Stacks, Queues, Priority Queues, Heaps

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Programming and Data Structures with Python

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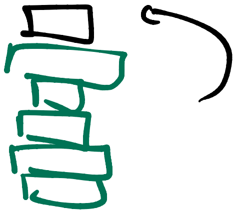
Stack

- Stack is a last-in, first-out sequence
- push(s,x) — add x to stack s
- pop(s) — return most recently added element



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- `push(s,x)` — add `x` to stack `s`
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- Maintain stack as list, push and pop from the right
  - `push(s,x)` is `s.append(x)`
  - `s.pop()` — Python built-in, returns last element

check empty stack

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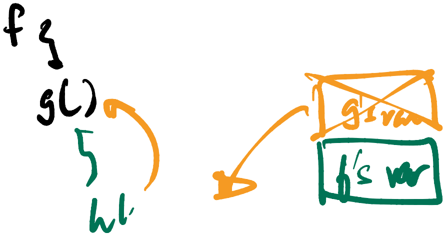
- Stacks are natural to keep track of local variables through function calls
  - Each function call pushes current frame onto a stack
  - When function exits, pop its frame off the stack



Queue

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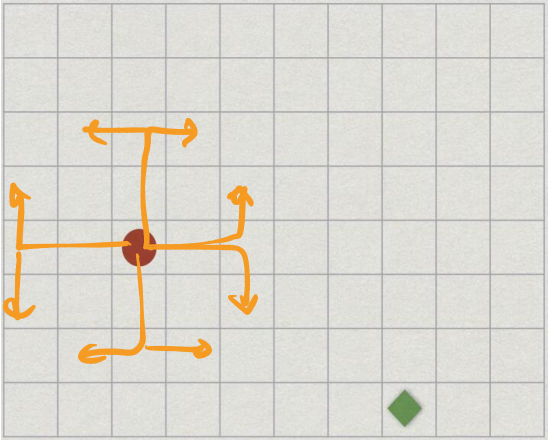
- `removeq(q)` — removes element at head of `q`
- Using Python lists, left is rear, right is front
  - `addq(q,x)` is `q.insert(0,x)`
    - `insert(j,x)`, insert `x` before position `j`
  - `removeq(q)` is `q.pop()`

Systematic exploration

- Rectangular $m \times n$ grid
- Chess knight starts at (sx, sy) ●
- Usual knight moves
- Can it reach a target square (tx, ty)? ♦



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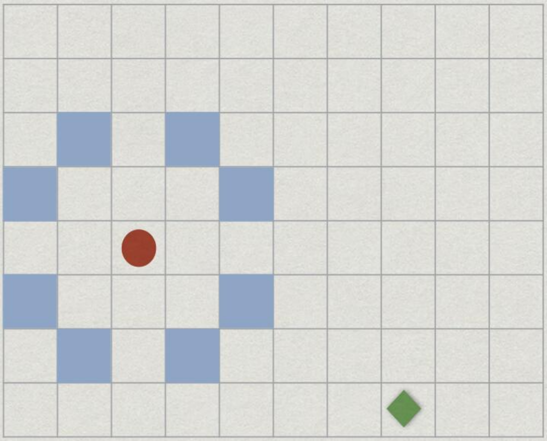

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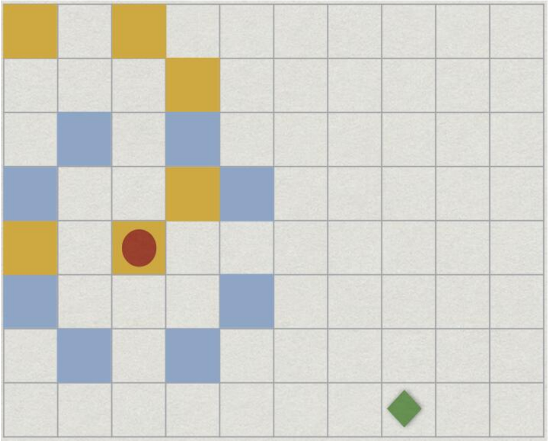

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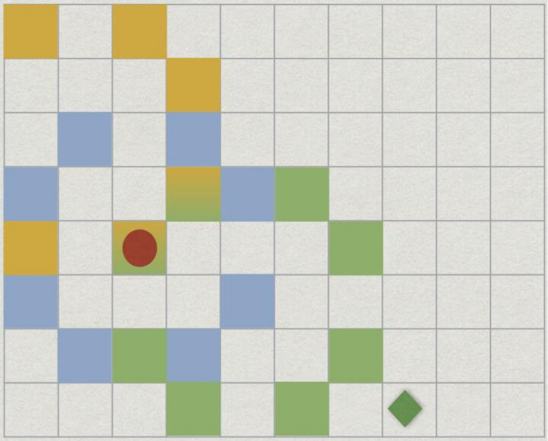

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- Don't explore an already marked square

