Lecture 23, 19 November 2024

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Programming and Data Structures with Python

Lecture 23, 19 Nov 2024

Inductive definitions, recursive programs, subproblems

- Factorial
  - $fact(0) = 1$
  - $fact(n) = n \times fact(n-1)$

Inductive definitions, recursive programs, subproblems

- Factorial
  - $fact(0) = 1$ — Base Case
  - $fact(n) = n \times fact(n-1)$ - Inductive step

```
def fact(n):
  if n <= 0:
    return(1)
  else:
    return(n * fact(n-1))
```

return(1) — Base Case

return(n * fact(n-1)) - recursive call

Inductive definitions, recursive programs, subproblems

- Factorial
  - $fact(0) = 1$
  - $fact(n) = n \times fact(n-1)$

```
def fact(n):
  if n <= 0:
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```

- Insertion sort
  - $isort([\,]) = [\,]$ ← Base
  - $isort([x_0, x_1, \ldots, x_n]) =$
    $insert(isort([x_0, x_1, \ldots, x_{n-1}]), x_n)$
    Inductive

Inductive definitions, recursive programs, subproblems

- Factorial
  - $fact(0) = 1$
  - $fact(n) = n \times fact(n-1)$

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def fact(n):
  if n <= 0:
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- Insertion sort
  - $isort([\,]) = [\,]$
  - $isort([x_0, x_1, \ldots, x_n]) = insert(isort([x_0, x_1, \ldots, x_{n-1}]), x_n)$

- $fact(n-1)$ is a subproblem of $fact(n)$
  - So are $fact(n-2)$, $fact(n-3)$, ..., $fact(0)$
- $isort([x_0, x_1, \ldots, x_{n-1}])$ is a subproblem of $isort([x_0, x_1, \ldots, x_n])$
  - So is $isort([x_0, \ldots, x_j])$ for any $0 < j < n$
- Solution to original problem can be derived by combining solutions to subproblems

Evaluating subproblems

- Fibonacci numbers
  - $fib(0) = 0$
  - $fib(1) = 1$
  - $fib(n) = fib(n-1) + fib(n-2)$

Base cases

Inductive

0 1 1 2 3 5 8 13

```
def fib(n):
  if n <= 1:
    value = n
  else:
    value = fib(n-1) + fib(n-2)
  return(value)
```

Recursive

Doesn't work, in practice

Why?

- Fibonacci numbers
  - $fib(0) = 0$
  - $fib(1) = 1$
  - $fib(n) = fib(n-1) + fib(n-2)$

```
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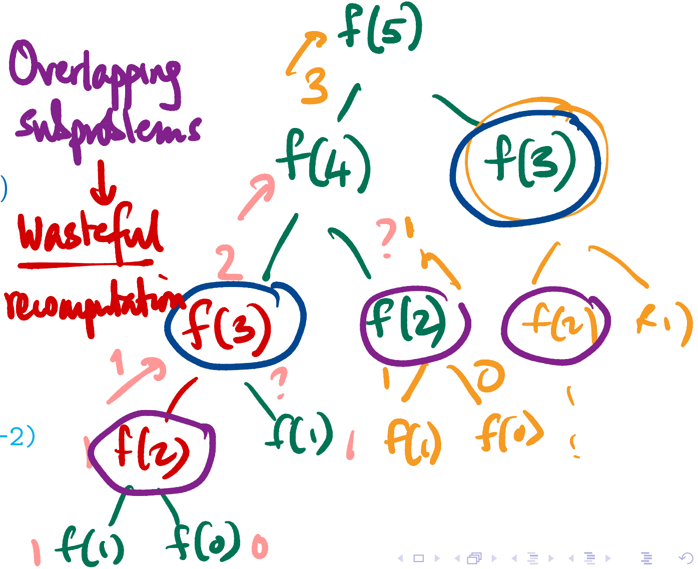

Evaluating subproblems

- Build a table of values already computed
  - Memory table

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- Memoization
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- Build a table of values already computed
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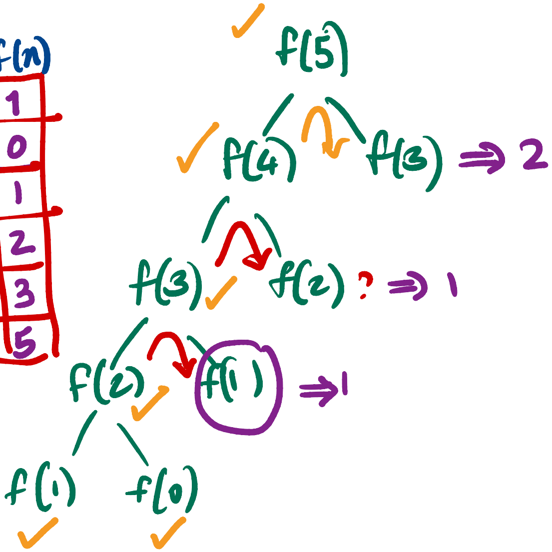
- Memoization
  - Check if the value to be computed was already seen before
- Store each newly computed value in a table
- Look up the table before making a recursive call
- Computation tree becomes linear

Table

n	f(n)
1	1
0	0
2	1
3	2
4	3
5	5



Memoizing recursive implmentations

```
def fib(n):
  if n <= 1:
    value = n
  else:
    value = fib(n-1) + fib(n-2)
  return(value)
```

Memoizing recursive implmentations

```
def fib(n):
  if n in fibtable.keys():          - Check table
    return(fibtable[n])
  if n <= 1:
    value = n
  else:
    value = fib(n-1) + fib(n-2)
  fibtable[n] = value               - Update table
  return(value)
```

Memoizing recursive implmentations

```
def fib(n):
  if n in fibtable.keys():
    return(fibtable[n])
  if n <= 1:
    value = n
  else:
    value = fib(n-1) + fib(n-2)
  fibtable[n] = value
  return(value)
```

In general

```
def f(x,y,z):
  if (x,y,z) in ftable.keys():
    return(ftable[(x,y,z)])
  recursively compute value
    from subproblems
  ftable[(x,y,z)] = value
  return(value)
```

Dynamic programming

- Anticipate the structure of subproblems
  - Derive from inductive definition
  - Dependencies are acyclic

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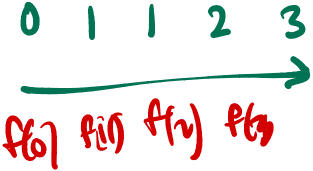
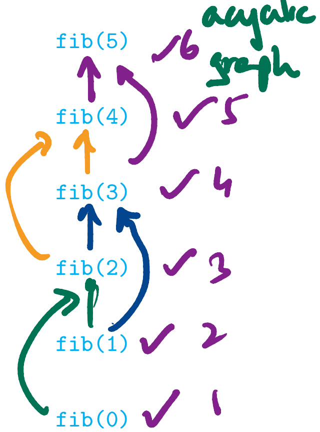


Evaluating `fib(5)`



Dynamic programming

- Anticipate the structure of subproblems
  - Derive from inductive definition
  - Dependencies are acyclic
- Solve subproblems in appropriate order
  - Start with base cases — no dependencies
  - Evaluate a value after all its dependencies are available
  - Fill table iteratively
  - Never need to make a recursive call

