Dynamic Programming

Madhavan Mukund

https://www.cmi.ac.in/~madhavan

Programming and Data Structures with Python

Lecture 22, 03 Nov 2022

Memoizing recursive implementations

```
def fib(n):
  if n in fibtable.keys():
    return(fibtable[n])
  if n <= 1:
    value = n
  else:
    value = fib(n-1) + fib(n-2)
  fibtable[n] = value
  return(value)
```

In general

```
def f(x,y,z):
  if (x,y,z) in ftable.keys():
    return(ftable[(x,y,z)])
  recursively compute value
    from subproblems
  ftable[(x,y,z)] = value
  return(value)
```

Dynamic programming

- Anticipate the structure of subproblems
  - Derive from inductive definition
  - Dependencies are acyclic

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Evaluating `fib(5)`

`fib(5)`

`fib(4)`

`fib(3)`

`fib(2)`

`fib(1)`

`fib(0)`

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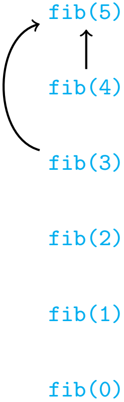

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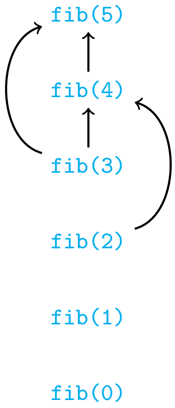

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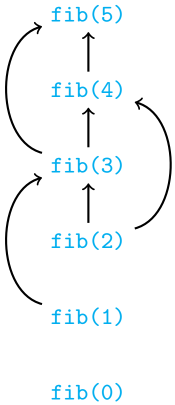

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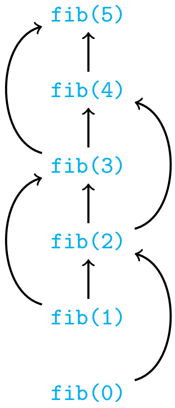

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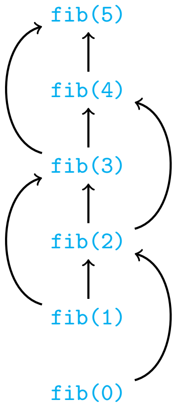

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- Solve subproblems in appropriate order
  - Start with base cases — no dependencies

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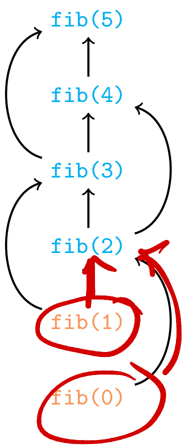

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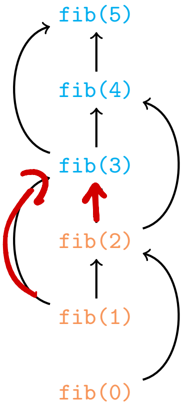

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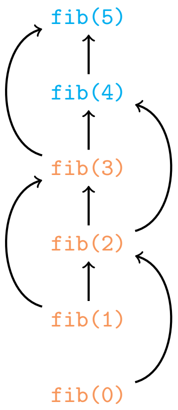

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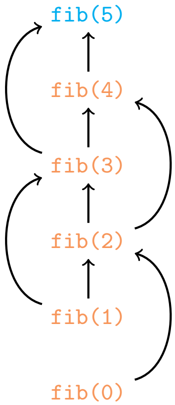

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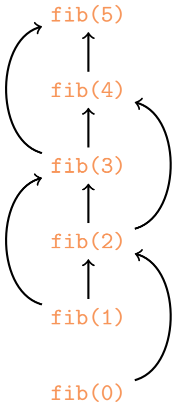
fib(8)

0 1 2 3 4 5 6 7 8

0 1 1 2 3 5 8 13 21

Evaluating `fib(5)`



Dynamic programming

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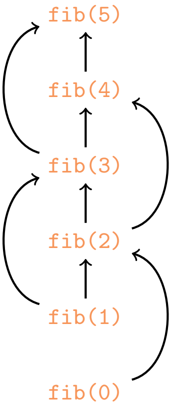
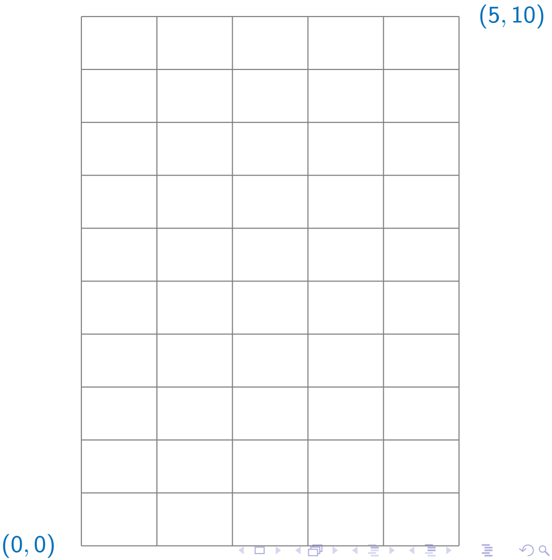
  - Dependencies are acyclic
- Solve subproblems in appropriate order
  - Start with base cases — no dependencies
  - Evaluate a value after all its dependencies are available
  - Fill table iteratively
  - Never need to make a recursive call

Evaluating `fib(5)`



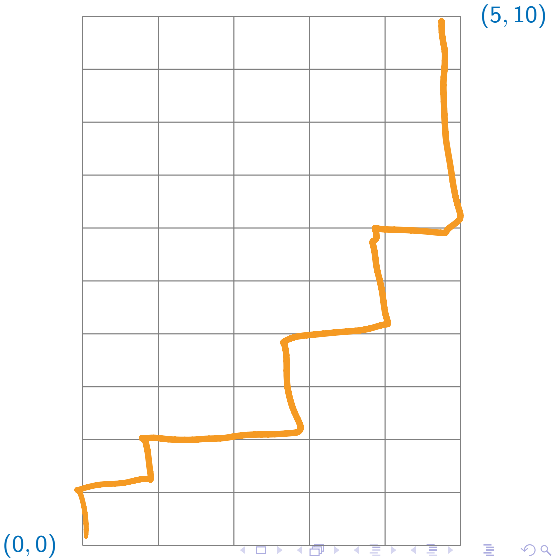
Grid paths

- Rectangular grid of one-way roads



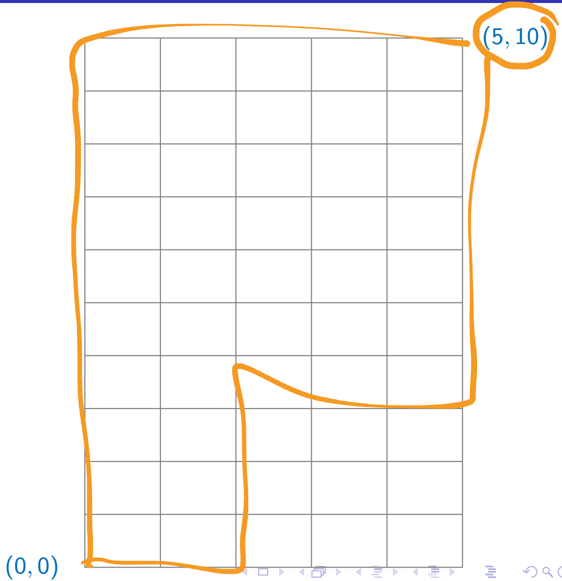
Grid paths

- Rectangular grid of one-way roads
- Can only go up and right



Grid paths

- Rectangular grid of one-way roads
- Can only go up and right
- How many paths from $(0,0)$ to (m,n)?