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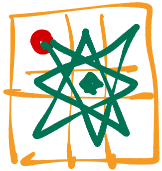

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26 .. 22 21 15 14 13 12 11 8 7 6 5 4 3

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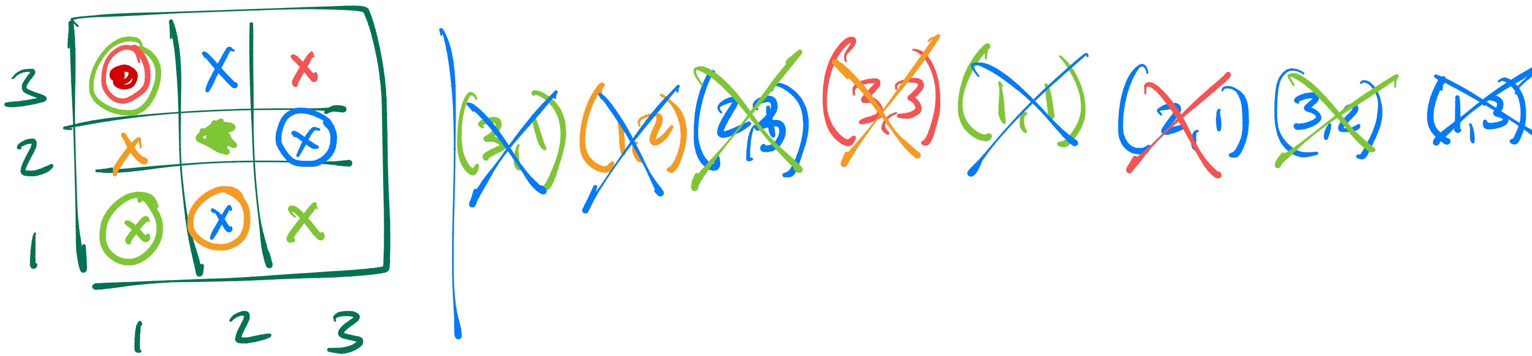
- Initially Q contains only start node (sx, sy)
  - Remove (ax, ay) from head of queue
  - Mark all squares reachable in one step from (ax, ay)
  - Add all newly marked squares to the queue
- When the queue is empty, we have finished



(3,1) (1,2) (2,3) (3,3) (1,1) (2,1) (3,2) (1,3)

Dealing with priorities

Job scheduler

- A job scheduler maintains a list of pending jobs with their priorities

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- New jobs may join the list at any time
- How should the scheduler maintain the list of pending jobs and their priorities?

Priority queue

- Need to maintain a collection of items with priorities to optimise the following operations
- `delete_max()`
  - Identify and remove item with highest priority
  - Need not be unique
- `insert()`
  - Add a new item to the collection

Implementing priority queues with one dimensional structures

- `delete_max()`
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- Unsorted list
  - `insert()` is $O(1)$
  - `delete_max()` is $O(n)$

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Implementing priority queues with one dimensional structures

- Unsorted list
  - `insert()` is $O(1)$
  - `delete_max()` is $O(n)$
- Sorted list
  - `delete_max()` is $O(1)$
  - `insert()` is $O(n)$
- Processing n items requires $O(n^2)$

- `delete_max()`
  - Identify and remove item with highest priority
  - Need not be unique
- `insert()`
  - Add a new item to the list

Moving to two dimensions

First attempt

- Assume N processes enter/leave the queue

$N = 25$

3	19	23	35	58
12	17	25	43	67
10	13	20		
11	16	28	49	
6	14			

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First attempt

- Assume N processes enter/leave the queue
- Maintain a $\sqrt{N} \times \sqrt{N}$ array

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Moving to two dimensions

First attempt

- Assume N processes enter/leave the queue
- Maintain a $\sqrt{N} \times \sqrt{N}$ array
- Each row is in sorted order

$N = 25$

3	19	23	35	58
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insert()

- Keep track of the size of each row

$N = 25$

3	19	23	35	58		5
12	17	25	43	67		5
10	13	20				3
11	16	28	49			4
6	14					2

insert()

- Keep track of the size of each row
- Insert into the first row that has space
  - Use size of row to determine

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- Keep track of the size of each row
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- Insert 15

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insert()

- Keep track of the size of each row
- Insert into the first row that has space
  - Use size of row to determine
- Insert 15
- Takes time $O(\sqrt{N})$
  - Scan size column to locate row to insert, $O(\sqrt{N})$
  - Insert into the first row with free space, $O(\sqrt{N})$

$N = 25$

						size
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delete_max()

- Maximum in each row is the last element

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delete_max()

- Maximum in each row is the last element
- Position is available through size column

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- Maximum in each row is the last element
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delete_max()

- Maximum in each row is the last element
- Position is available through size column
- Identify the maximum amongst these
- Delete it
- Again $O(\sqrt{N})$
  - Find the maximum among last entries, $O(\sqrt{N})$
  - Delete it, $O(1)$

