Dynamic Programming

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Programming and Data Structures with Python

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Memoizing recursive implementations

```
def fib(n):
  if n in fibtable.keys():
    return(fibtable[n])
  if n <= 1:
    value = n
  else:
    value = fib(n-1) + fib(n-2)
  fibtable[n] = value
  return(value)
```

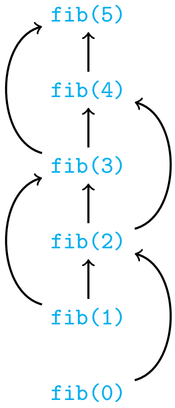
In general

```
def f(x,y,z):
  if (x,y,z) in ftable.keys():
    return(ftable[(x,y,z)])
  recursively compute value
    from subproblems
  ftable[(x,y,z)] = value
  return(value)
```

Dynamic programming

- Anticipate the structure of subproblems
  - Derive from inductive definition
  - Dependencies are acyclic

Evaluating `fib(5)`



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  - Fill table iteratively
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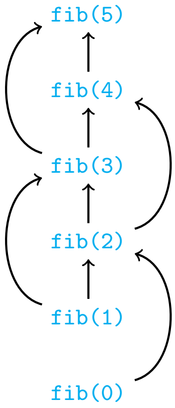

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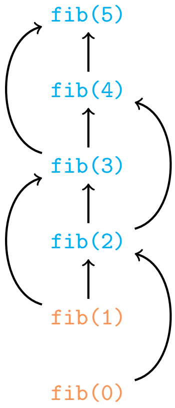

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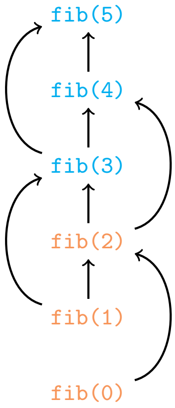

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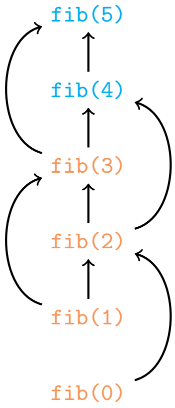

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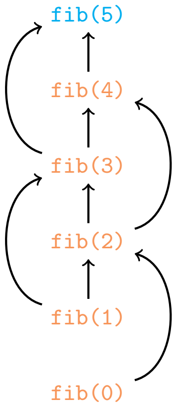

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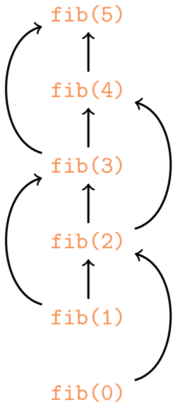

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- In our example, 24 characters inserted, 18 ~~deleted~~, 2 substituted

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Edit distance

- Minimum number of edit operations needed
- In our example, 24 characters inserted, 18 ~~deleted~~, 2 substituted
- Edit distance is at most 44

Edit distance

- Minimum number of editing operations needed to transform one document to the other
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- From LCS, we can compute edit distance without substitution

Inductive structure for edit distance

- $u = a_0 a_1 \ldots a_{m-1}$
- $v = b_0 b_1 \ldots b_{n-1}$

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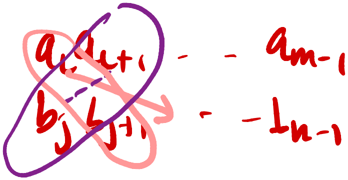

- Recall LCS
- If $a_i = b_j$,
  $LCS(i,j) = 1 + LCS(i{+}1, j{+}1)$
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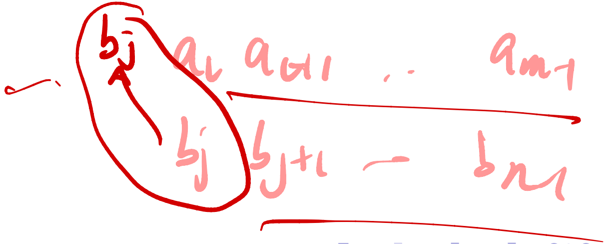