Evaluating `fib(5)`

`fib(5)`

`fib(4)`

`fib(3)`

`fib(2)`

`fib(1)`

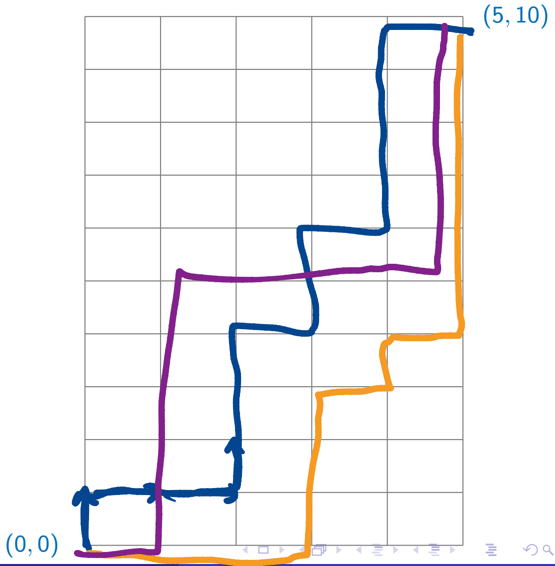
`fib(0)`

Topological Sort

```
f = {}
f[0] = 0
f[1] = 1
for j in range(2,n+1)
  f[j] = f[j-1] + f[j-2]
```

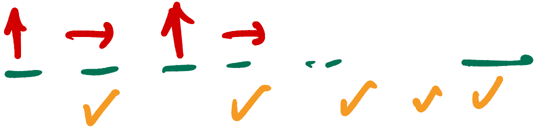
Grid paths

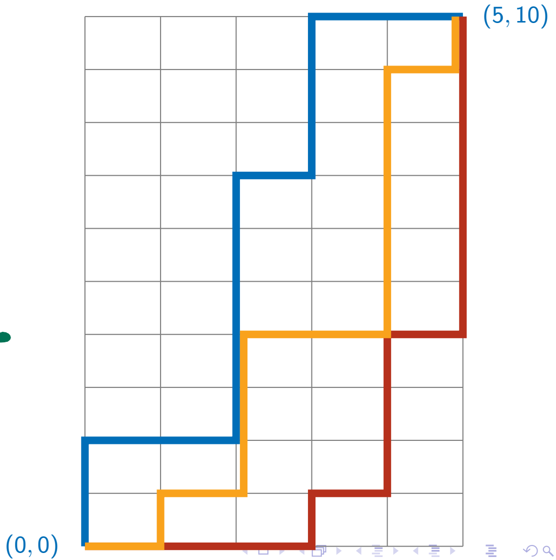
- Rectangular grid of one-way roads
- Can only go up and right
- How many paths from $(0, 0)$ to (m, n)?



Combinatorial solution

- Every path from $(0,0)$ to $(5,10)$ has 15 segments
  - Out of 15, exactly 5 are right moves, 10 are up moves
  - Fix the positions of the 5 right moves among the 15 positions overall





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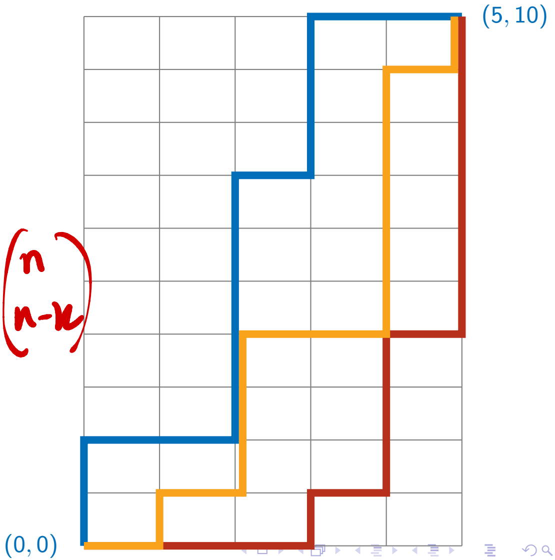
  - $\binom{15}{5} = \frac{15!}{10! \cdot 5!} = 3003$
  - Same as $\binom{15}{10}$ — fix the 10 up moves

$$\binom{n}{k} = \binom{n}{n-k}$$



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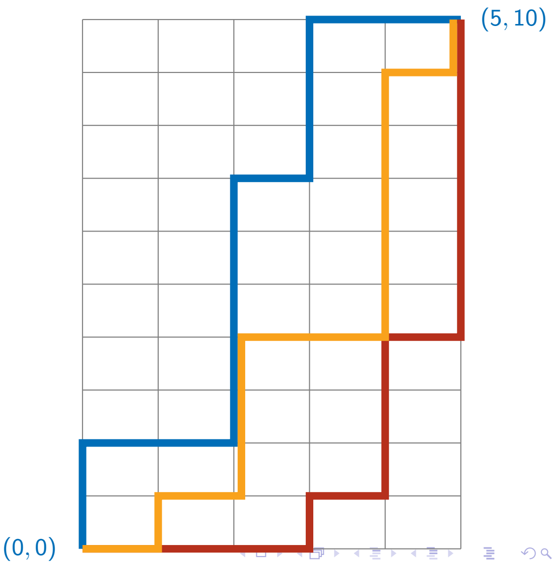
  - $\binom{15}{5} = \frac{15!}{10! \cdot 5!} = 3003$
  - Same as $\binom{15}{10}$ — fix the 10 up moves
- In general $m+n$ segments from $(0,0)$ to (m,n)

$\binom{m+n}{m}$ $\binom{m+n}{n}$



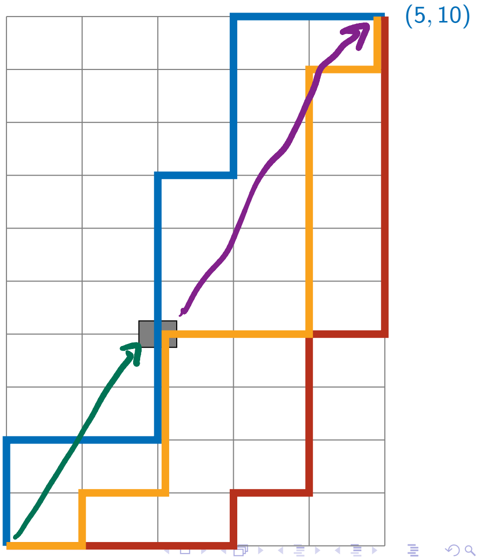
- What if an intersection is blocked?
  - For instance, $(2,4)$

