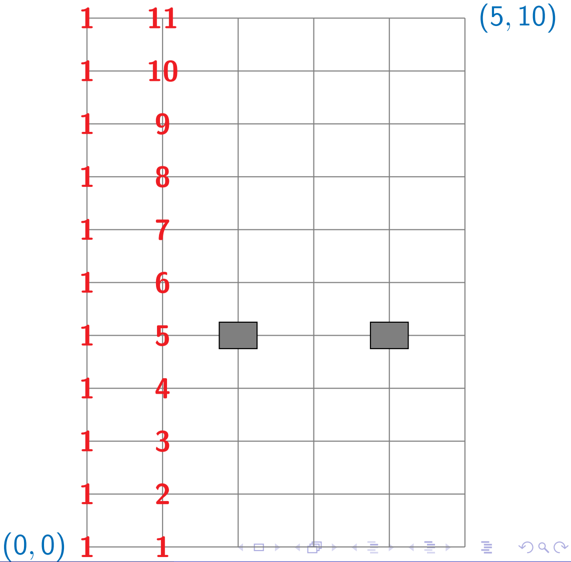

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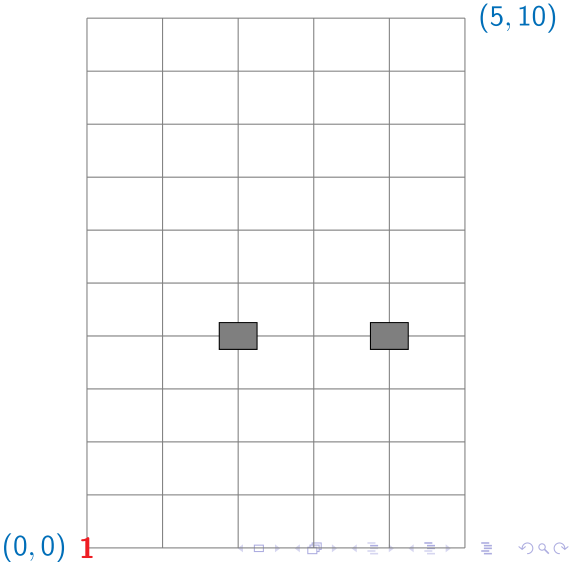
	1	5	0	20	0	56
	1	4	10	20	35	56
	1	3	6	10	15	21
	1	2	3	4	5	6
$(0,0)$	1	1	1	1	1	1

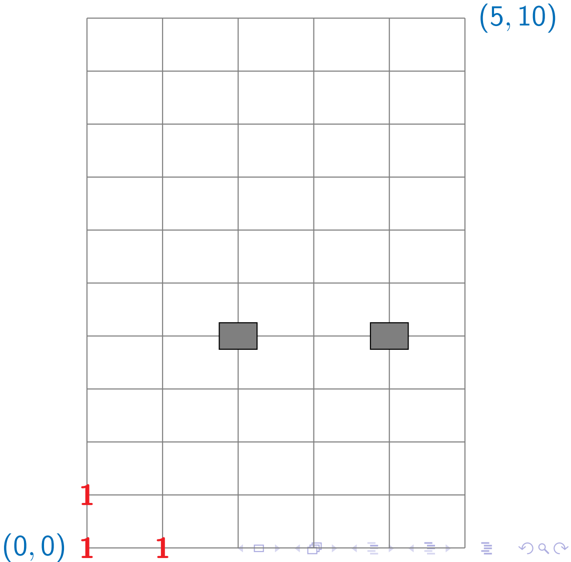
Dynamic programming

- Identify subproblem structure
- $P(0,0)$ has no dependencies
- Start at $(0,0)$
- Fill row by row
- Fill column by column
- Fill diagonal by diagonal



Dynamic programming

- Identify subproblem structure
- $P(0,0)$ has no dependencies
- Start at $(0,0)$
- Fill row by row
- Fill column by column
- Fill diagonal by diagonal