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- Base cases
  - $ED(m,n) = 0$
  - $ED(i,n) = m - i$ for all $0 \leq i \leq m$
    Delete $a_i a_{i+1} \dots a_{m-1}$ from u

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  - $ED(m,j) = n - j$ for all $0 \leq j \leq n$
    Insert $b_j b_{j+1} \ldots b_{n-1}$ into u

Subproblem dependency

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- Table of $(m+1) \cdot (n+1)$ values

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							
1	i							
2	s							
3	e							
4	c							
5	t							
6	•							

Subproblem dependency

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- No dependency for $ED(m,n)$ — start at bottom right and fill by row, column or diagonal

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b					**4**	**5**	**6**
1	i					**3**	**4**	**5**
2	s					**2**	**3**	**4**
3	e					**1**	**2**	**3**
4	c					**1**	**1**	**2**
5	t					**1**	**0**	**1**
6	•					**2**	**1**	**0**

Subproblem dependency

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b				4	4	5	6
1	i				3	3	4	5
2	s				2	2	3	4
3	e				2	1	2	3
4	c				2	1	1	2
5	t				2	1	0	1
6	•				3	2	1	0

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
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1	i			**3**	**3**	**3**	**4**	**5**
2	s			**3**	**2**	**2**	**3**	**4**
3	e			**3**	**2**	**1**	**2**	**3**
4	c			**2**	**2**	**1**	**1**	**2**
5	t			**3**	**2**	**1**	**0**	**1**
6	•			**4**	**3**	**2**	**1**	**0**

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b		4	4	4	4	5	6
1	i		4	3	3	3	4	5
2	s		3	3	2	2	3	4
3	e		2	3	2	1	2	3
4	c		3	2	2	1	1	2
5	t		4	3	2	1	0	1
6	•		5	4	3	2	1	0

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	4	4	4	4	4	5	6
1	i	3	4	3	3	3	4	5
2	s	2	3	3	2	2	3	4
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Reading off the solution

- Transform `bisect` to `secret`

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	4	4	4	4	4	5	6
1	i	3	4	3	3	3	4	5
2	s	2	3	3	2	2	3	4
3	e	3	2	3	2	1	2	3
4	c	4	3	2	2	1	1	2
5	t	5	4	3	2	1	0	1
6	•	6	5	4	3	2	1	0

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Reading off the solution

- Transform `bisect` to `secret`
- Delete `b`

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	4	4	4	4	4	5	6
1	i	3	4	3	3	3	4	5
2	s	2	3	3	2	2	3	4
3	e	3	2	3	2	1	2	3
4	c	4	3	2	2	1	1	2
5	t	5	4	3	2	1	0	1
6	•	6	5	4	3	2	1	0

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Reading off the solution

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- Delete `b` , Delete `i`

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	4	4	4	4	4	5	6
1	i	3	4	3	3	3	4	5
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Reading off the solution