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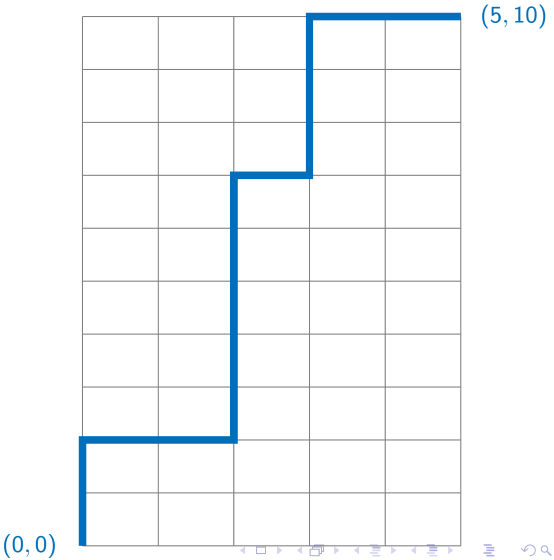

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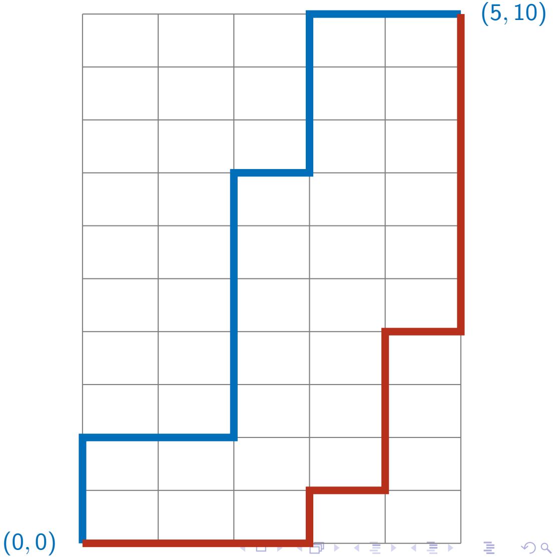

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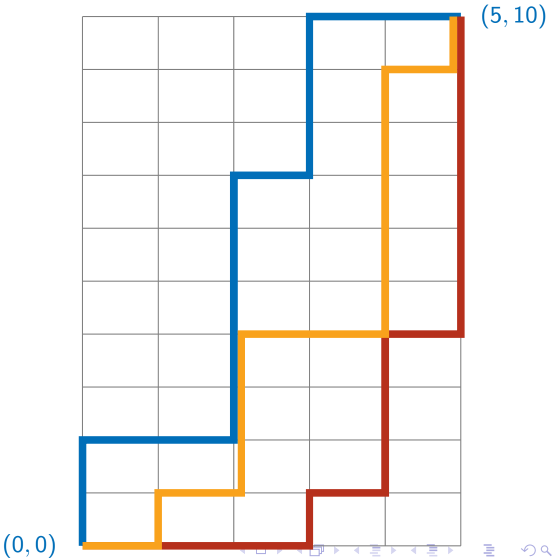
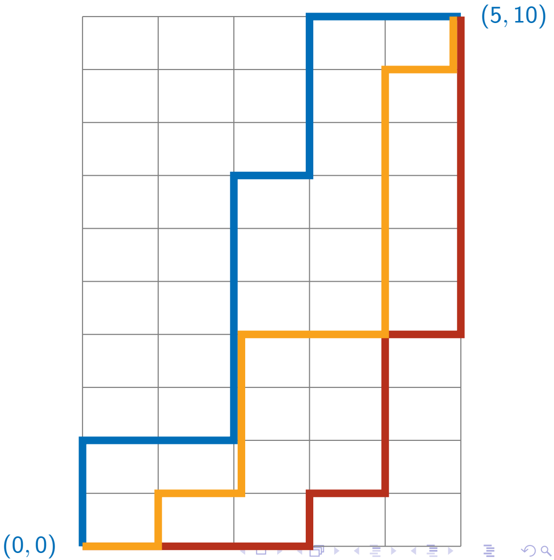


Combinatorial solution

- Every path from $(0,0)$ to $(5,10)$ has 15 segments
  - In general $m+n$ segments from $(0,0)$ to (m,n)



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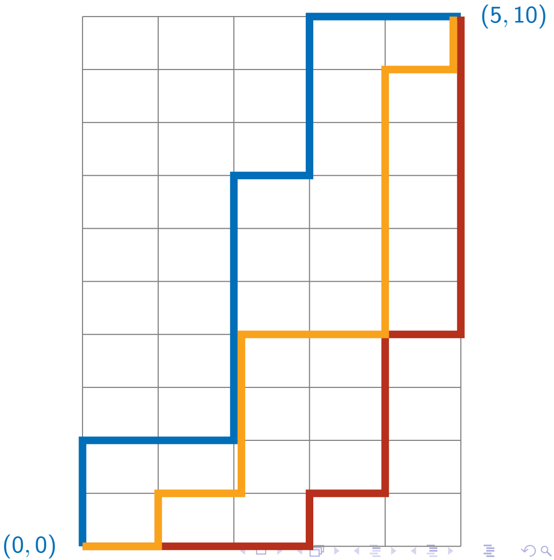
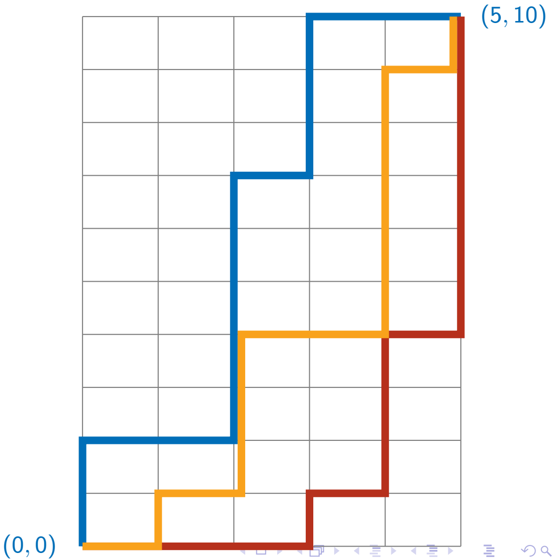
- Out of 15, exactly 5 are right moves, 10 are up moves