P_1: $(0,0) \rightarrow (2,4)$? $\binom{6}{4}$ or $\binom{6}{2}$

P_2: $(2,4) \rightarrow (5,10)$ $\binom{9}{3}$ or $\binom{9}{6}$

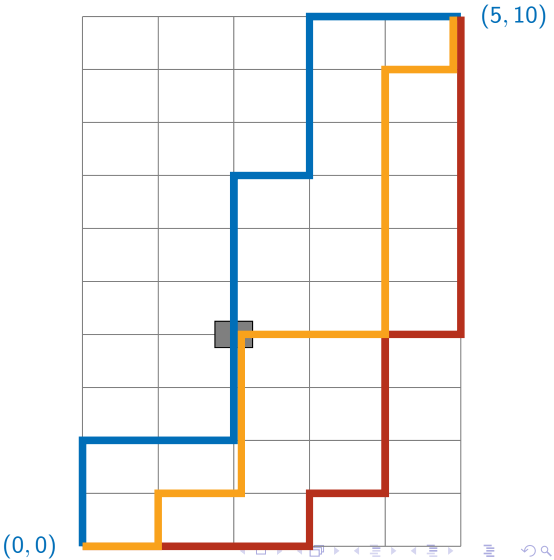
$P_1 \times P_2$ bad paths

$\binom{15}{5} - (P_1 \times P_2)$



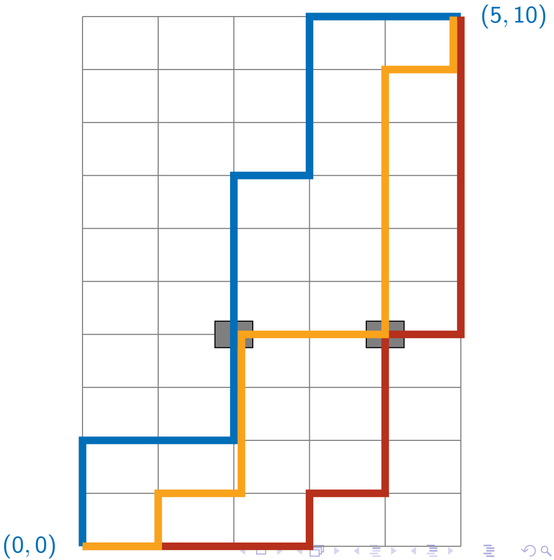
Combinatorial solution for holes

- Discard paths passing through $(2,4)$



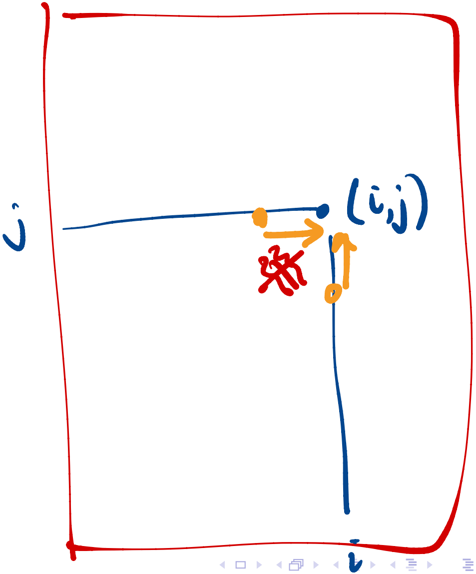
More holes

- What if two intersections are blocked?
- Discard paths via $(2,4)$, $(4,4)$
  - Some paths are counted twice
- Add back the paths that pass through both holes
- Inclusion-exclusion — counting is messy



Inductive formulation

- How can a path reach (i, j)



Inductive formulation

- How can a path reach (i, j)
  - Move up from $(i, j-1)$

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- How can a path reach (i, j)
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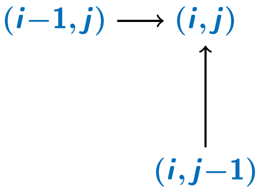

Inductive formulation

- How can a path reach (i, j)
  - Move up from $(i, j-1)$
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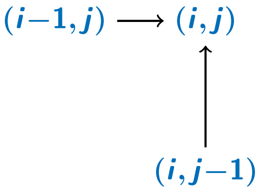
- Each path to these neighbours extends to a unique path to (i, j)



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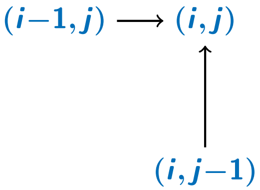
- Recurrence for $P(i, j)$, number of paths from $(0, 0)$ to (i, j)
  - $P(i, j) = P(i-1, j) + P(i, j-1)$



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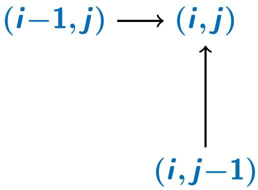
  - $P(0, 0) = 1$ — base case



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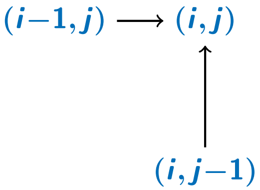
  - $P(i, 0) = P(i-1, 0)$ — bottom row



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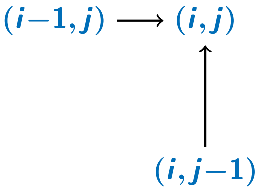

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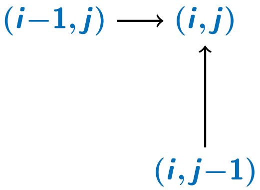
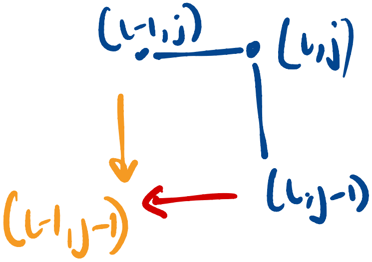
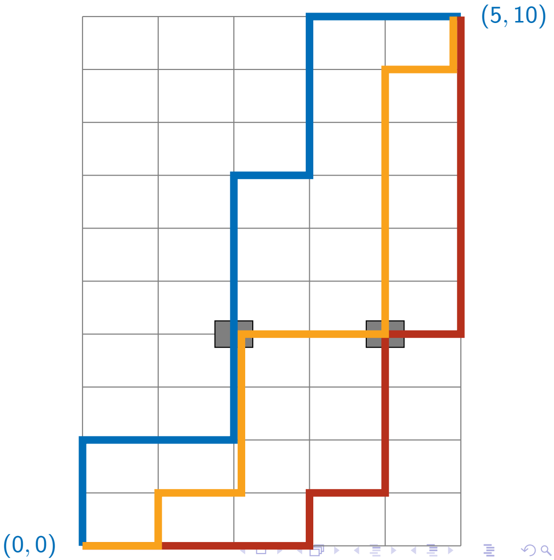
- $P(i, j) = 0$ if there is a hole at (i, j)



Computing $P(i, j)$

- Naive recursion recomputes same subproblem repeatedly





Computing $P(i,j)$

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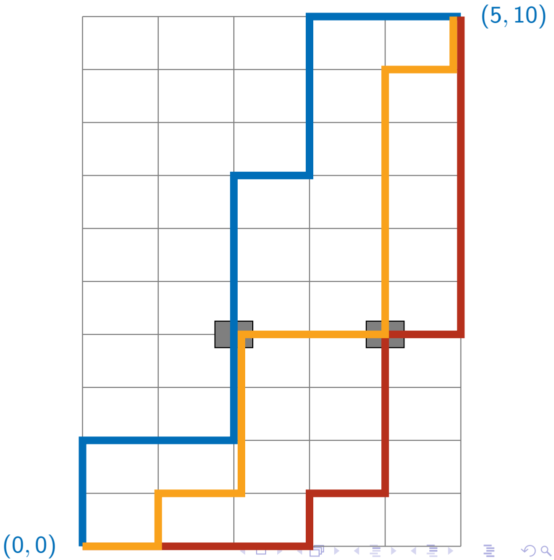
  - $P(5,10)$ requires $P(4,10)$, $P(5,9)$