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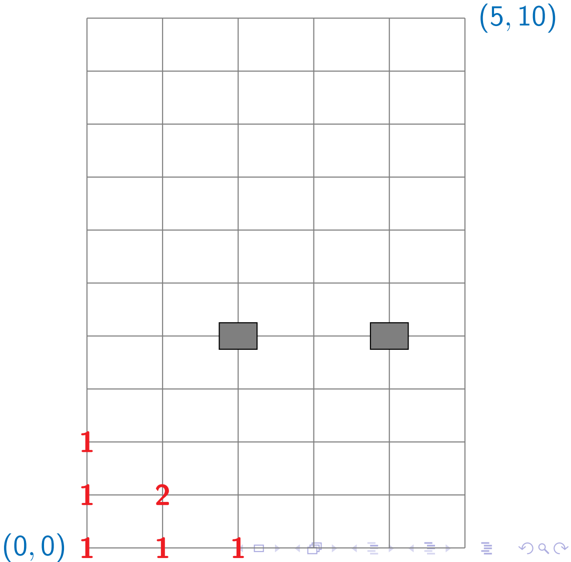

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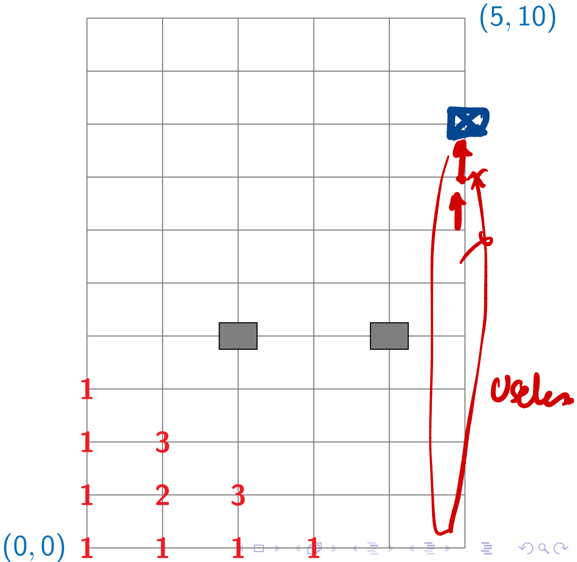
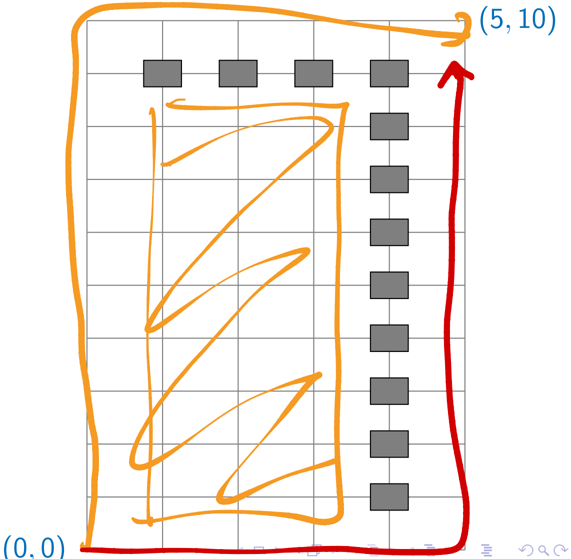