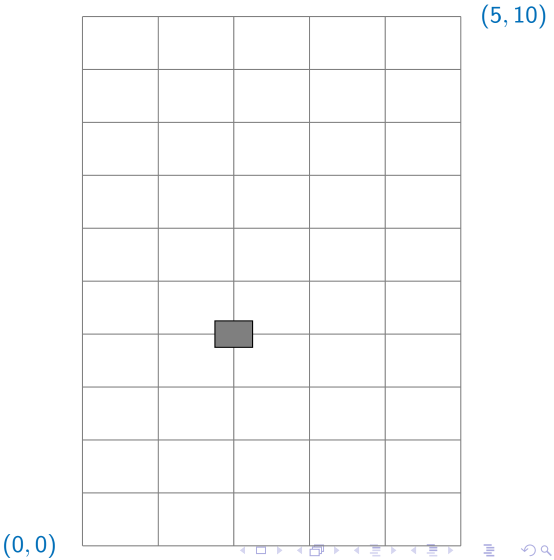
Combinatorial solution

- Every path from $(0,0)$ to $(5,10)$ has 15 segments
  - In general $m+n$ segments from $(0,0)$ to (m,n)
- Out of 15, exactly 5 are right moves, 10 are up moves
- Fix the positions of the 5 right moves among the 15 positions overall
  - $\binom{15}{5} = \frac{15!}{10! \cdot 5!} = 3003$
  - Same as $\binom{15}{10}$ — fix the 10 up moves



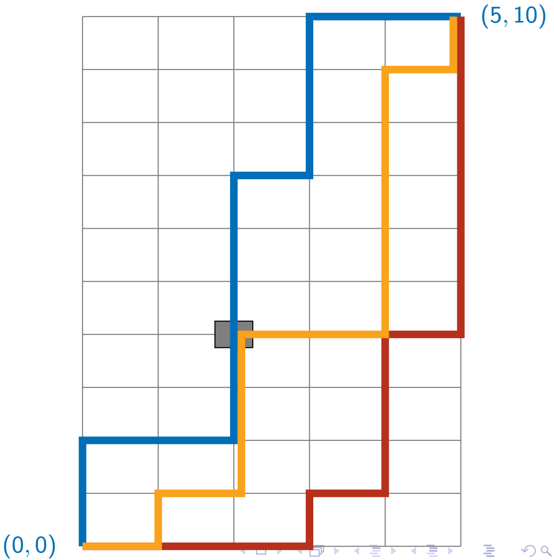
Holes

- What if an intersection is blocked?
  - For instance, $(2, 4)$



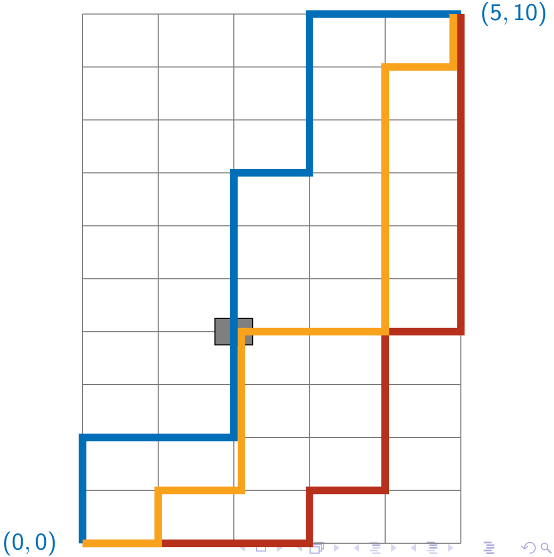
Holes

- What if an intersection is blocked?
  - For instance, $(2,4)$
- Need to discard paths passing through $(2,4)$
  - Two of our earlier examples are invalid paths



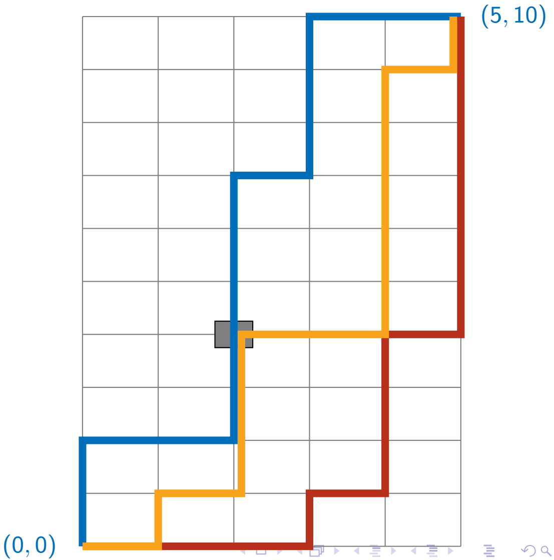
Combinatorial solution for holes

- Discard paths passing through $(2,4)$



Combinatorial solution for holes

- Discard paths passing through $(2,4)$
- Every path via $(2,4)$ combines a path from $(0,0)$ to $(2,4)$ with a path from $(2,4)$ to $(5,10)$
  - Count these separately
  - $\binom{2+4}{2} = 15$ paths $(0,0)$ to $(2,4)$
  - $\binom{3+6}{3} = 84$ paths $(2,4)$ to $(5,10)$



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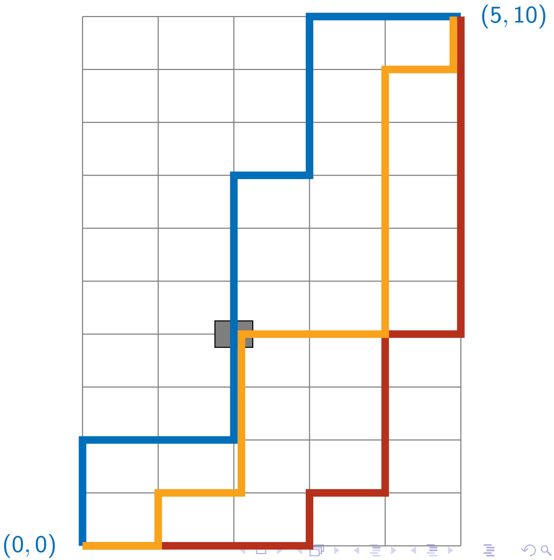
- $15 \times 84 = 1260$ paths via $(2, 4)$