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Summary

- 2D $\sqrt{N} \times \sqrt{N}$ array with sorted rows
  - `insert()` is $O(\sqrt{N})$
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  - Processing N items is $O(N\sqrt{N})$

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- Flexible — need not fix N in advance

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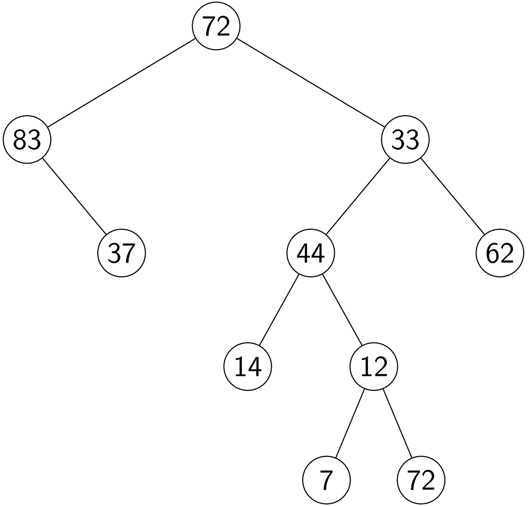
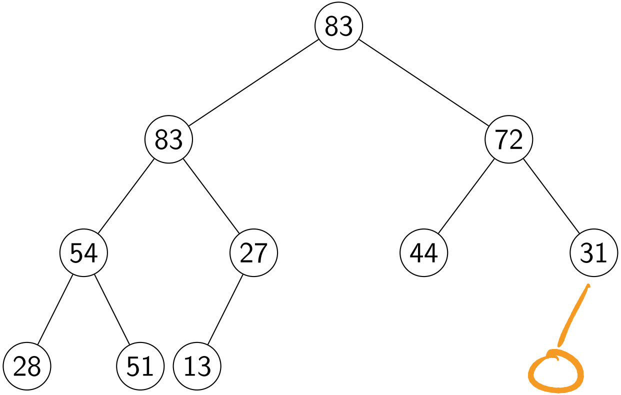
Binary trees

- Values are stored as nodes in a rooted tree
- Each node has up to two children
  - Left child and right child
  - Order is important
- Other than the root, each node has a unique parent
- Leaf node — no children
- Size — number of nodes
- Height — number of levels



Heap

- Binary tree filled level by level, left to right
- The value at each node is at least as big the values of its children
  - max-heap



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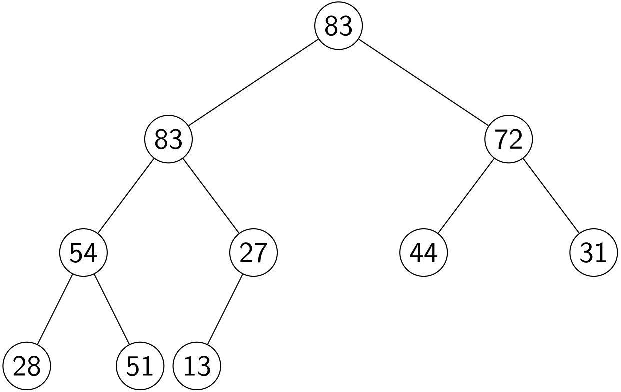

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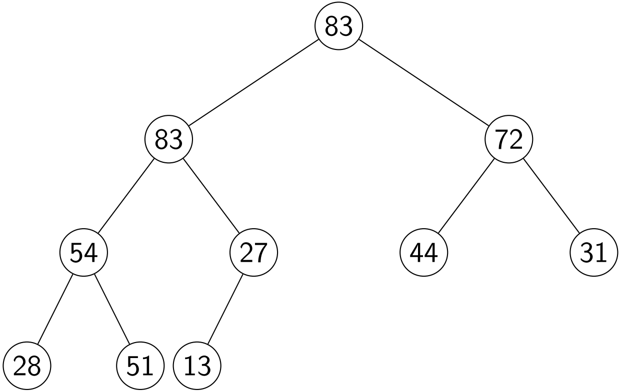
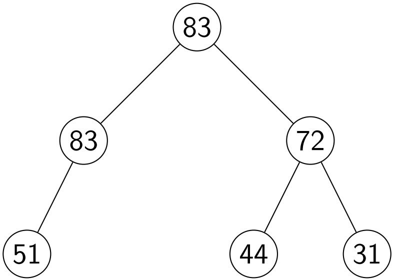
- Binary tree on the right is an example of a heap
- Root always has the largest value
  - By induction, because of the max-heap property