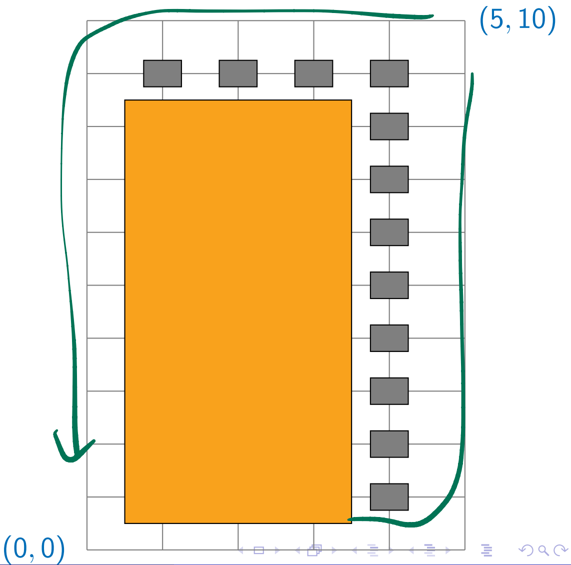
Memoization vs dynamic programming

- Barrier of holes just inside the border



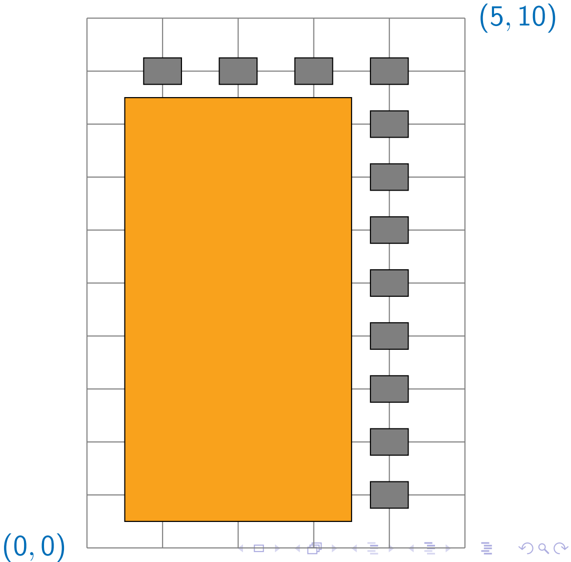
Memoization vs dynamic programming

- Barrier of holes just inside the border
- Memoization never explores the shaded region



Memoization vs dynamic programming

- Barrier of holes just inside the border
- Memoization never explores the shaded region
- Memo table has $O(m+n)$ entries



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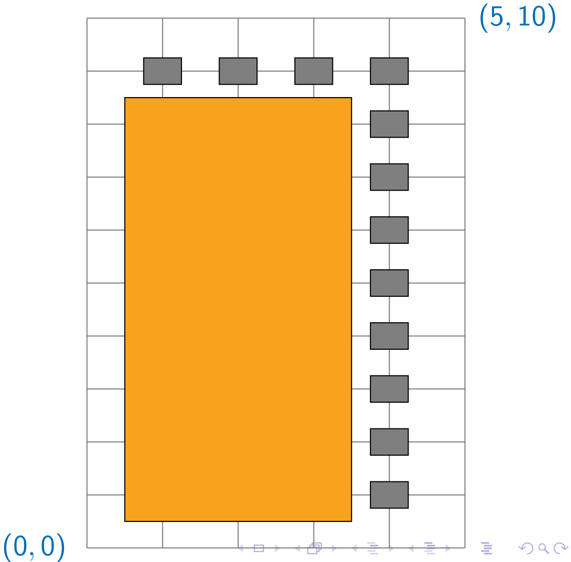
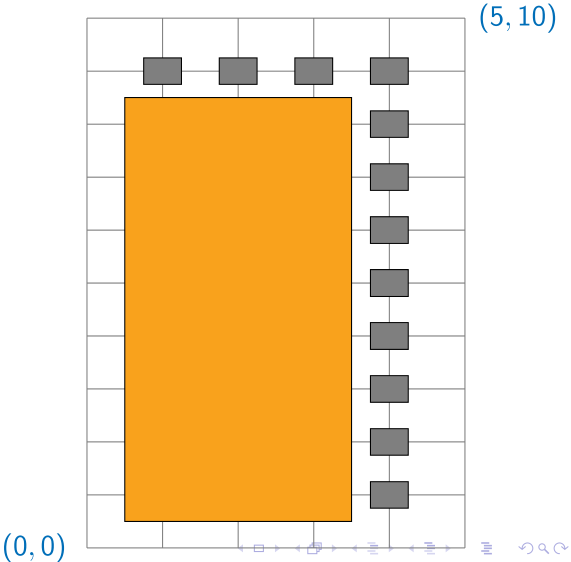


Memoization vs dynamic programming

- Barrier of holes just inside the border
- Memoization never explores the shaded region
- Memo table has $O(m + n)$ entries
- Dynamic programming blindly fills all mn cells of the table
- Tradeoff between recursion and iteration
  - "Wasteful" dynamic programming still better in general



Longest common subword

- Given two strings, find the (length of the) longest common subword
  - `"secret"`, `"secretary"` — `"secret"`, length 6
  - `"bisect"`, `"trisect"` — `"isect"`, length 5
  - `"bisect"`, `"secret"` — `"sec"`, length 3
  - `"director"`, `"secretary"` — `"ec"`, `"re"`, length 2

ect

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  - `"director"`, `"secretary"` — `"ec"`, `"re"`, length 2
- Formally
  - $u = a_0 a_1 \ldots a_{m-1}$
  - $v = b_0 b_1 \ldots b_{n-1}$