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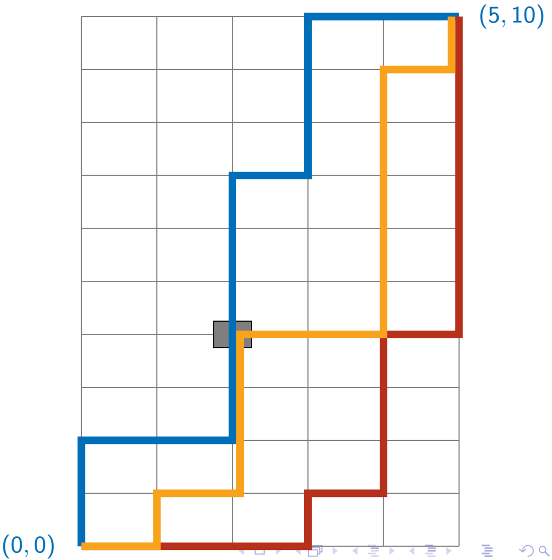
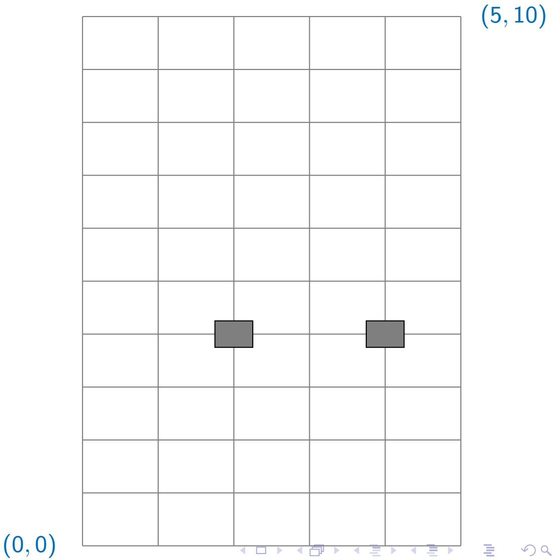
- $15 \times 84 = 1260$ paths via $(2,4)$
- $3003 - 1260 = 1743$ valid paths avoiding $(2,4)$



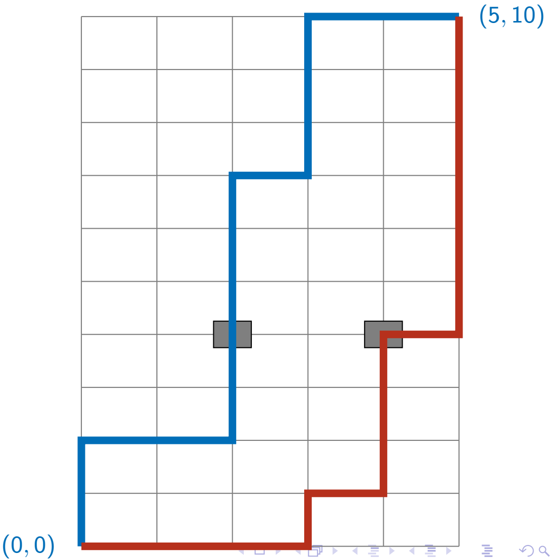
More holes

- What if two intersections are blocked?



More holes

- What if two intersections are blocked?
- Discard paths via $(2,4)$, $(4,4)$



More holes

- What if two intersections are blocked?
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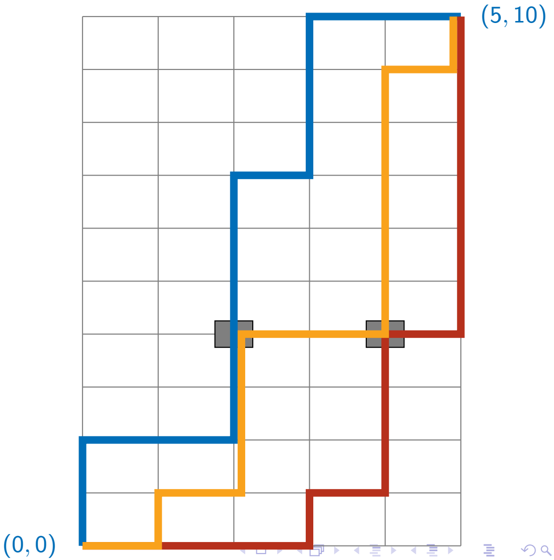

More holes

- What if two intersections are blocked?
- Discard paths via $(2,4)$, $(4,4)$
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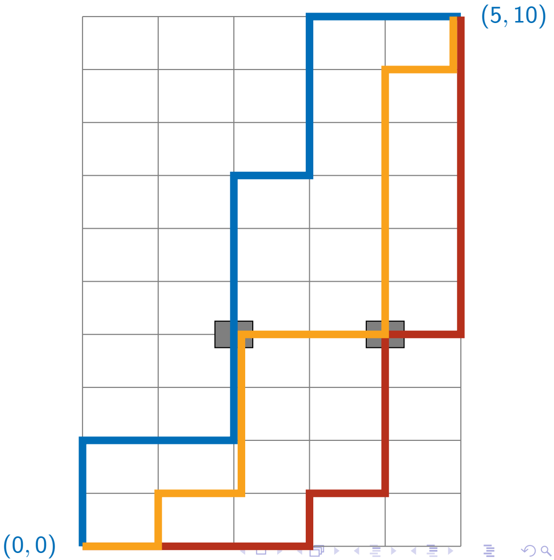
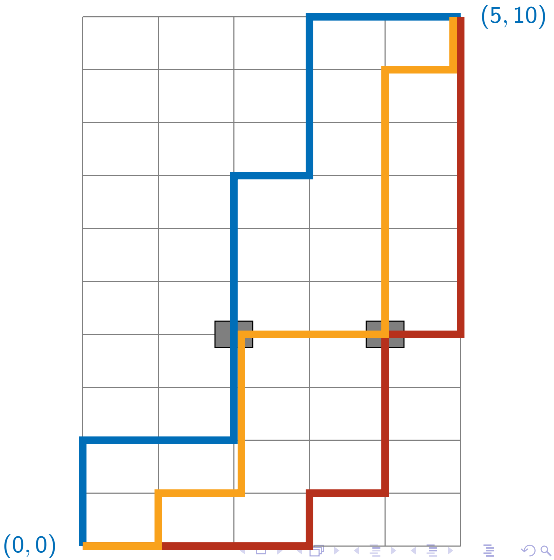
- Add back the paths that pass through both holes



More holes

- What if two intersections are blocked?
- Discard paths via $(2,4)$, $(4,4)$
  - Some paths are counted twice
- Add back the paths that pass through both holes
- Inclusion-exclusion — counting is messy



Inductive formulation

- How can a path reach (i,j)