Non-examples

No "holes" allowed



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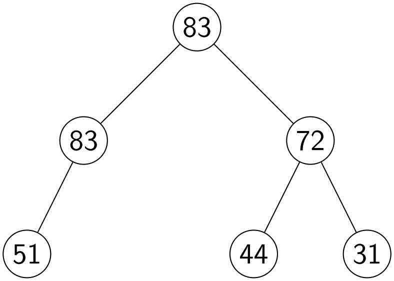
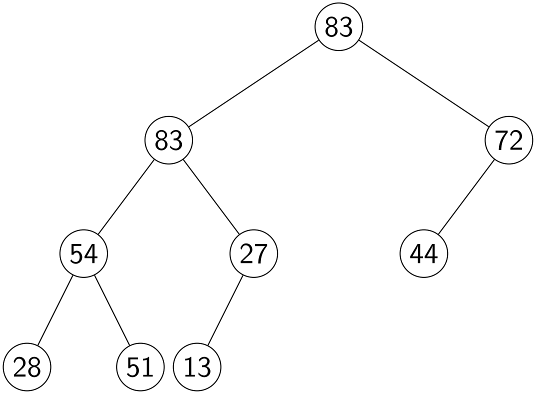
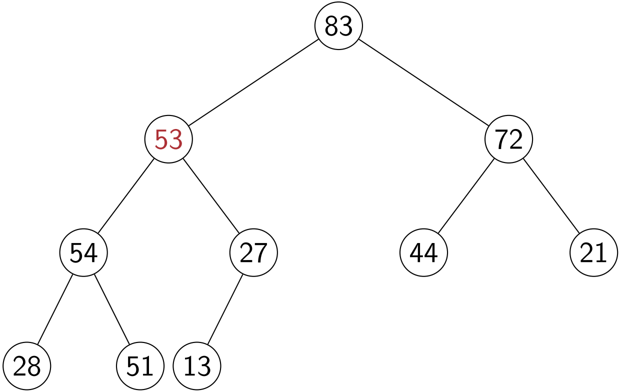


Cannot leave a level incomplete



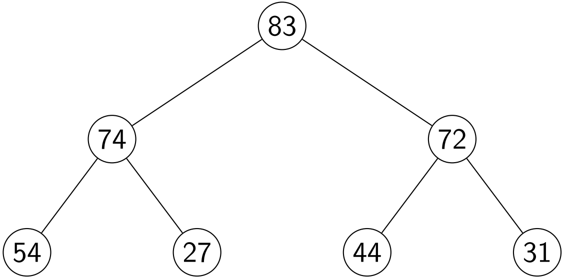
Non-examples

Heap property is violated



insert()

- `insert(77)`



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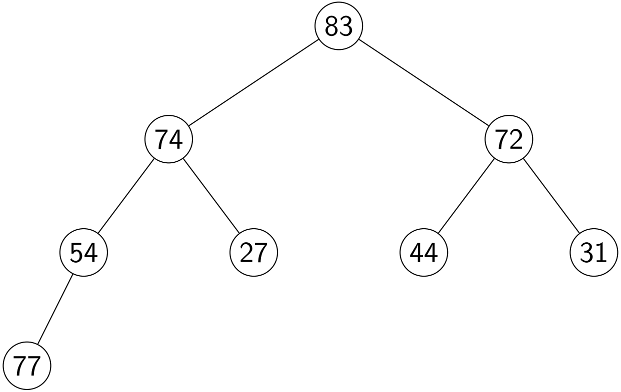
- `insert(77)`
- Add a new node at dictated by heap structure



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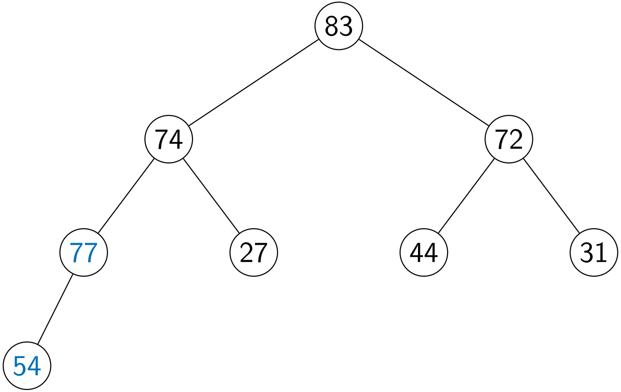
- Add a new node at dictated by heap structure
- Restore the heap property along path to the root



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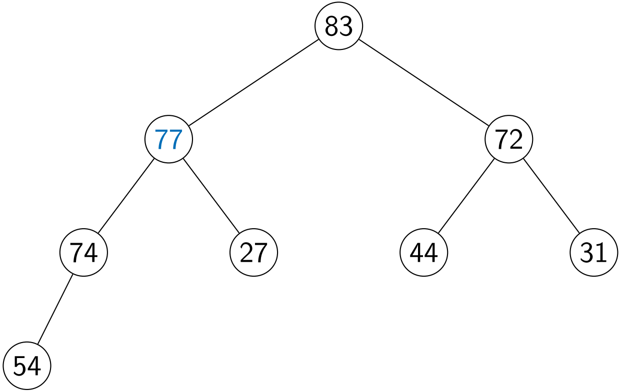

insert()