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- Naive recursion recomputes same subproblem repeatedly
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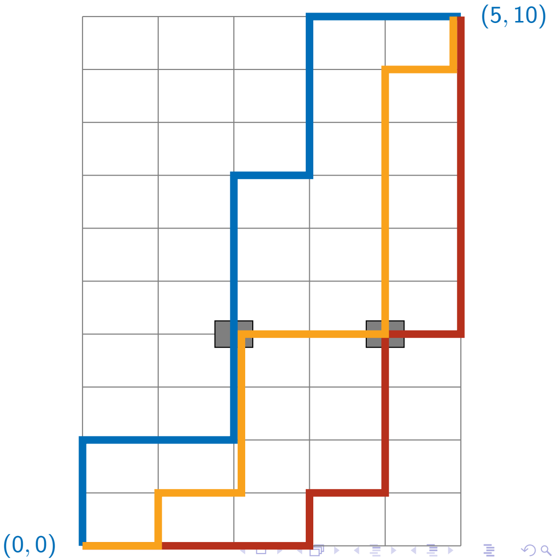
  - Both $P(4,10)$, $P(5,9)$ require $P(4,9)$



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- Naive recursion recomputes same subproblem repeatedly
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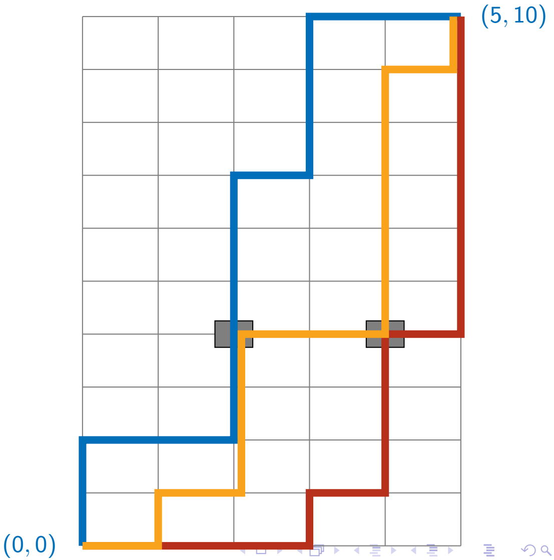
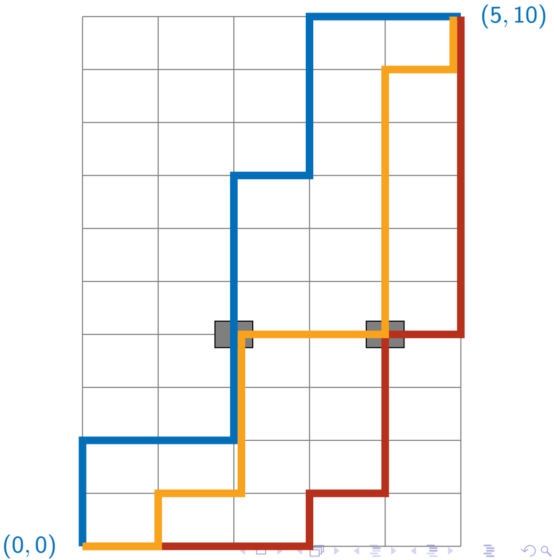
- Use memoization . . .

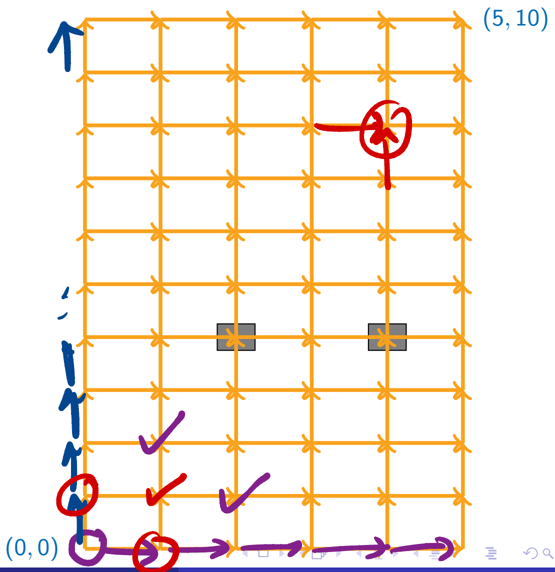


Computing $P(i,j)$

- Naive recursion recomputes same subproblem repeatedly
  - $P(5,10)$ requires $P(4,10)$, $P(5,9)$
  - Both $P(4,10)$, $P(5,9)$ require $P(4,9)$
- Use memoization ...
- ... or find a suitable order to compute the subproblems

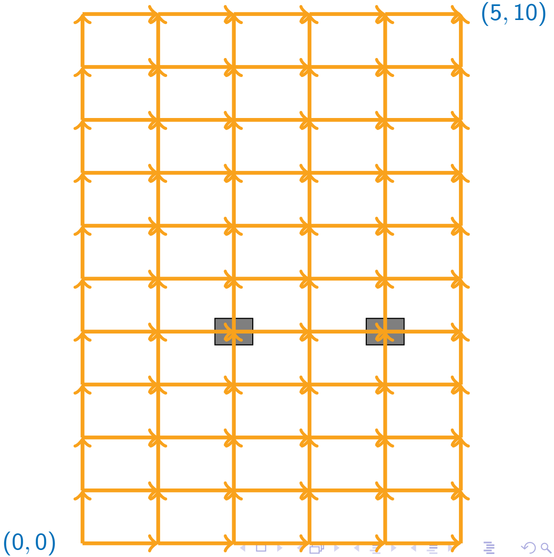


- Identify subproblem structure



Dynamic programming

- Identify subproblem structure
- $P(0,0)$ has no dependencies



Dynamic programming

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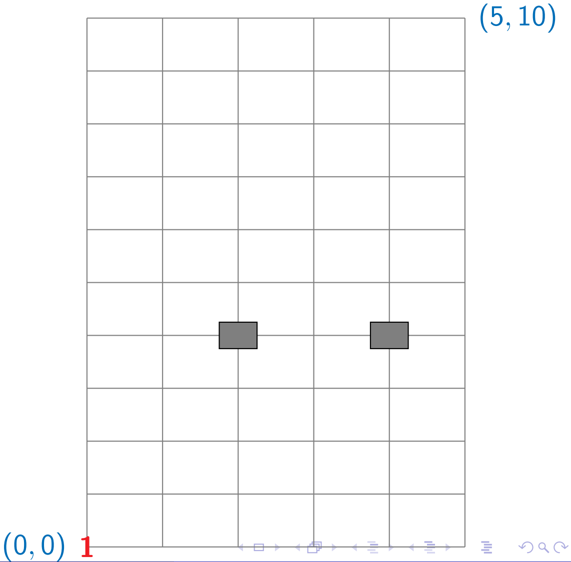
- Start at $(0,0)$



Dynamic programming

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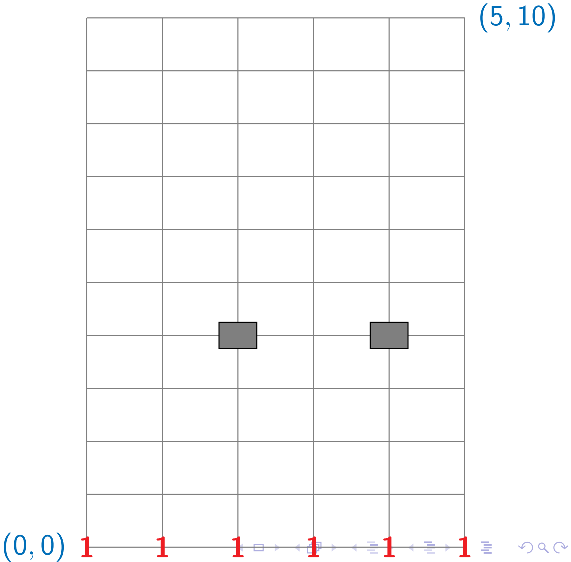
- $P(0,0)$ has no dependencies
- Start at $(0,0)$
- Fill row by row