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(\boldsymbol{i-1,j}) \arrow[r] & (\boldsymbol{i,j}) \\
 & (\boldsymbol{i,j-1}) \arrow[u]
\end{tikzcd}$$

Inductive formulation

- How can a path reach (i, j)
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- Each path to these neighbours extends to a unique path to (i, j)

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  - $P(i,j) = P(i-1,j) + P(i,j-1)$

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  - $P(0,0) = 1$ — base case

$$
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$$\begin{tikzcd}
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- $P(i,j) = 0$ if there is a hole at (i,j)

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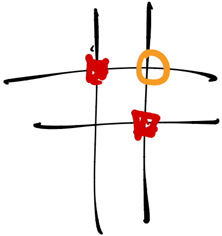
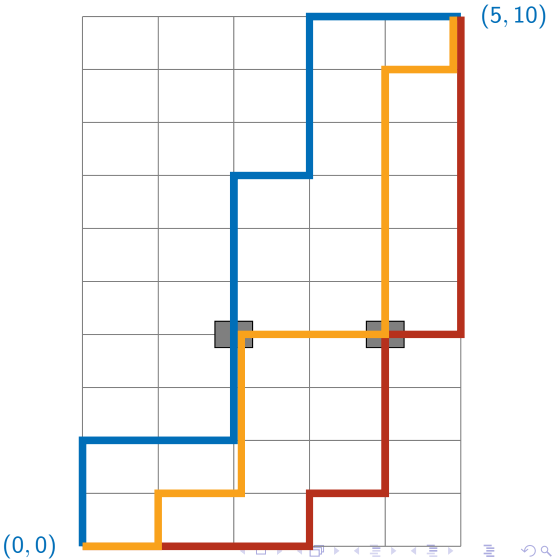


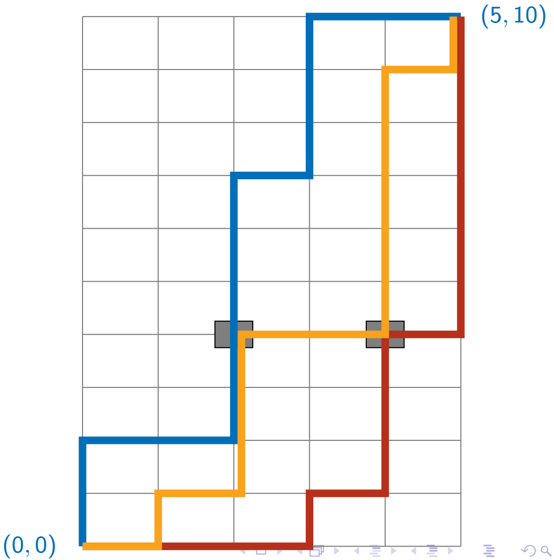
Computing $P(i,j)$

- Naive recursion recomputes same subproblem repeatedly



Computing $P(i, j)$

- Naive recursion recomputes same subproblem repeatedly
  - $P(5, 10)$ requires $P(4, 10)$, $P(5, 9)$



Computing $P(i, j)$

- Naive recursion recomputes same subproblem repeatedly
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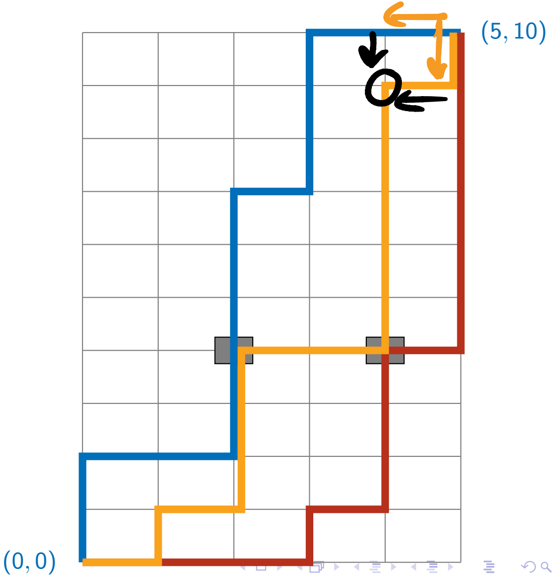
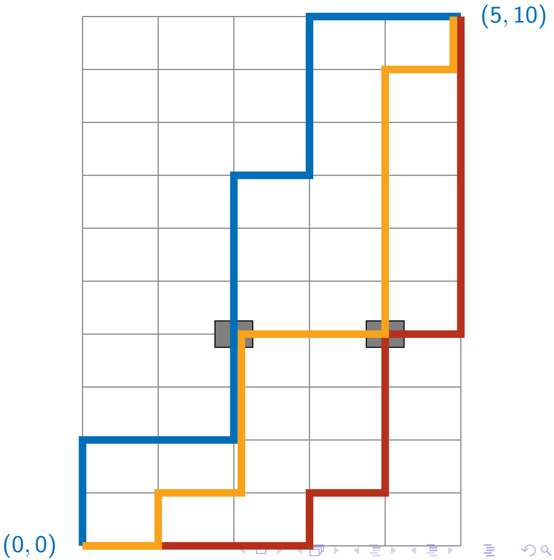
  - Both $P(4, 10)$, $P(5, 9)$ require $P(4, 9)$



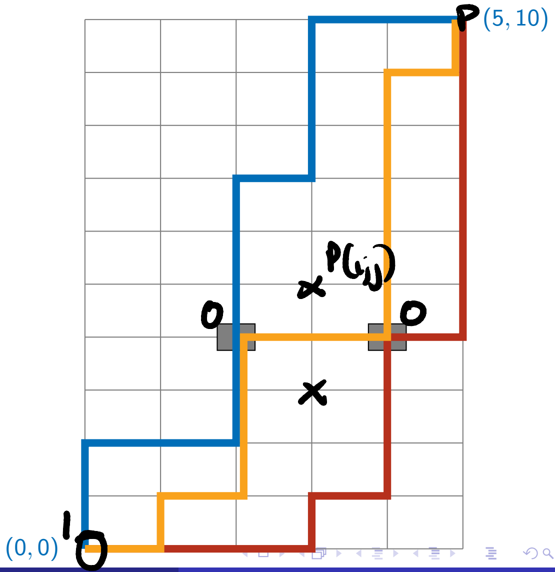
Computing $P(i, j)$

- Naive recursion recomputes same subproblem repeatedly
  - $P(5, 10)$ requires $P(4, 10)$, $P(5, 9)$
  - Both $P(4, 10)$, $P(5, 9)$ require $P(4, 9)$
- Use memoization ...



