Analysis of algorithms

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Programming and Data Structures with Python

Lecture 15, 6 Oct 2022

A real world problem

- Every SIM card needs to be linked to an Aadhaar card
- Validate Aadhaar number for each SIM card

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- What are M and N
  - Almost everyone in India has an Aadhaar card: $N > 10^9$
  - Number of SIM cards registered is similar: $M > 10^9$

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- How can we fix this?

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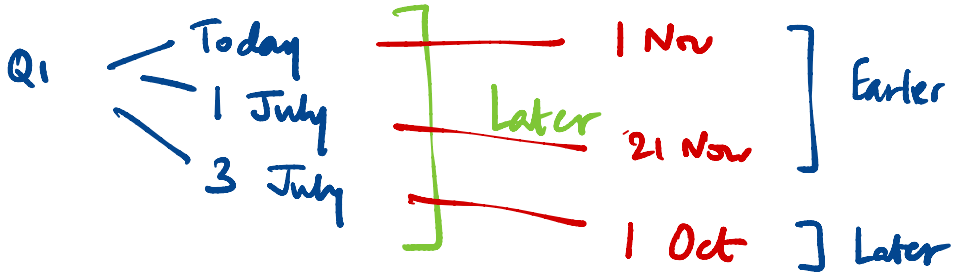
Guess ~~my birthday~~ my daughter's birthday

- You propose a date
- I answer, *Yes*, *Earlier*, *Later*



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  - September 12? Earlier
  - February 23? Later
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  - ...

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- Query midpoint — halves the interval
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  - March 31? Later
  - May 15? Earlier
  - April 22? Earlier
  - April 11? Later
  - April 16? Earlier
  - April 13? Earlier
  - April 12? Yes

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- Interval shrinks from $365 \rightarrow 182 \rightarrow 91 \rightarrow 45 \rightarrow 22 \rightarrow 11 \rightarrow 5 \rightarrow 2 \rightarrow 1$
- Under 10 questions

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- Use the halving strategy to check each SIM card

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- Assume Aadhaar details are sorted by Aadhaar number
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- Halving 10 times reduces the interval by a factor of 1000, because $2^{10} = 1024$

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- After 20 queries, interval shrinks to 10^3
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$2^{10} = 10^3$

$10^9 = 10^3 \times 10^3 \times 10^3$

$= 2^{10} \times 2^{10} \times 2^{10}$

$= 2^{30}$

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- From 3200 years to 50 minutes!

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- 3000 seconds, or 50 minutes
- From 3200 years to 50 minutes!
- Of course, to achieve this we have to first sort the Aadhaar cards
- Arranging the data results in a much more efficient solution
- Both algorithms and data structures matter

Comparing orders of magnitude

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Orders of magnitude

- When comparing $t(n)$, focus on orders of magnitude
  - Ignore constant factors
- $f(n) = n^3$ eventually grows faster than $g(n) = 5000n^2$

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- When comparing $t(n)$, focus on orders of magnitude
  - Ignore constant factors
- $f(n) = n^3$ eventually grows faster than $g(n) = 5000n^2$
- How do we compare functions with respect to orders of magnitude?

Upper bounds

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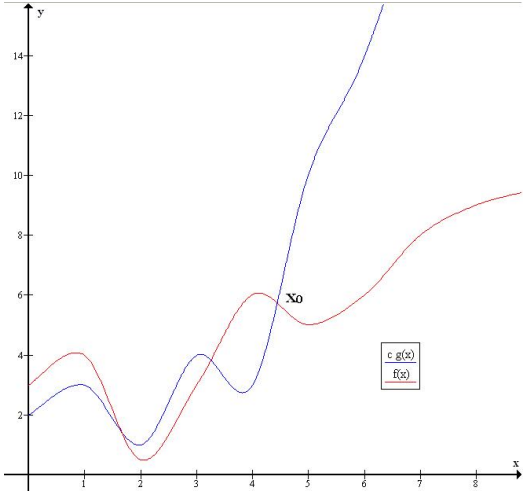