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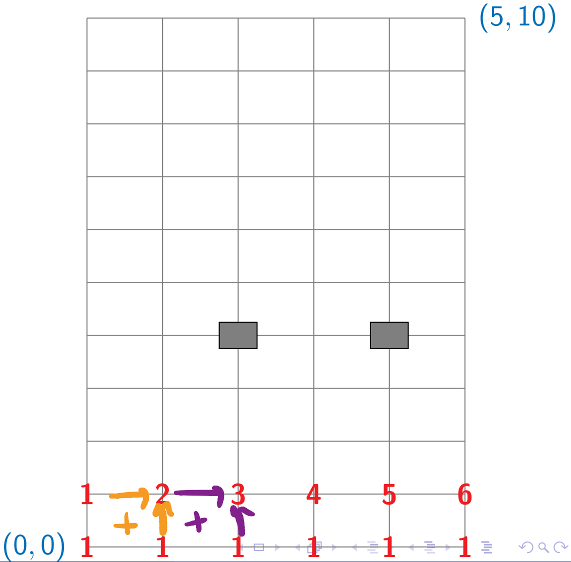

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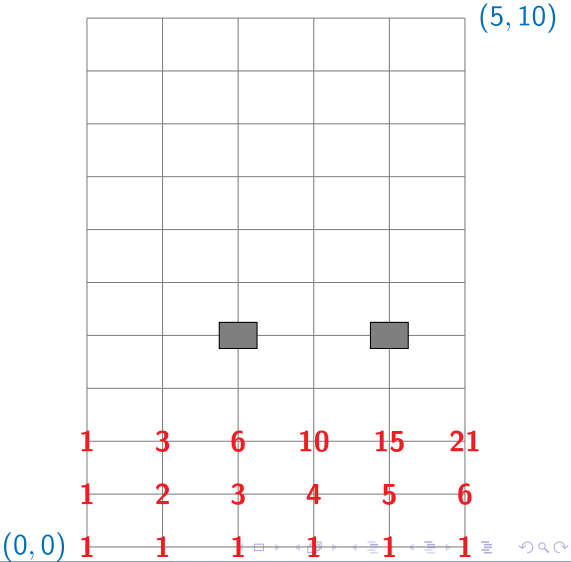

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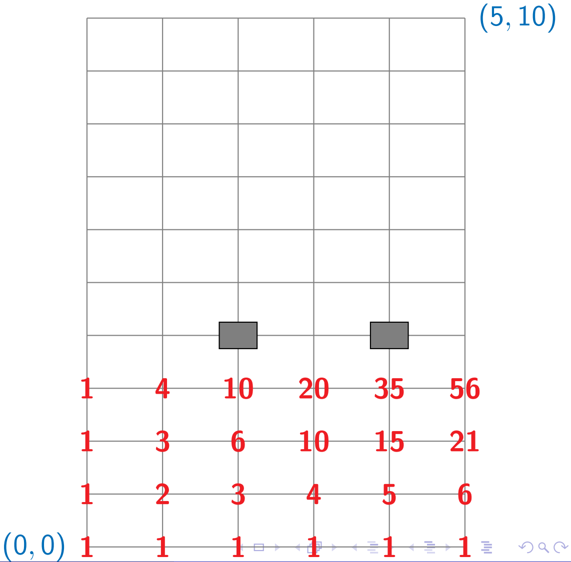

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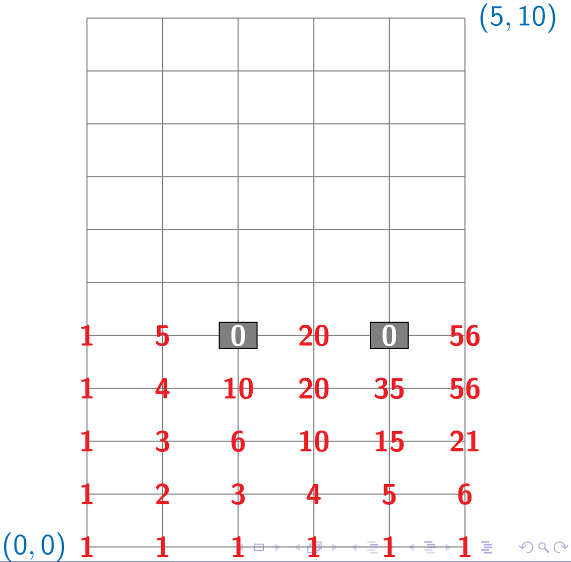

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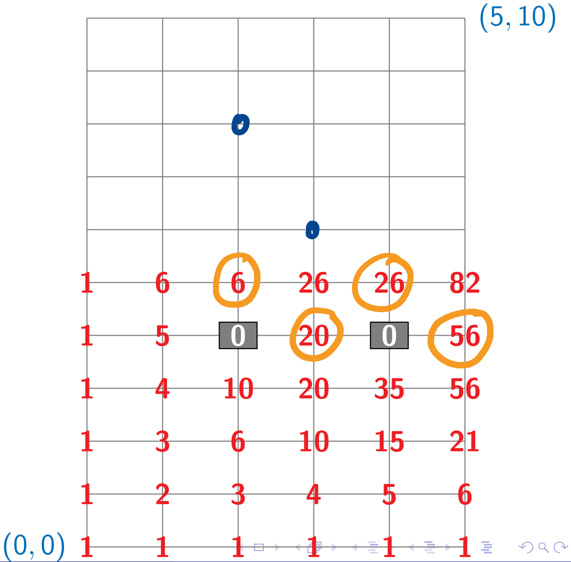

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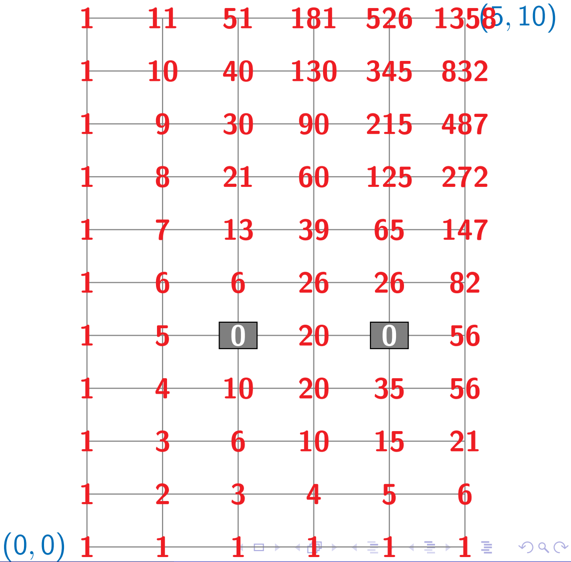