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- Restore the heap property along path to the root
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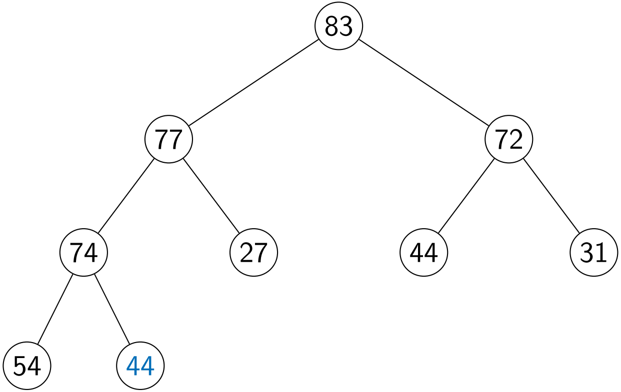

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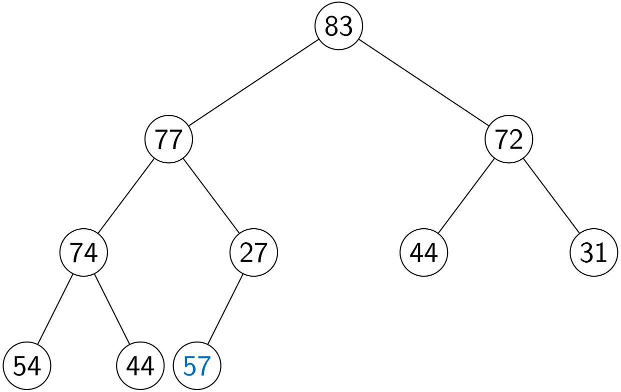

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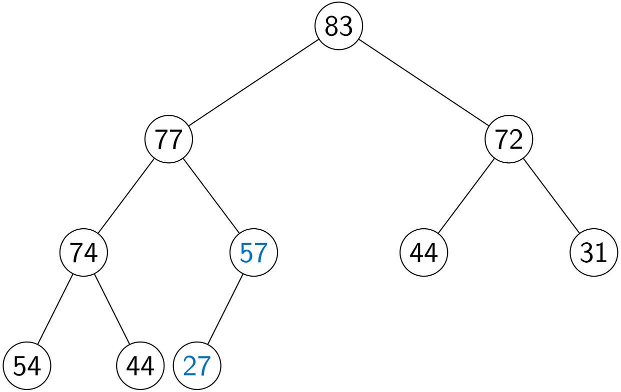
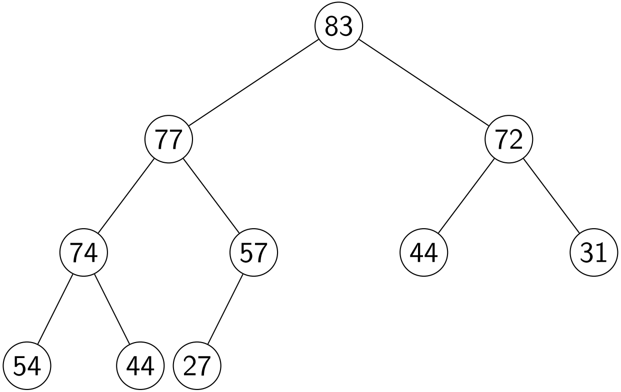


Complexity of `insert()`

- Need to walk up from the leaf to the root
  - Height of the tree



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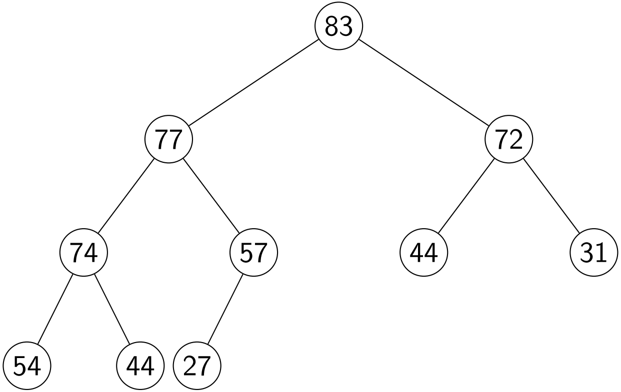
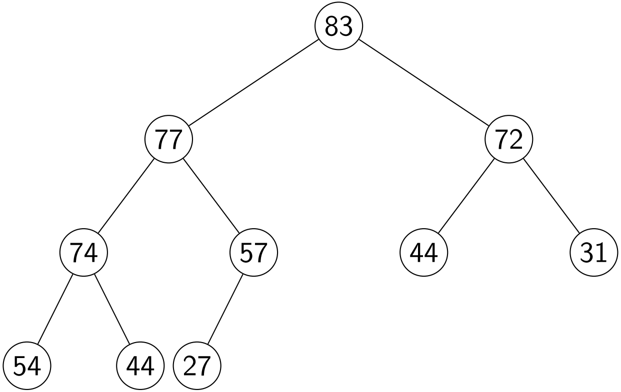
- Number of nodes at level 0 is $2^0 = 1$