- Transform `bisect` to `secret`
- Delete `b` , Delete `i` , Insert `r`

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	4	4	4	4	4	5	6
1	i	3	4	3	3	3	4	5
2	s	2	3	3	2	2	3	4
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		s	e	c	r	e	t	•
0	b	4	4	4	4	4	5	6
1	i	3	4	3	3	3	4	5
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Implementation

```
def ED(u,v):
  import numpy as np
  (m,n) = (len(u),len(v))
  ed = np.zeros((m+1,n+1))

  for i in range(m-1,-1,-1):
    ed[i,n] = m-i
  for j in range(n-1,-1,-1):
    ed[m,j] = n-j

  for j in range(n-1,-1,-1):
    for i in range(m-1,-1,-1):
      if u[i] == v[j]:
        ed[i,j] = ed[i+1,j+1]
      else:
        ed[i,j] = 1 + min(ed[i+1,j+1],
                          ed[i,j+1],
                          ed[i+1,j])
  return(ed[0,0])
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Complexity

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Complexity

- Again $O(mn)$, using dynamic programming or memoization

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Complexity

- Again $O(mn)$, using dynamic programming or memoization
  - Fill a table of size $O(mn)$
  - Each table entry takes constant time to compute

Multiplying matrices

- Multiply matrices A, B
  - $AB[i,j] = \sum_{k=0}^{n-1} A[i,k]B[k,j]$

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- Let $A : 1 \times 100$, $B : 100 \times 1$, $C : 1 \times 100$