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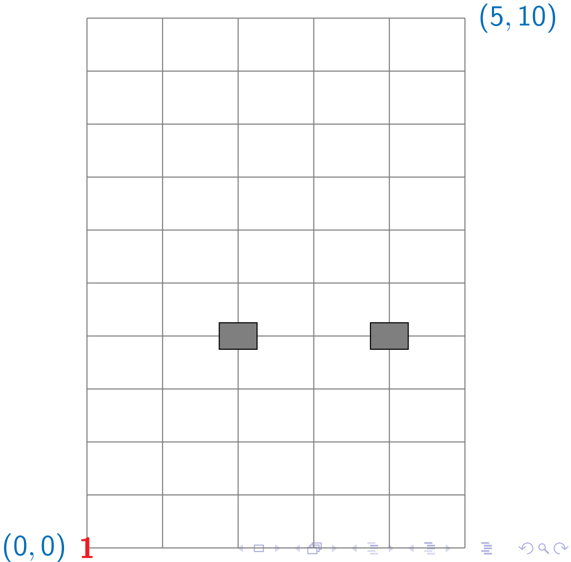

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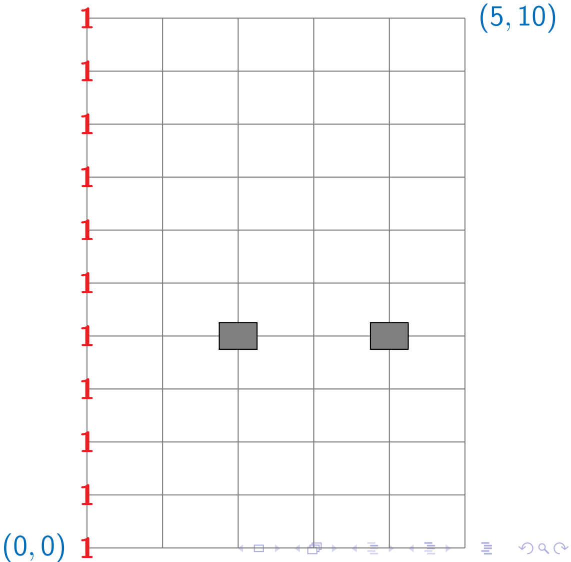

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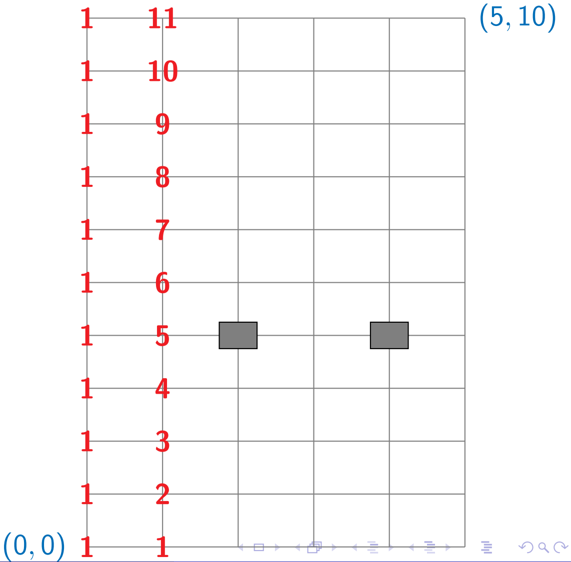
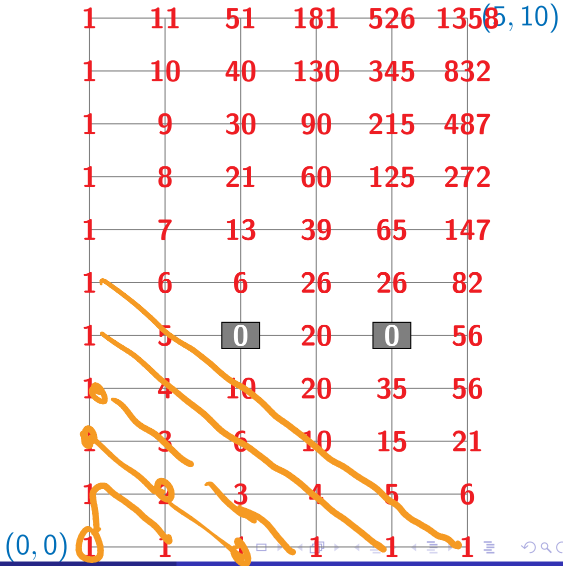


Dynamic programming

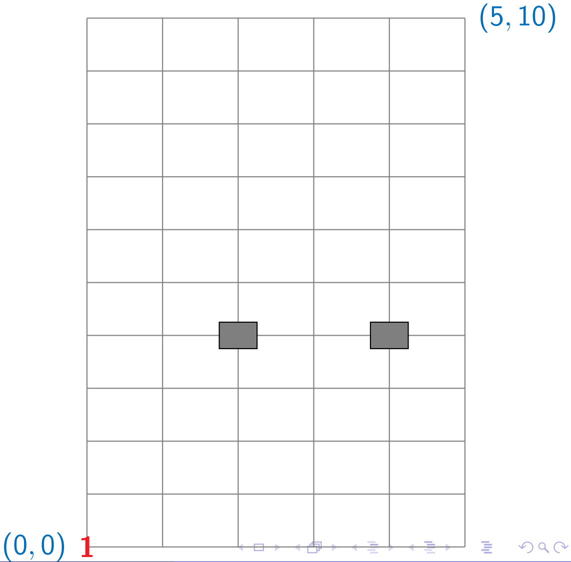
- Identify suproblem structure
- $P(0,0)$ has no dependencies
- Start at $(0,0)$
- Fill row by row
- Fill column by column

Yet another order



Dynamic programming

- Identify subproblem structure
- $P(0,0)$ has no dependencies
- Start at $(0,0)$
- Fill row by row
- Fill column by column
- Fill diagonal by diagonal



Dynamic programming

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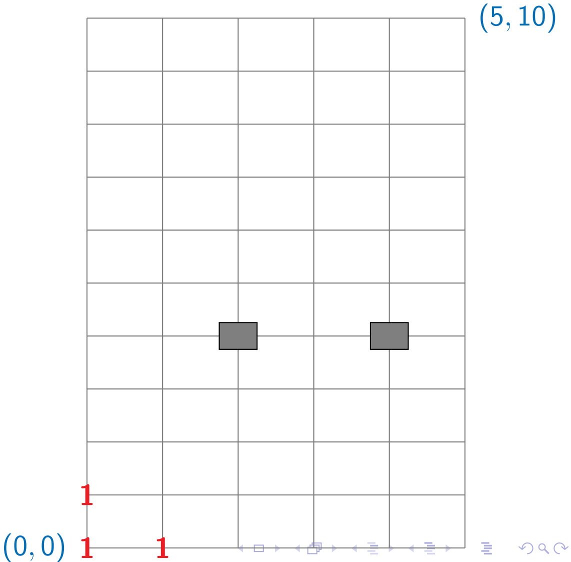

Dynamic programming

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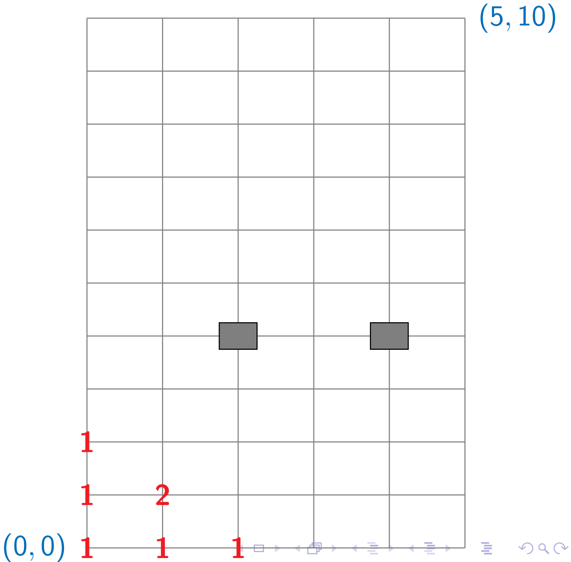