Complexity of `insert()`

- Need to walk up from the leaf to the root
  - Height of the tree
- Number of nodes at level 0 is $2^0 = 1$
- Number of nodes at level j is 2^j



Complexity of `insert()`

- Need to walk up from the leaf to the root
  - Height of the tree
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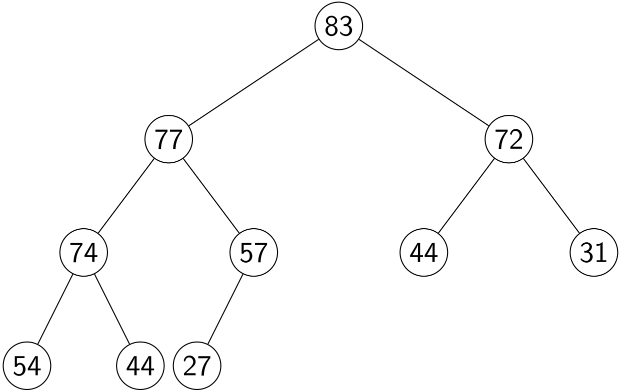

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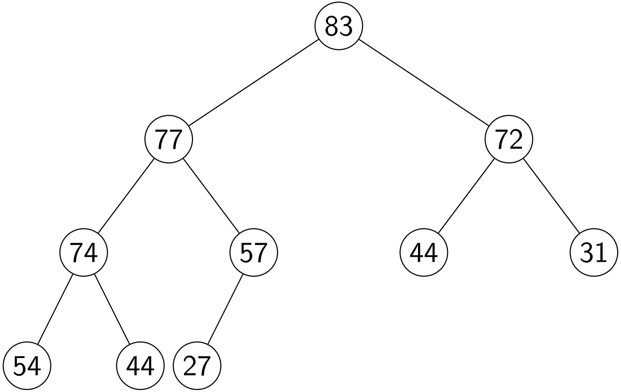

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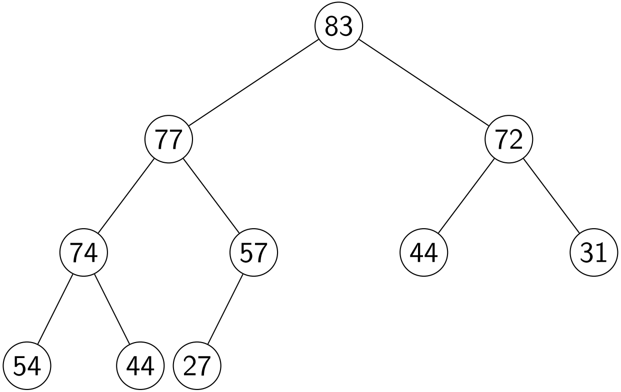
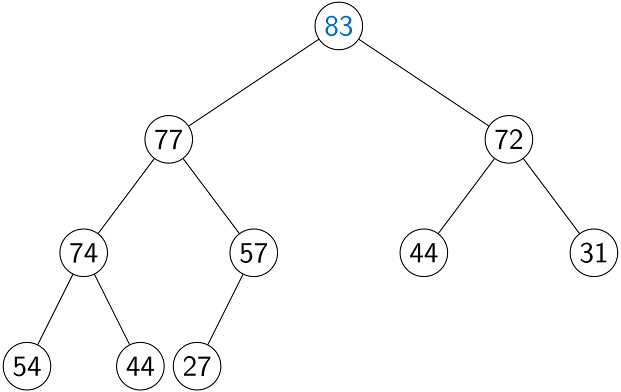


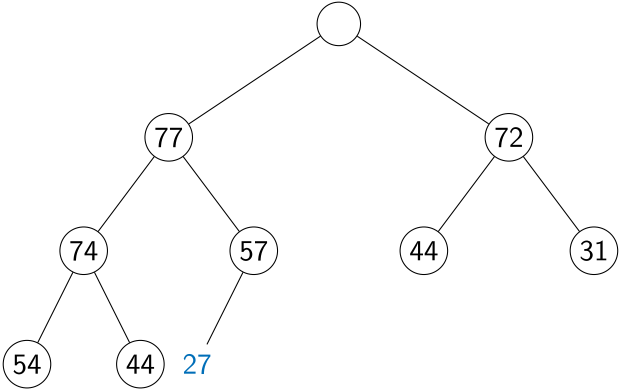
delete_max()

- Maximum value is always at the root



delete_max()

- Maximum value is always at the root
- After we delete one value, tree shrinks
  - Node to delete is rightmost at lowest level