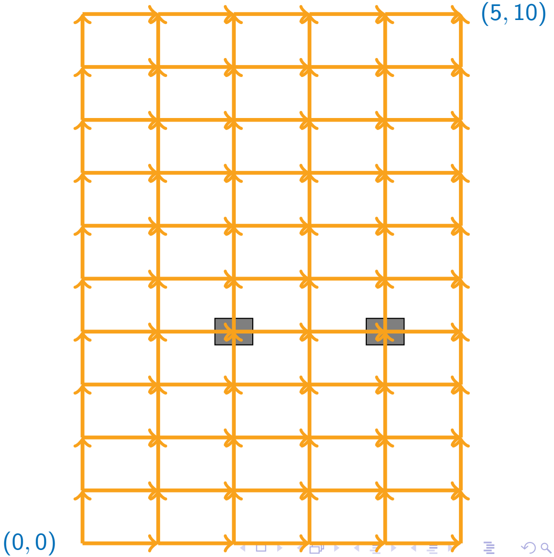
Computing $P(i,j)$

- Naive recursion recomputes same subproblem repeatedly
  - $P(5,10)$ requires $P(4,10)$, $P(5,9)$
  - Both $P(4,10)$, $P(5,9)$ require $P(4,9)$
- Use memoization …
- …or find a suitable order to compute the subproblems



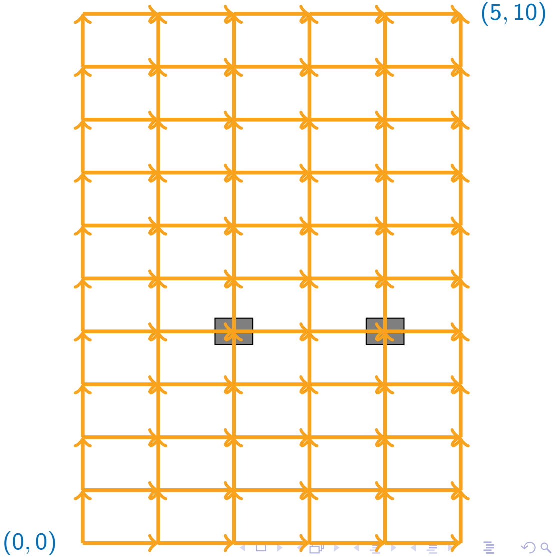
Dynamic programming

- Identify subproblem structure



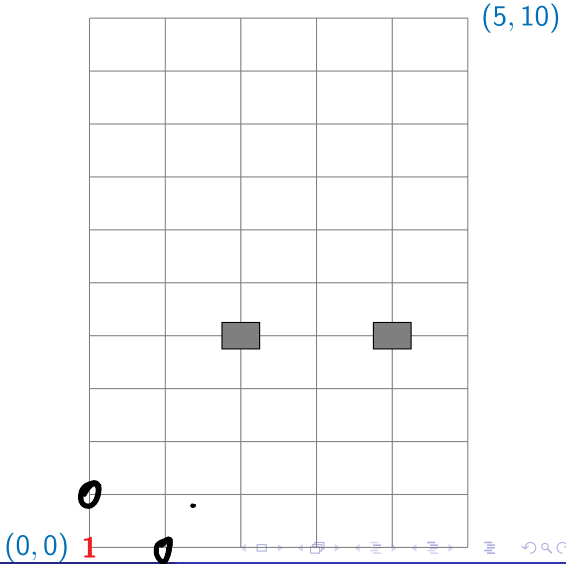
Dynamic programming

- Identify subproblem structure
- $P(0,0)$ has no dependencies



Dynamic programming

- Identify subproblem structure
- $P(0,0)$ has no dependencies
- Start at $(0,0)$



Dynamic programming

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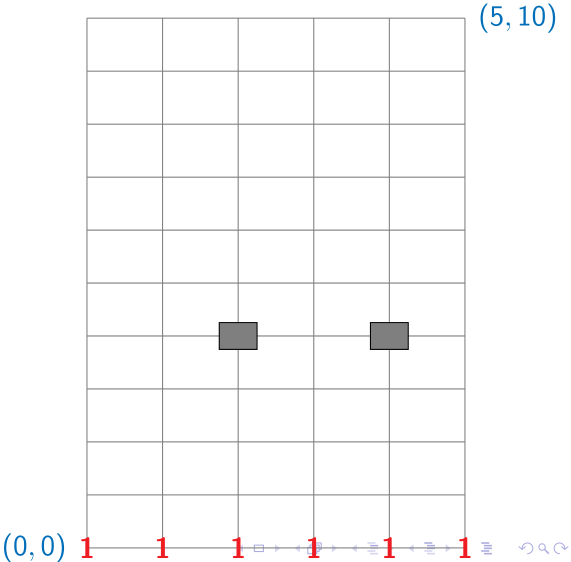
- Fill row by row



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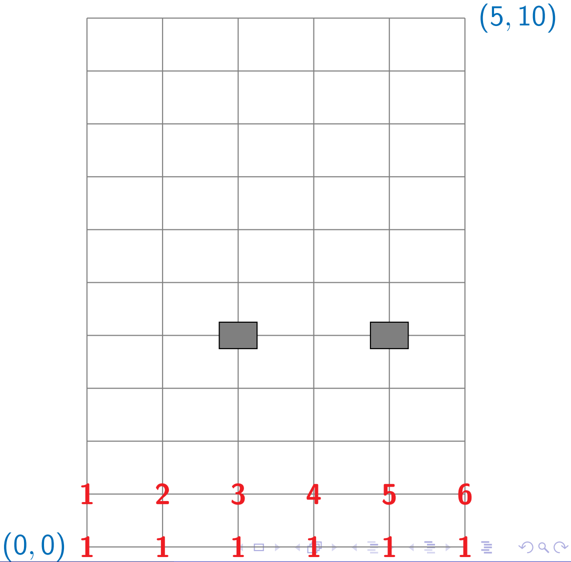

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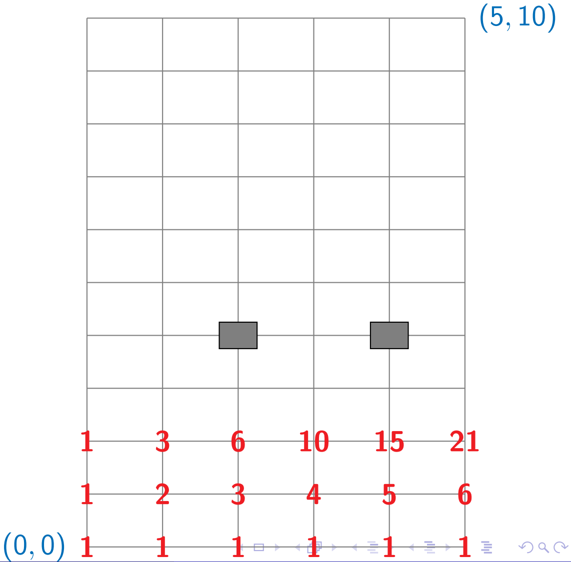