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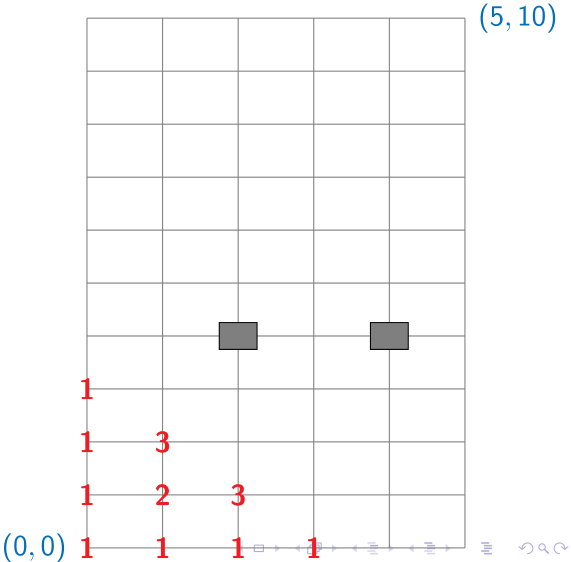
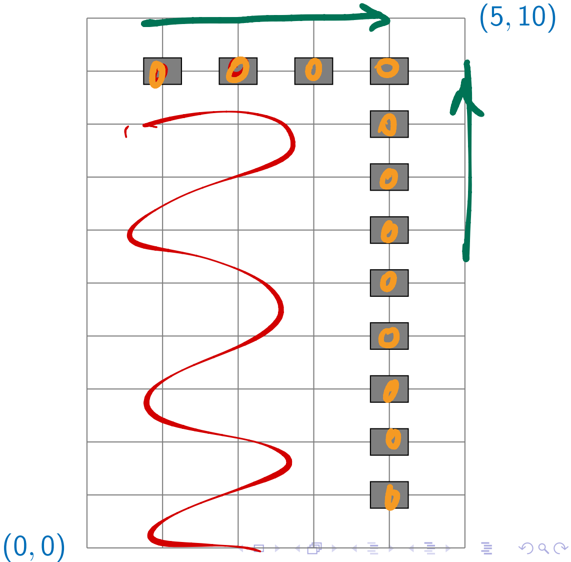


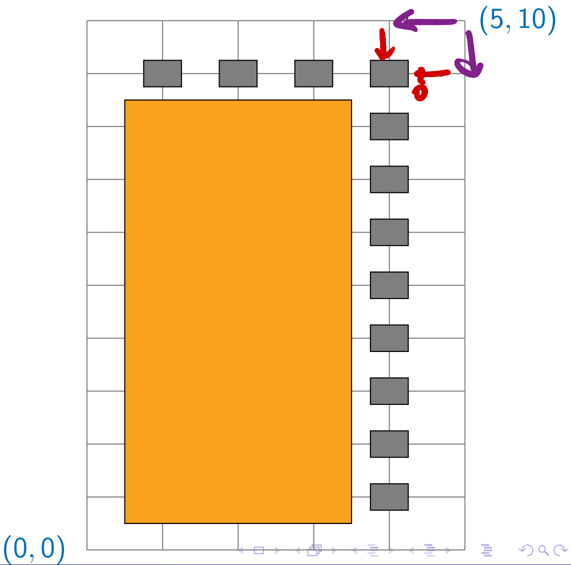
Memoization vs dynamic programming

- Barrier of holes just inside the border



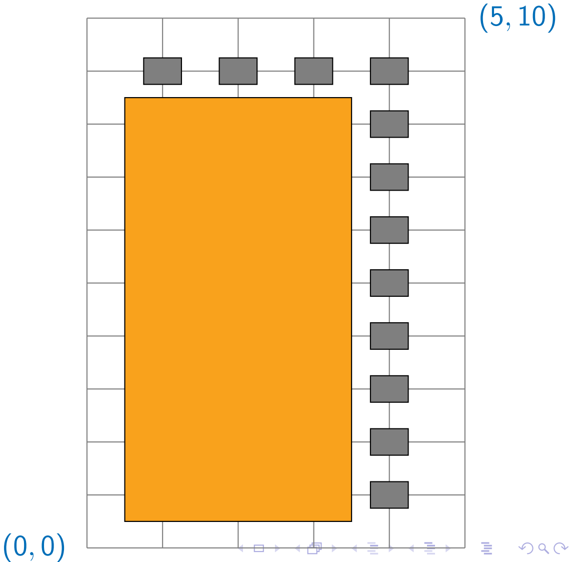
Memoization vs dynamic programming

- Barrier of holes just inside the border
- Memoization never explores the shaded region



Memoization vs dynamic programming

- Barrier of holes just inside the border
- Memoization never explores the shaded region
- Memo table has $O(m+n)$ entries



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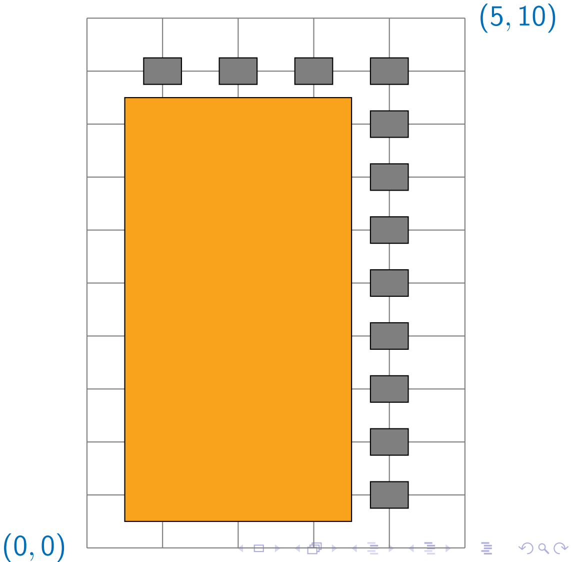
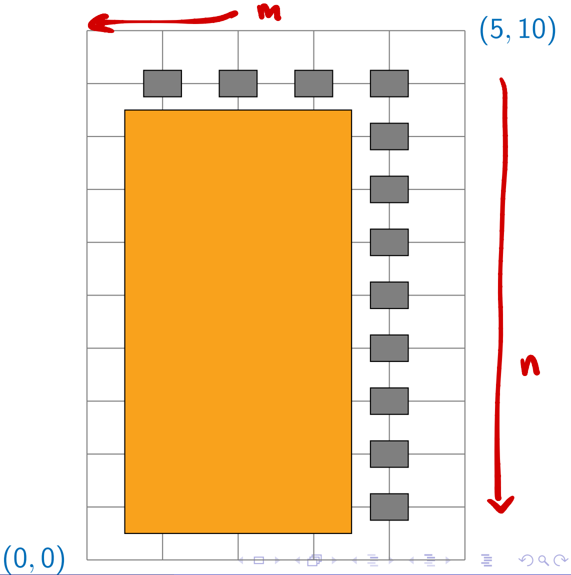