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    2 0
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- Formally
  - $u = a_0 a_1 \ldots a_{m-1}$
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  - Common subword of length k — for some positions i and j, $a_i a_{i+1} a_{i+k-1} = b_j b_{j+1} b_{j+k-1}$

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  - Common subword of length k — for some positions i and j, $a_i a_{i+1} a_{i+k-1} = b_j b_{j+1} b_{j+k-1}$
  - Find the largest such k — length of the longest common subword

Brute force

- $u = a_0 a_1 \ldots a_{m-1}$
- $v = b_0 b_1 \ldots b_{n-1}$
- Find the largest k such that for some positions i and j, $a_i a_{i+1} a_{i+k-1} = b_j b_{j+1} b_{j+k-1}$

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- Try every pair of starting positions i in u, j in v
  - Match $(a_i, b_j), (a_{i+1}, b_{j+1}), \dots$ as far as possible
  - Keep track of longest match

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- Try every pair of starting positions i in u, j in v
  - Match $(a_i, b_j), (a_{i+1}, b_{j+1}), \ldots$ as far as possible
  - Keep track of longest match
- Assuming $m > n$, this is $O(mn^2)$
  - mn pairs of starting positions
  - From each starting position, scan could be $O(n)$

aa-- a~~b~~a

aa - - - a~~b~~a

$\underline{m \cdot n}$ starting pos

n^3 if $m \simeq n$

Inductive structure

- $u = a_0 a_1 \ldots a_{m-1}$
- $v = b_0 b_1 \ldots b_{n-1}$
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- Find the largest k such that for some positions i and j, $a_i a_{i+1} a_{i+k-1} = b_j b_{j+1} b_{j+k-1}$
- $LCW(i,j)$ — length of longest common subword in $a_i a_{i+1} \ldots a_{m-1}$, $b_j b_{j+1} \ldots b_{n-1}$
  - If $a_i \neq b_j$, $LCW(i,j) = 0$
  - If $a_i = b_j$, $LCW(i,j) = 1 + LCW(i{+}1, j{+}1)$

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  - In general, $LCW(i,n) = 0$ for all $0 \leq i \leq m$
  - In general, $LCW(m,j) = 0$ for all $0 \leq j \leq n$

$(i,j) \leftarrow (i+1,j+1)$

Subproblem dependency

- Subproblems are $LCW(i,j)$, for $0 \leq i \leq m$, $0 \leq j \leq n$

Subproblem dependency

- Subproblems are $LCW(i, j)$, for $0 \leq i \leq m$, $0 \leq j \leq n$
- Table of $(m + 1) \cdot (n + 1)$ values

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	?						
1	i							
2	s							
3	e							
4	c							
5	t							
6	•							

Subproblem dependency

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- $LCW(i,j)$ depends on $LCW(i{+}1,j{+}1)$

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	?						
1	i		✓					
2	s			?				
3	e							
4	c							
5	t							
6	•							

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- $LCW(i, j)$ depends on $LCW(i+1, j+1)$
- Start at bottom right and fill row by row or column by column

$LCW(m, n) = 0$

$(i, n) = 0$

$(m, j) = 0$

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							**0**
1	i							**0**
2	s							**0**
3	e							**0**
4	c							**0**
5	t							**0**
6	•			0				**0**

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b						**0**	**0**
1	i						**0**	**0**
2	s						**0**	**0**
3	e						**0**	**0**
4	c						**0**	**0**
5	t						**1**	**0**
6	•					0	**0**	**0**

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b					0	0	0
1	i					0	0	0
2	s					0	0	0
3	e					1	0	0
4	c					0	0	0
5	t					0	1	0
6	•					0	0	0

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no r

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b				**0**	**0**	**0**	**0**
1	i				**0**	**0**	**0**	**0**
2	s				**0**	**0**	**0**	**0**
3	e				**0**	**1**	**0**	**0**
4	c				**0**	**0**	**0**	**0**
5	t				**0**	**0**	**1**	**0**
6	•				**0**	**0**	**0**	**0**

Subproblem dependency

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- Start at bottom right and fill row by row or column by column

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b			0	0	0	0	0
1	i			0	0	0	0	0
2	s			0	0	0	0	0
3	e			0	0	1	0	0
4	c			1	0	0	0	0
5	t			0	0	0	1	0
6	•			0	0	0	0	0