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- Computing $A(BC)$
  - $BC : 100 \times 100$, takes $100 \cdot 1 \cdot 100 = 10000$ steps to compute

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- Computing $(AB)C$

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- Given n matrices $M_0 : r_0 \times c_0$, $M_1 : r_1 \times c_1$, . . . , $M_{n-1} : r_{n-1} \times c_{n-1}$

Multiplying matrices

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  - Dimensions match: $r_j = c_{j-1}$, $0 < j < n$

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  - Dimensions match: $r_j = c_{j-1}$, $0 < j < n$
  - Product $M_0 \cdot M_1 \cdots M_{n-1}$ can be computed

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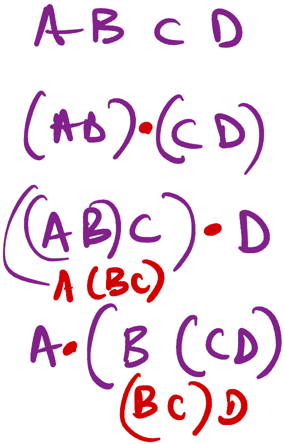
  - Bracketing does not change answer
  - ... but can affect the complexity!

- Given n matrices $M_0 : r_0 \times c_0$, $M_1 : r_1 \times c_1$, ..., $M_{n-1} : r_{n-1} \times c_{n-1}$
  - Dimensions match: $r_j = c_{j-1}$, $0 < j < n$
  - Product $M_0 \cdot M_1 \cdots M_{n-1}$ can be computed
- Find an optimal order to compute the product
  - Multiply two matrices at a time
  - Bracket the expression optimally

Inductive structure

- Final step combines two subproducts $(M_0 \cdot M_1 \cdots M_{k-1}) \cdot (M_k \cdot M_{k+1} \cdots M_{n-1})$ for some $0 < k < n$



Inductive structure

- Final step combines two subproducts