Memoization vs dynamic programming

- Barrier of holes just inside the border
- Memoization never explores the shaded region
- Memo table has $O(m+n)$ entries
- Dynamic programming blindly fills all mn cells of the table
- Tradeoff between recursion and iteration
  - "Wasteful" dynamic programming still better in general



Longest common subword

- Given two strings, find the (length of the) longest common subword
  - `"secret"`, `"secretary"` — `"secret"`, length 6
  - `"bisect"`, `"trisect"` — `"isect"`, length 5
  - `"bisect"`, `"secret"` — `"sec"`, length 3
  - `"director"`, `"secretary"` — `"ec"`, `"re"`, length 2

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- Formally
  - $u = a_0 a_1 \ldots a_{m-1}$
  - $v = b_0 b_1 \ldots b_{n-1}$

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  - $u = a_0 a_1 \ldots a_{m-1}$
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  - Common subword of length k — for some positions i and j, $a_i a_{i+1} a_{i+k-1} = b_j b_{j+1} b_{j+k-1}$

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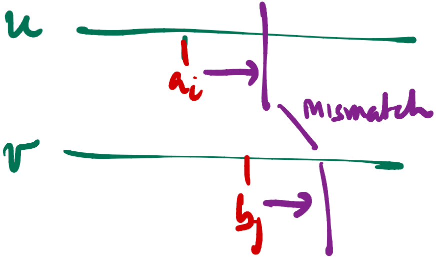
  - Find the largest such k — length of the longest common subword

Brute force

- $u = a_0 a_1 \dots a_{m-1}$
- $v = b_0 b_1 \dots b_{n-1}$
- Find the largest k such that for some positions i and j, $a_i a_{i+1} a_{i+k-1} = b_j b_{j+1} b_{j+k-1}$



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- Try every pair of starting positions i in u, j in v
  - Match $(a_i, b_j), (a_{i+1}, b_{j+1}), \dots$ as far as possible
  - Keep track of longest match

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  - Match $(a_i, b_j), (a_{i+1}, b_{j+1}), \ldots$ as far as possible
  - Keep track of longest match
- Assuming $m > n$, this is $O(mn^2)$
  - mn pairs of starting positions
  - From each starting position, scan could be $O(n)$

Inductive structure

- $u = a_0 a_1 \ldots a_{m-1}$
- $v = b_0 b_1 \ldots b_{n-1}$
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- $u = a_0 a_1 \ldots a_{m-1}$
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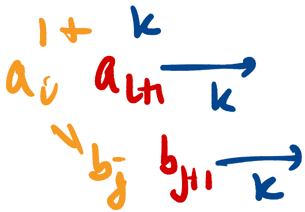
- Find the largest k such that for some positions i and j, $a_i a_{i+1} a_{i+k-1} = b_j b_{j+1} b_{j+k-1}$
- $LCW(i,j)$ — length of longest common subword in $a_i a_{i+1} \ldots a_{m-1}$, $b_j b_{j+1} \ldots b_{n-1}$
  - If $a_i \neq b_j$, $LCW(i,j) = 0$
  - If $a_i = b_j$, $LCW(i,j) = 1 + LCW(i{+}1, j{+}1)$

LCW(i,j) is longest ending at a_i, b_j



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  - In general, $LCW(m,j) = 0$ for all $0 \leq j \leq n$

Subproblem dependency

- Subproblems are $LCW(i, j)$, for $0 \leq i \leq m$, $0 \leq j \leq n$

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- Table of $(m+1) \cdot (n+1)$ values

		0	1	2	3	4	5	6
		s	e	c	r	e	t	●
0	b							
1	i							
2	s							
3	e							
4	c							
5	t							
6	●							

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							
1	i							
2	s							
3	e							
4	c							
5	t							
6	•							

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- Start at bottom right and fill row by row or column by column

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							**0**
1	i							**0**
2	s							**0**
3	e							**0**
4	c							**0**
5	t							**0**
6	•							**0**

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$1 + LCW(i+1, j+1)$

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b						**0**	**0**
1	i						**0**	**0**
2	s						**0**	**0**
3	e						**0**	**0**
4	c						**0**	**0**
5	t						**1**	**0**
6	•					0	**0**	**0**

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b					0	0	0
1	i					0	0	0
2	s					0	0	0
3	e					1	0	0
4	c					0	0	0
5	t					0	1	0
6	•					0	0	0

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b				0	0	0	0
1	i				0	0	0	0
2	s				0	0	0	0
3	e				0	1	0	0
4	c				0	0	0	0
5	t				0	0	1	0
6	•				0	0	0	0

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b			0	0	0	0	0
1	i			0	0	0	0	0
2	s			0	0	0	0	0
3	e			0	0	1	0	0
4	c			1	0	0	0	0
5	t			0	0	0	1	0
6	•			0	0	0	0	0

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b		0	0	0	0	0	0
1	i		0	0	0	0	0	0
2	s		0	0	0	0	0	0
3	e		2	0	0	1	0	0
4	c		0	1	0	0	0	0
5	t		0	0	0	0	1	0
6	•		0	0	0	0	0	0

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	0	0	0	0	0	0	0
1	i	0	0	0	0	0	0	0
2	s	3	0	0	0	0	0	0
3	e	0	2	0	0	1	0	0
4	c	0	0	1	0	0	0	0
5	t	0	0	0	0	0	1	0
6	•	0	0	0	0	0	0	0

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- Start at bottom right and fill row by row or column by column

Reading off the solution

- Find entry (i,j) with largest LCW value

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	0	0	0	0	0	0	0
1	i	0	0	0	0	0	0	0
2	s	3	0	0	0	0	0	0
3	e	0	2	0	0	1	0	0
4	c	0	0	1	0	0	0	0
5	t	0	0	0	0	0	1	0
6	•	0	0	0	0	0	0	0

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Reading off the solution

- Find entry (i,j) with largest LCW value
- Read off the actual subword diagonally

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1	i	**0**	**0**	**0**	**0**	**0**	**0**	**0**
2	s	**3**	**0**	**0**	**0**	**0**	**0**	**0**
3	e	**0**	**2**	**0**	**0**	**1**	**0**	**0**
4	c	**0**	**0**	**1**	**0**	**0**	**0**	**0**
5	t	**0**	**0**	**0**	**0**	**0**	**1**	**0**
6	•	**0**	**0**	**0**	**0**	**0**	**0**	**0**

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	0	0	0	0	0	0	0
1	i	0	0	0	0	0	0	0
2	s	3	0	0	0	0	0	0
3	e	0	2	0	0	1	0	0
4	c	0	0	1	0	0	0	0
5	t	0	0	0	0	0	1	0
6	•	0	0	0	0	0	0	0

Implementation

```
def LCW(u,v):
  import numpy as np
  (m,n) = (len(u),len(v))
  lcw = np.zeros((m+1,n+1))

  maxlcw = 0

  for j in range(n-1,-1,-1):
    for i in range(m-1,-1,-1):
      if u[i] == v[j]:
        lcw[i,j] = 1 + lcw[i+1,j+1]
      else:
        lcw[i,j] = 0
      if lcw[i,j] > maxlcw:
        maxlcw = lcw[i,j]

  return(maxlcw)
```

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      else:
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  return(maxlcw)
```

Complexity

- Recall that brute force was $O(mn^2)$
- Inductive solution is $O(mn)$, using dynamic programming or memoization
  - Fill a table of size $O(mn)$
  - Each table entry takes constant time to compute