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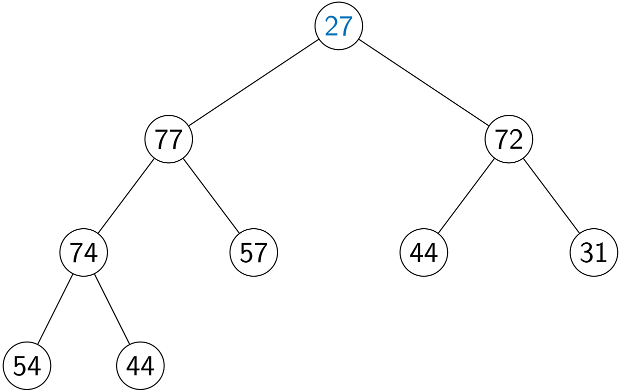
- Move "homeless" value to the root



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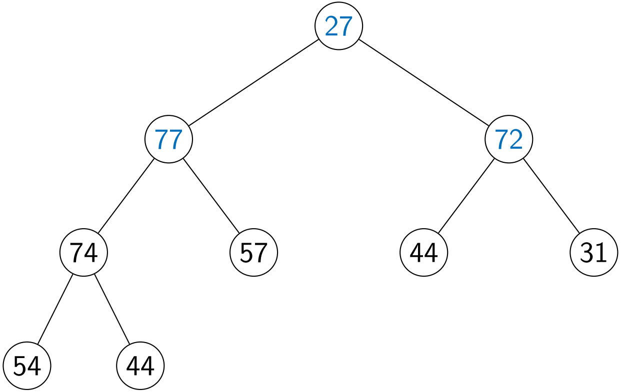
- Restore the heap property downwards



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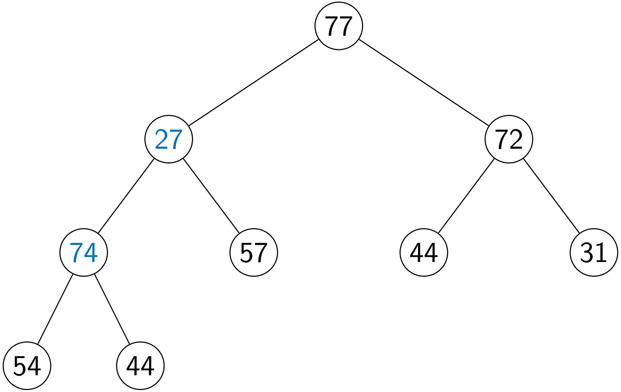

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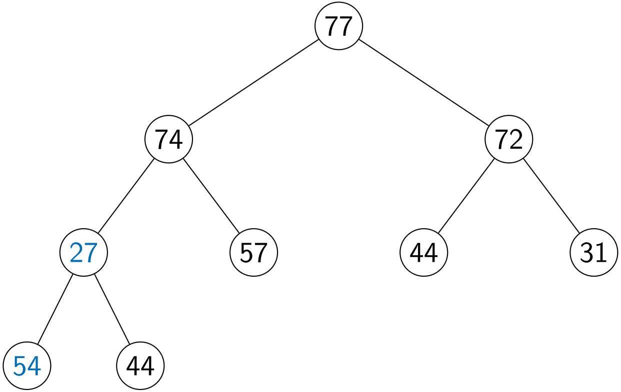

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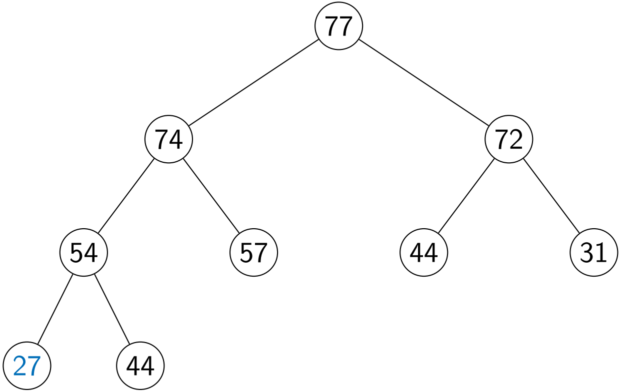
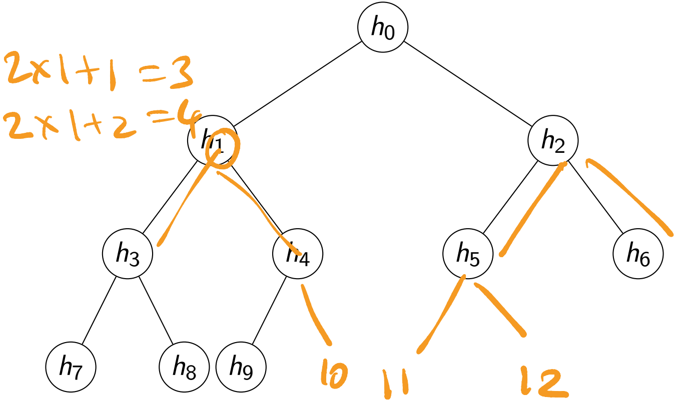