  $(M_0 \cdot M_1 \cdots M_{k-1}) \cdot (M_k \cdot M_{k+1} \cdots M_{n-1})$
  for some $0 < k < n$
- First factor is $r_0 \times c_{k-1}$, second is
  $r_k \times c_{n-1}$, where $r_k = c_{k-1}$

Inductive structure

- Final step combines two subproducts
  r_0 $(M_0 \cdot M_1 \cdots M_{k-1}) \cdot (M_k \cdot M_{k+1} \cdots M_{n-1})$ c_{n-1}
  for some $0 < k < n$ $c_{k-1} = r_k$
- First factor is $r_0 \times c_{k-1}$, second is $r_k \times c_{n-1}$, where $r_k = c_{k-1}$
- Let $C(0, n-1)$ denote the overall cost

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- $C(0, n-1) = C(0, k-1) + C(k, n-1) + r_0 r_k c_{n-1}$

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- Which k should we choose?

Inductive structure

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  - Try all and choose the minimum!
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  - $M_0 \cdot M_1 \cdots M_{k-1}$ would decompose as $(M_0 \cdots M_{j-1}) \cdot (M_j \cdots M_{k-1})$
  - Generic subproblem is $M_j \cdot M_{j+1} \cdots M_k$

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- $C(j,k) = \min_{j<\ell\leq k} [C(j, \ell{-}1) + C(\ell, k) + r_j r_\ell c_k]$

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- $C(j, k) =$
  $\min_{j < \ell \leq k} [C(j, \ell{-}1) + C(\ell, k) + r_j r_\ell c_k]$
- Base case: $C(j, j) = 0$ for $0 \leq j < n$

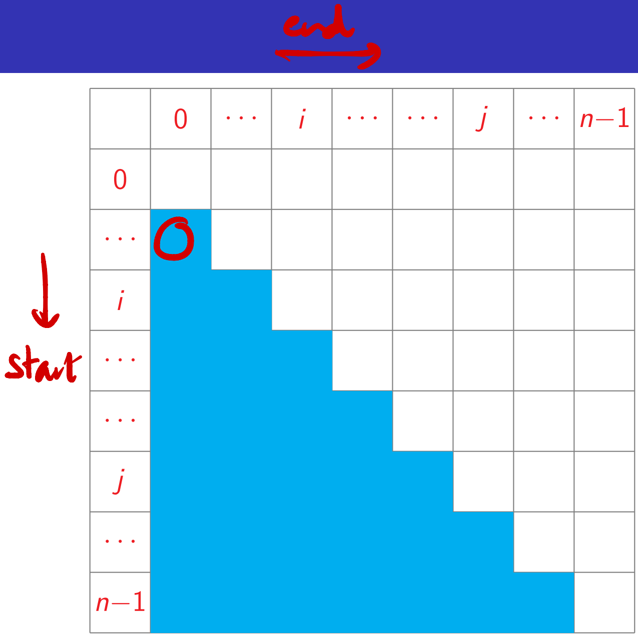
Subproblem dependency