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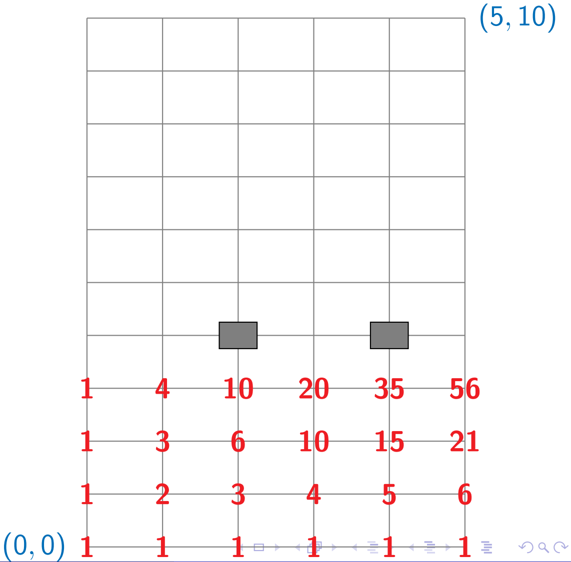

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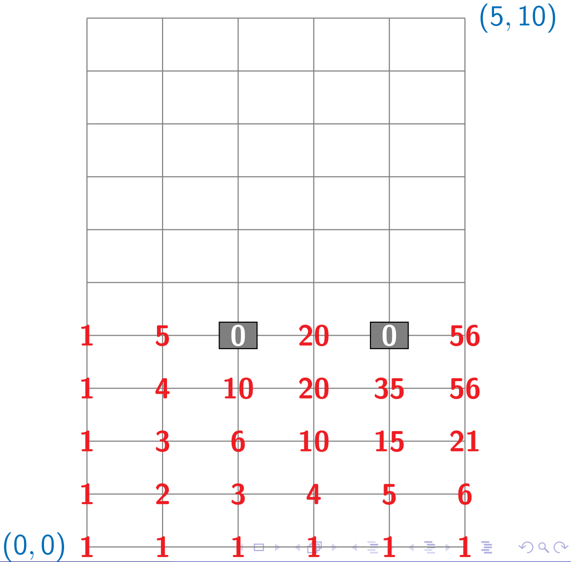

Dynamic programming

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$(5,10)$

1	6	6	26	26	82
1	5	$\boxed{0}$	20	$\boxed{0}$	56
1	4	10	20	35	56
1	3	6	10	15	21
1	2	3	4	5	6
1	1	1	1	1	1

$(0,0)$

Dynamic programming

- Identify subproblem structure
- $P(0,0)$ has no dependencies
- Start at $(0,0)$
- Fill row by row

	1	11	51	181	526	1358 $(5,10)$
	1	10	40	130	345	832
	1	9	30	90	215	487
	1	8	21	60	125	272
	1	7	13	39	65	147
	1	6	6	26	26	82
	1	5	0	20	0	56
	1	4	10	20	35	56
	1	3	6	10	15	21
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$(0,0)$	1	1	1	1	1	1

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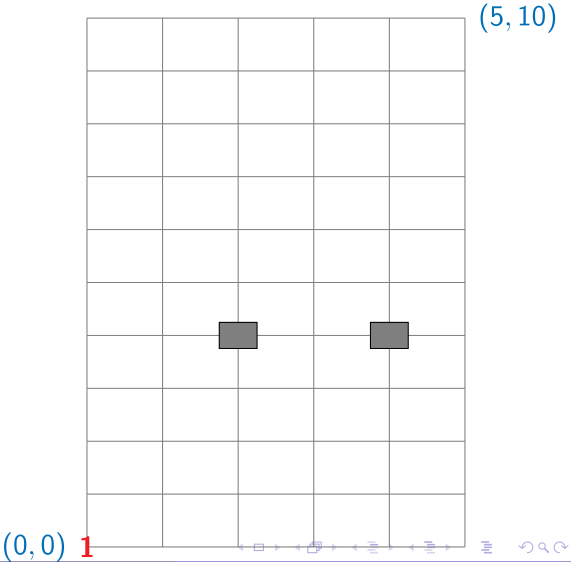

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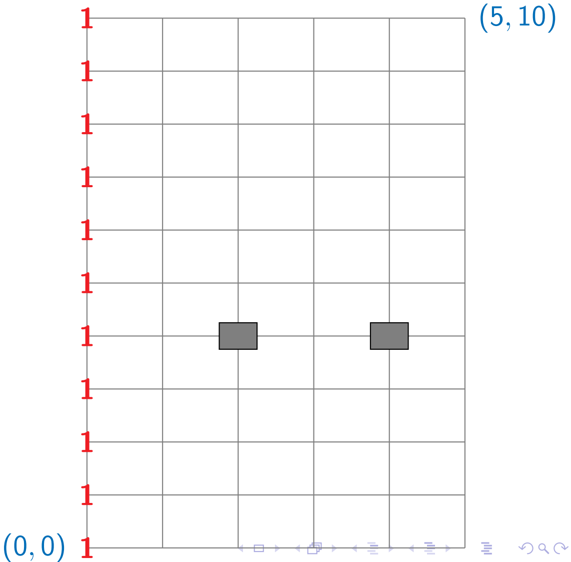

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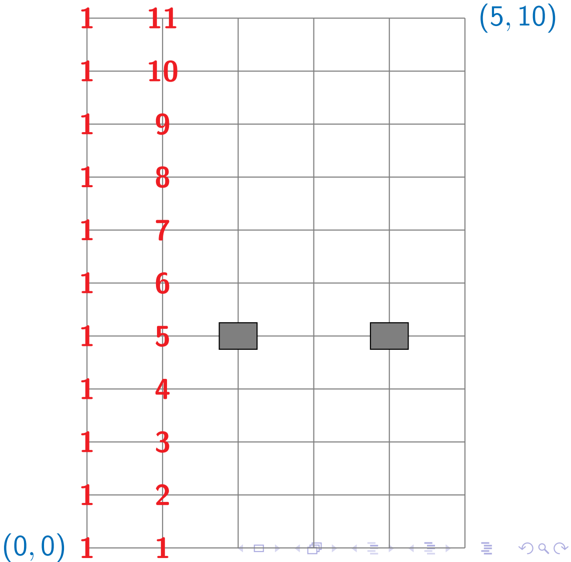