Subproblem dependency

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b		0	0	0	0	0	0
1	i		0	0	0	0	0	0
2	s		0	0	0	0	0	0
3	e		2	0	0	1	0	0
4	c		0	1	0	0	0	0
5	t		0	0	0	0	1	0
6	•		0	0	0	0	0	0

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	0	0	0	0	0	0	0
1	i	0	0	0	0	0	0	0
2	s	3	0	0	0	0	0	0
3	e	0	2	0	0	1	0	0
4	c	0	0	1	0	0	0	0
5	t	0	0	0	0	0	1	0
6	•	0	0	0	0	0	0	0

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Reading off the solution

- Find entry (i,j) with largest LCW value

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1	i	**0**	**0**	**0**	**0**	**0**	**0**	**0**
2	s	**3**	**0**	**0**	**0**	**0**	**0**	**0**
3	e	**0**	**2**	**0**	**0**	**1**	**0**	**0**
4	c	**0**	**0**	**1**	**0**	**0**	**0**	**0**
5	t	**0**	**0**	**0**	**0**	**0**	**1**	**0**
6	•	**0**	**0**	**0**	**0**	**0**	**0**	**0**

Subproblem dependency

- Subproblems are $LCW(i,j)$, for $0 \leq i \leq m$, $0 \leq j \leq n$
- Table of $(m+1)\cdot(n+1)$ values
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Reading off the solution

- Find entry (i,j) with largest LCW value
- Read off the actual subword diagonally

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1	i	**0**	**0**	**0**	**0**	**0**	**0**	**0**
2	s	**3**	**0**	**0**	**0**	**0**	**0**	**0**
3	e	**0**	**2**	**0**	**0**	**1**	**0**	**0**
4	c	**0**	**0**	**1**	**0**	**0**	**0**	**0**
5	t	**0**	**0**	**0**	**0**	**0**	**1**	**0**
6	•	**0**	**0**	**0**	**0**	**0**	**0**	**0**

Subproblem dependency

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	0	0	0	0	0	0	0
1	i	0	0	0	0	0	0	0
2	s	3	0	0	0	0	0	0
3	e	0	2	0	0	1	0	0
4	c	0	0	1	0	0	0	0
5	t	0	0	0	0	0	1	0
6	•	0	0	0	0	0	0	0

Implementation

```
def LCW(u,v):
  import numpy as np
  (m,n) = (len(u),len(v))
  lcw = np.zeros((m+1,n+1))

  maxlcw = 0

  for j in range(n-1,-1,-1):
    for i in range(m-1,-1,-1):
      if u[i] == v[j]:
        lcw[i,j] = 1 + lcw[i+1,j+1]
      else:
        lcw[i,j] = 0
      if lcw[i,j] > maxlcw:
        maxlcw = lcw[i,j]

  return(maxlcw)
```

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Complexity

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Complexity

- Recall that brute force was $O(mn^2)$

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Complexity

- Recall that brute force was $O(mn^2)$
- Inductive solution is $O(mn)$, using dynamic programming or memoization

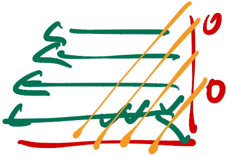
Implementation

```
def LCW(u,v):
  import numpy as np
  (m,n) = (len(u),len(v))
  lcw = np.zeros((m+1,n+1))

  maxlcw = 0

  for j in range(n-1,-1,-1):        Flip
    for i in range(m-1,-1,-1):
      if u[i] == v[j]:
        lcw[i,j] = 1 + lcw[i+1,j+1]
      else:
        lcw[i,j] = 0     NOT NEEDED
      if lcw[i,j] > maxlcw:
        maxlcw = lcw[i,j]

  return(maxlcw)
```



Complexity

- Recall that brute force was $O(mn^2)$
- Inductive solution is $O(mn)$, using dynamic programming or memoization
  - Fill a table of size $O(mn)$
  - Each table entry takes constant time to compute

Longest common subsequence

- Subsequence — can drop some letters in between
- Given two strings, find the (length of the) longest common subsequence
  - "secret", "secretary" — "secret", length 6
  - "bisect", "trisect" — "isect", length 5
  - "bisect", "secret" — "sect", length 4
  - "director", "secretary" — "ectr", "retr", length 4