- Compute $C(i,j)$, $0 \leq i,j < n$

	0	$\cdots$	i	$\cdots$	$\cdots$	j	$\cdots$	$n-1$
0								
$\cdots$								
i								
$\cdots$								
$\cdots$								
j								
$\cdots$								
$n-1$								

Subproblem dependency

- Compute $C(i, j)$, $0 \leq i, j < n$
  - Only for $i \leq j$
  - Entries above main diagonal



Subproblem dependency

- Compute $C(i,j)$, $0 \leq i,j < n$
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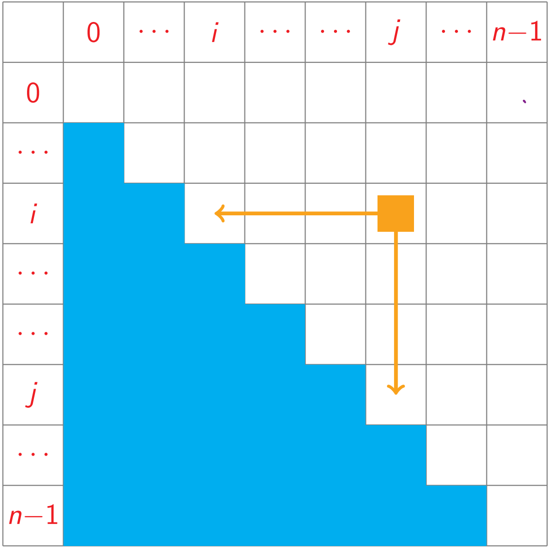
- $C(i,j)$ depends on $C(i,k-1)$, $C(k,j)$ for every $i < k \leq j$



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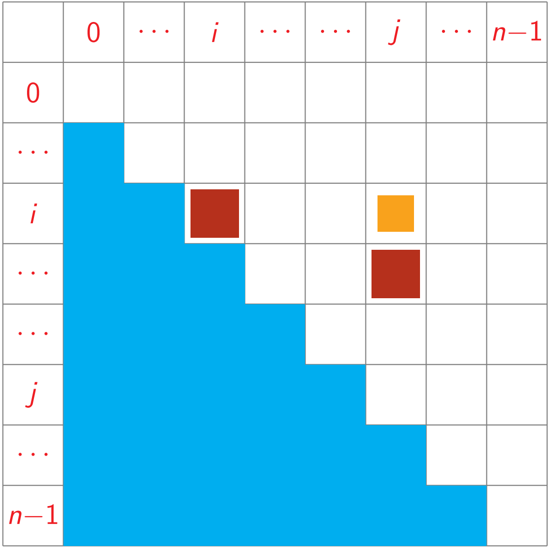

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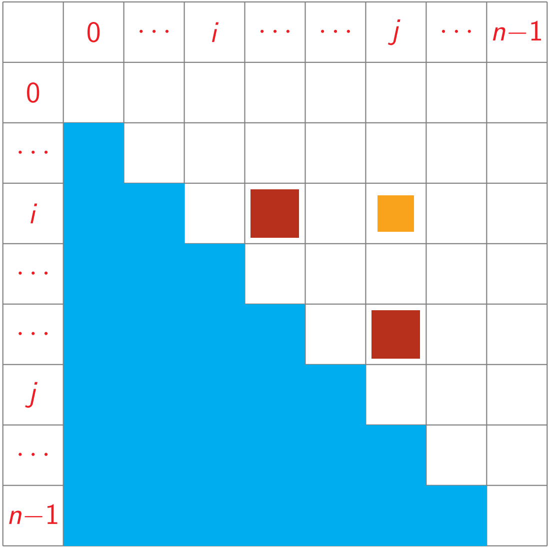

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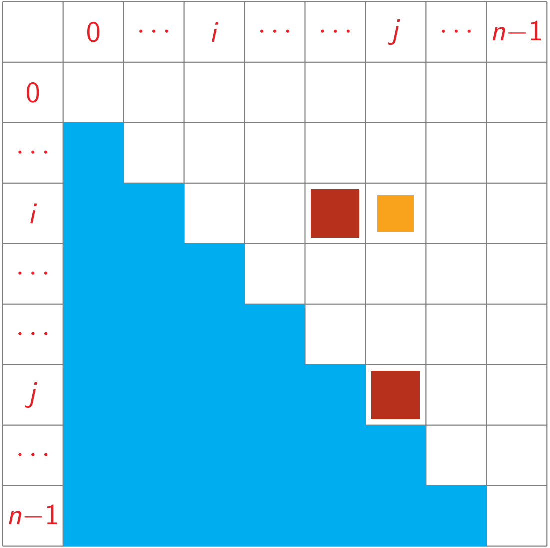

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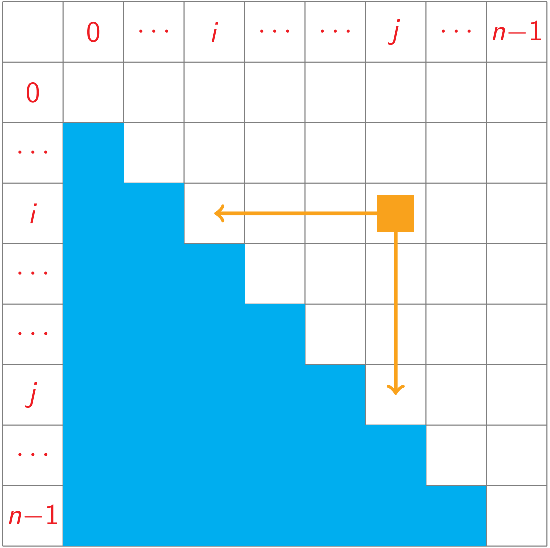
  - $O(n)$ dependencies per entry, unlike LCW, LCS and ED



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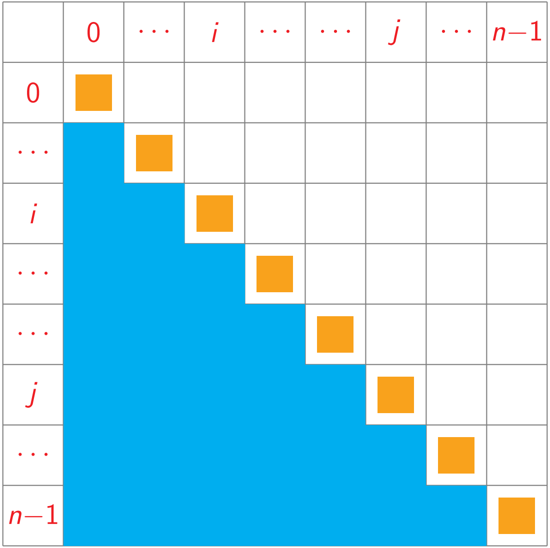

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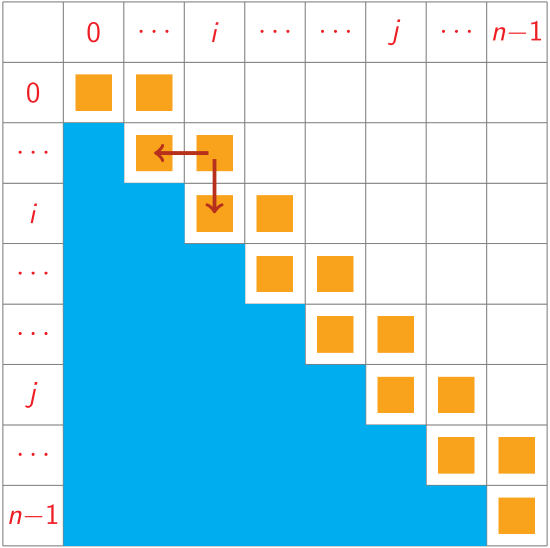
- Diagonal entries are base case
- Fill matrix by diagonal, from main diagonal



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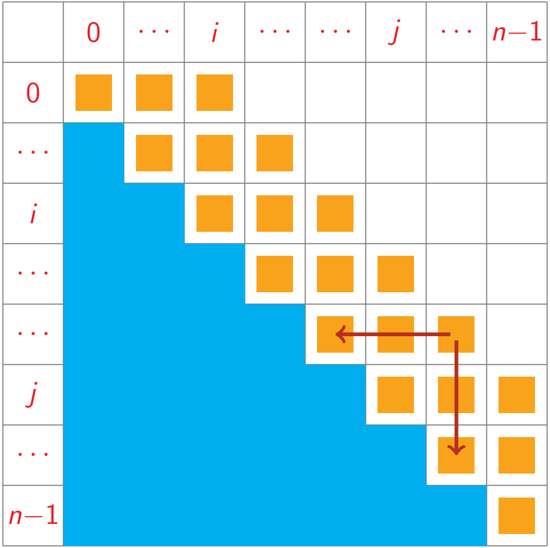

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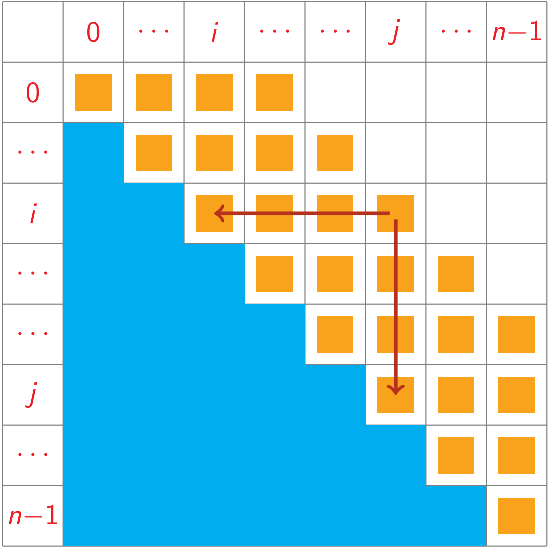