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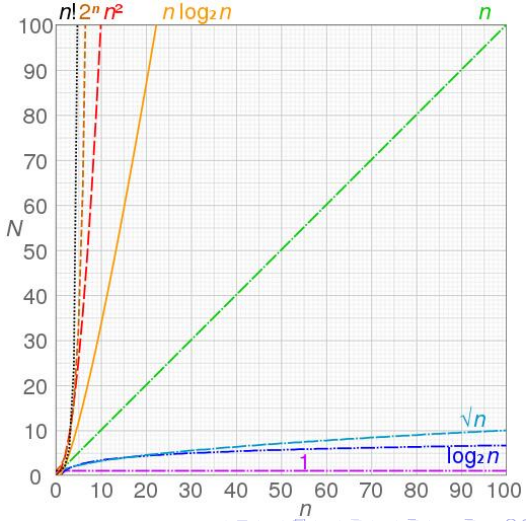
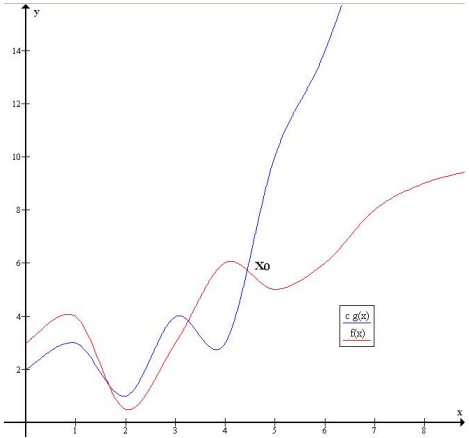
- Graphs of typical functions we have seen



Examples

- $100n + 5$ is $O(n^2)$
  - $100n + 5 \leq 100n + n = 101n$, for $n \geq 5$
  - $101n \leq 101n^2$
  - Choose $n_0 = 5$, $c = 101$



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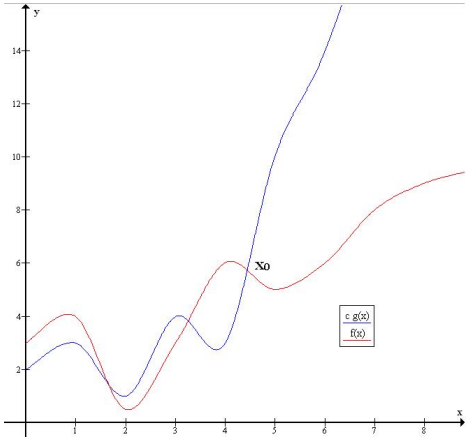
- Alternatively
  - $100n + 5 \leq 100n + 5n = 105n$, for $n \geq 1$
  - $105n \leq 105n^2$
  - Choose $n_0 = 1$, $c = 105$



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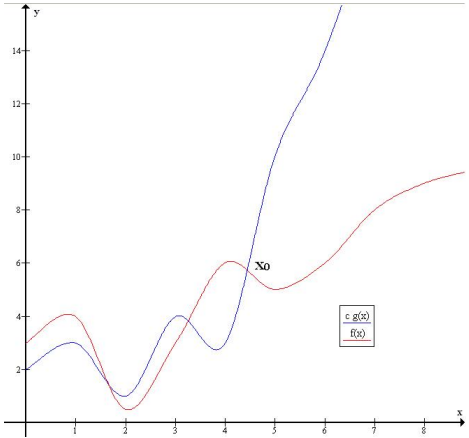
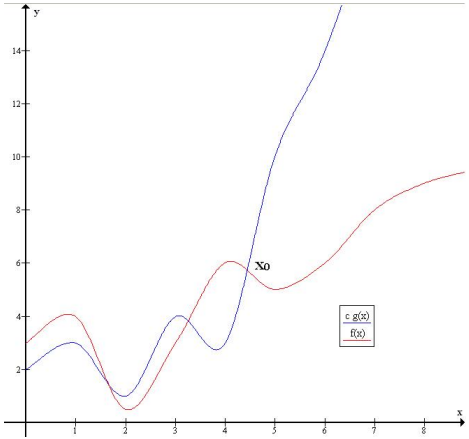
  - Choose $n_0 = 1$, $c = 105$
- Choose of n_0, c not unique



Examples . . .

- $100n^2 + 20n + 5$ is $O(n^2)$
  - $100n^2 + 20n + 5 \leq 100n^2 + 20n^2 + 5n^2$, for $n \geq 1$
  - $100n^2 + 20n + 5 \leq 125n^2$, for $n \geq 1$
  - Choose $n_0 = 1$, $c = 125$



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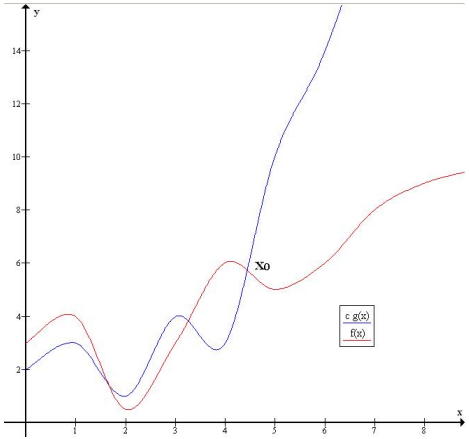
  - Choose $n_0 = 1$, $c = 125$
- What matters is the highest term
  - $20n + 5$ is dominated by $100n^2$