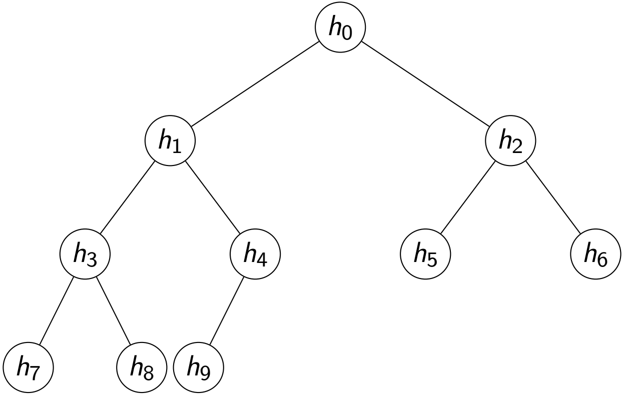
Implementation

- Number the nodes top to bottom left right
- Store as a list
  `H = [h0,h1,h2,...,h9]`
- Children of `H[i]` are at
  `H[2*i+1]`, `H[2*i+2]`
- Parent of `H[i]` is at
  `H[(i-1)//2]`,
  for `i > 0`



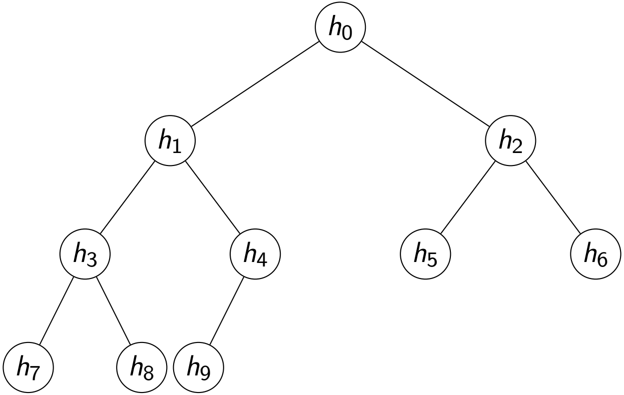
Building a heap — `heapify()`

- Convert a list `[v0,v1,...,vN]` into a heap



Building a heap — `heapify()`

- Convert a list `[v0,v1,...,vN]` into a heap
- Simple strategy
  - Start with an empty heap
  - Repeatedly apply `insert(vj)`
  - Total time is $O(N \log N)$



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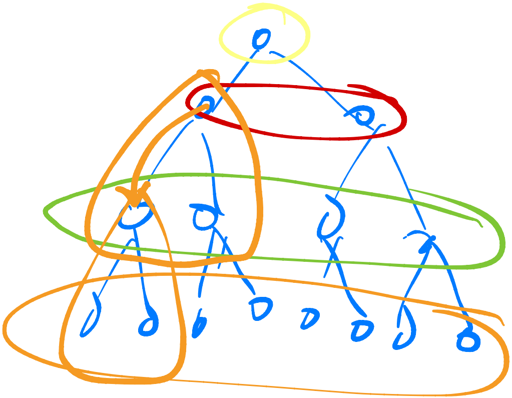
- `mid = len(L)//2`,
  Slice `L[mid:]` has only leaf nodes
  - Already satisfy heap condition



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- Cost turns out to be $O(n)$

Heap sort

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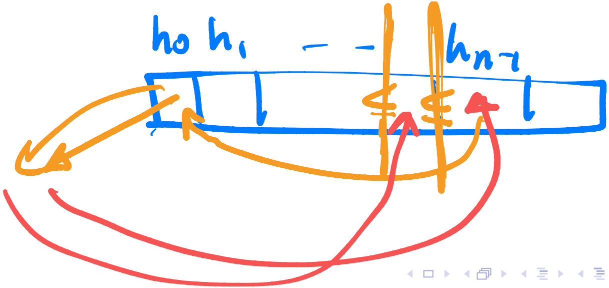
- In place $O(n \log n)$ sort



Summary

- Heaps are a tree implementation of priority queues
  - `insert()` is $O(\log N)$
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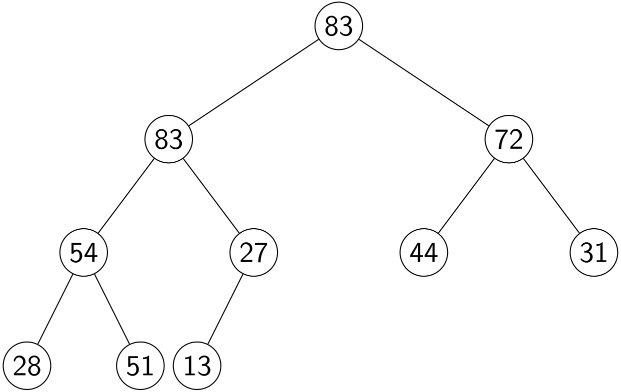

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  - `insert()` is $O(\log N)$
  - `delete_max()` is $O(\log N)$
  - `heapify()` builds a heap in $O(N)$
- Can invert the heap condition
  - Each node is smaller than its children
  - min-heap
  - `delete_min()` rather than `delete_max()`