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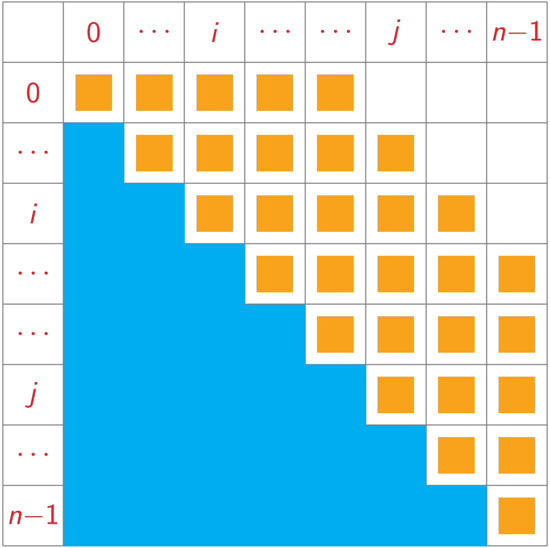

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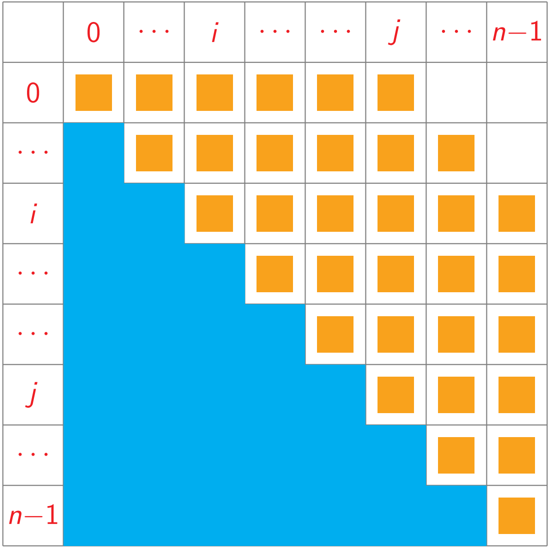

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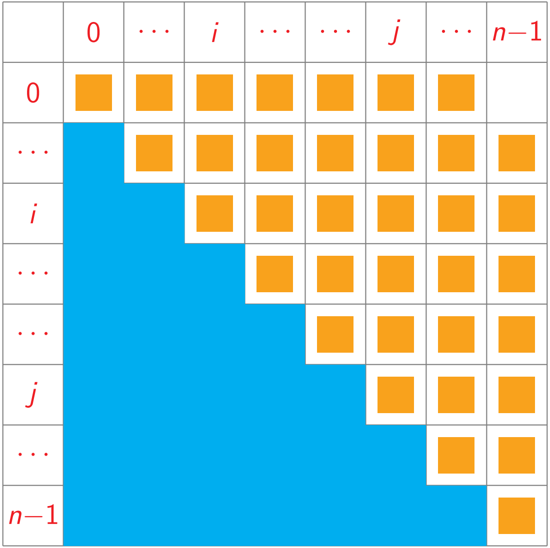

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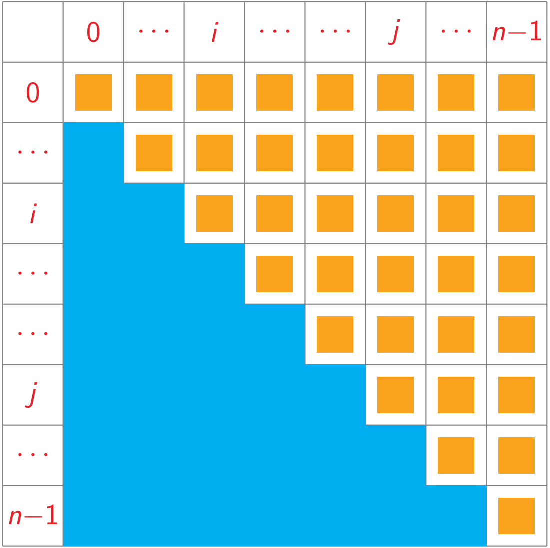

Implementation

```
def C(dim):
  # dim: dimension matrix,
  #      entries are pairs (r_i,c_i)
  import numpy as np
  n = dim.shape[0]
  C = np.zeros((n,n))
  for i in range(n):
    C[i,i] = 0
  for diff in range(1,n):
    for i in range(0,n-diff):
      j = i + diff
      C[i,j] = C[i,i] +
               C[i+1,j] +
               dim[i][0]*dim[i+1][0]*dim[j][1]
      for k in range(i+1,j+1):
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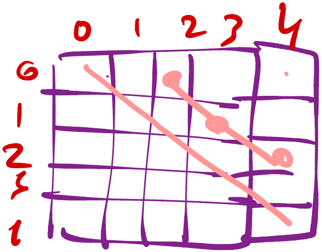
  return(C[0,n-1])
```



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Complexity

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Complexity

- We have to fill a table of size $O(n^2)$

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Complexity

- We have to fill a table of size $O(n^2)$
- Filling each entry takes $O(n)$

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```

$n^2(1 + - n)$

Complexity

- We have to fill a table of size $O(n^2)$
- Filling each entry takes $O(n)$
- Overall, $O(n^3)$