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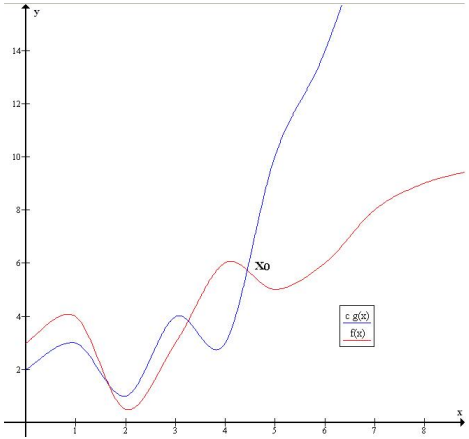
  - Choose $n_0 = 1$, $c = 125$
- What matters is the highest term
  - $20n + 5$ is dominated by $100n^2$
- n^3 is not $O(n^2)$
  - No matter what c we choose, cn^2 will be dominated by n^3 for $n \geq c$



Useful properties

- If $f_1(n)$ is $O(g_1(n))$ and $f_2(n)$ is $O(g_2(n))$, then $f_1(n) + f_2(n)$ is $O(\max(g_1(n), g_2(n)))$

$m + n$

$m > n$

$\leq 2m$

$n > m$

$\leq 2n$

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- Algorithm as a whole takes time $\max(O(g_A(n), g_B(n)))$
- Least efficient phase is the upper bound for the whole algorithm

Lower bounds

- $f(x)$ is said to be $\Omega(g(x))$ if we can find constants c and x_0 such that $cg(x)$ is a lower bound for $f(x)$ for x beyond x_0
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- n^3 is $\Omega(n^2)$
  - $n^3 > n^2$ for all n, so $n_0 = 1$, $c = 1$
- Typically we establish lower bounds for a problem rather than an individual algorithm
  - If we sort a list by comparing elements and swapping them, we require $\Omega(n \log n)$ comparisons
  - This is independent of the algorithm we use for sorting

Tight bounds

- $f(x)$ is said to be $\Theta(g(x))$ if it is both $O(g(x))$ and $\Omega(g(x))$
  - Find constants c_1, c_2, x_0 such that $c_1 g(x) \leq f(x) \leq c_2 g(x)$ for every $x \geq x_0$

Tight bounds

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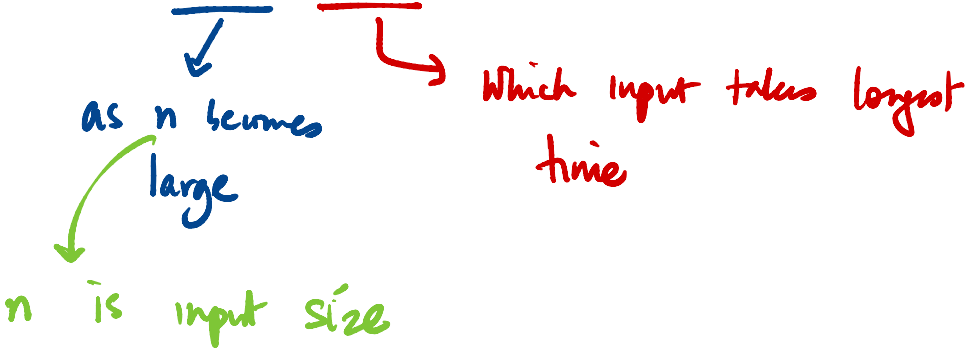
  - Choose $n_0 = 2$, $c_1 = 1/4$, $c_2 = 1/2$

Summary

- $f(n)$ is $O(g(n))$ means $g(n)$ is an upper bound for $f(n)$
  - Useful to describe asymptotic worst case running time



Multiply $a \times b$ What is input size?

Cost grows with values of a & b

$2^x = 10^y$

$x \log_2 2 = y \log_2 10$

$x = y \cdot \frac{\log_2 10}{\log_2 2}$

3 5 7 8

x 7 6 4 3 9

add ↓ x x x

N. of digit

$\log_{10} n$

$= c \cdot \log_2 n$

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  - Typically used for a problem as a whole, rather than an individual algorihm
- $f(n)$ is $\Theta(g(n))$: matching upper and lower bounds
  - We have found an optimal algorithm for a problem