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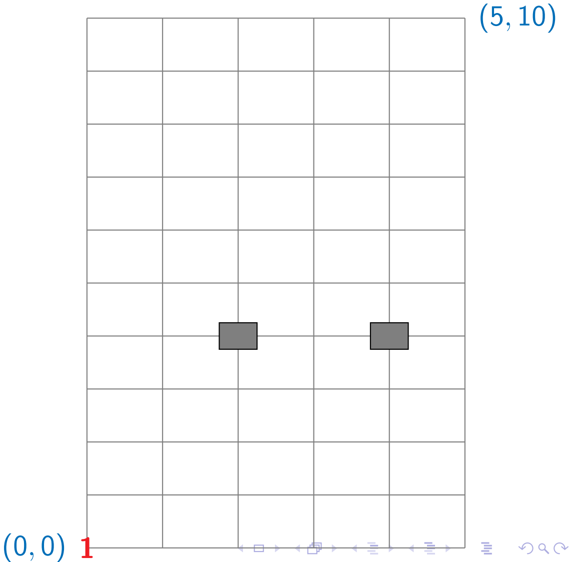
$(0,0)$

Dynamic programming

- Identify subproblem structure
- $P(0,0)$ has no dependencies
- Start at $(0,0)$
- Fill row by row
- Fill column by column
- Fill diagonal by diagonal



Dynamic programming

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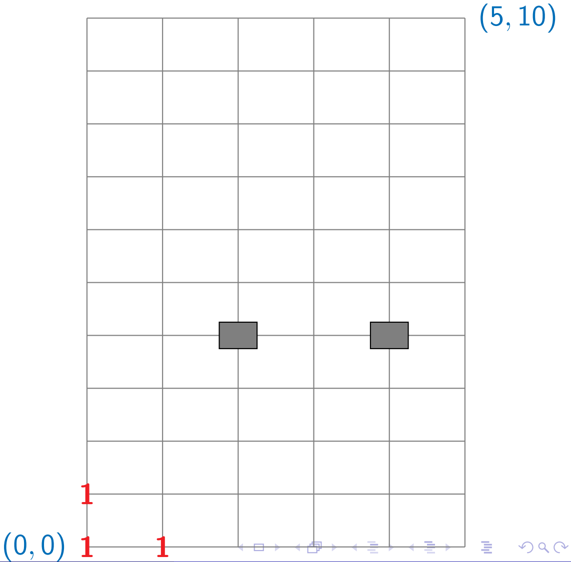
- Fill column by column
- Fill diagonal by diagonal



Dynamic programming

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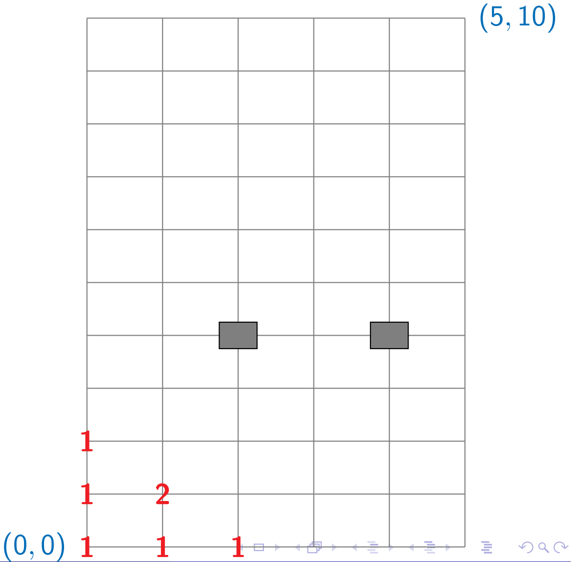

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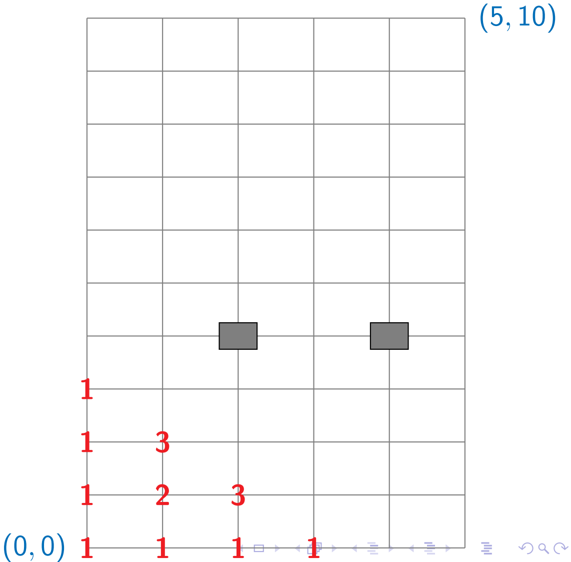
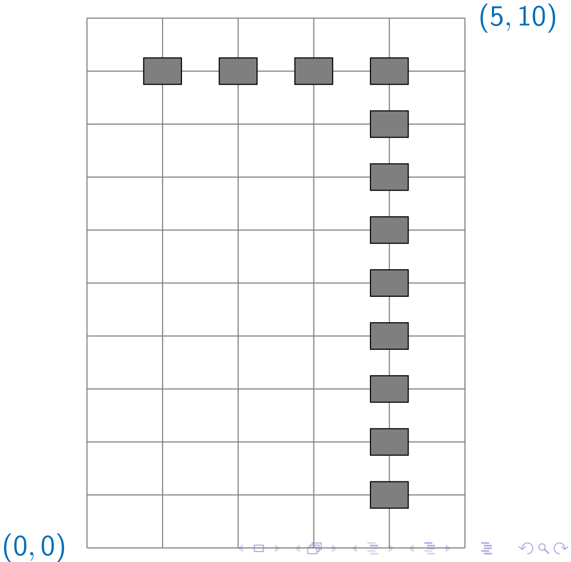


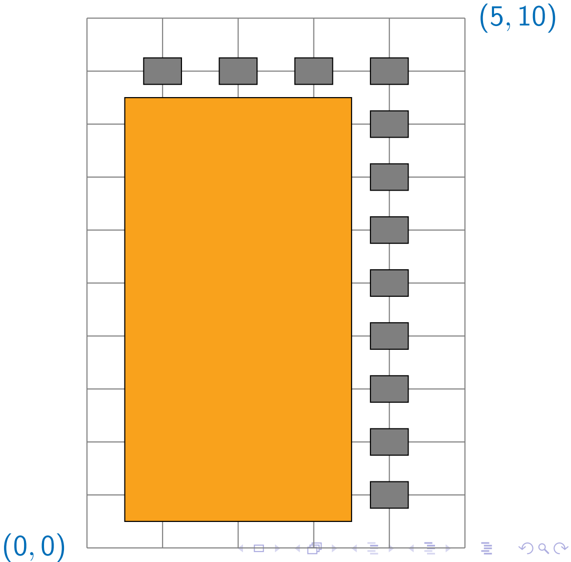
Memoization vs dynamic programming

- Barrier of holes just inside the border



Memoization vs dynamic programming

- Barrier of holes just inside the border
- Memoization never explores the shaded region



Memoization vs dynamic programming

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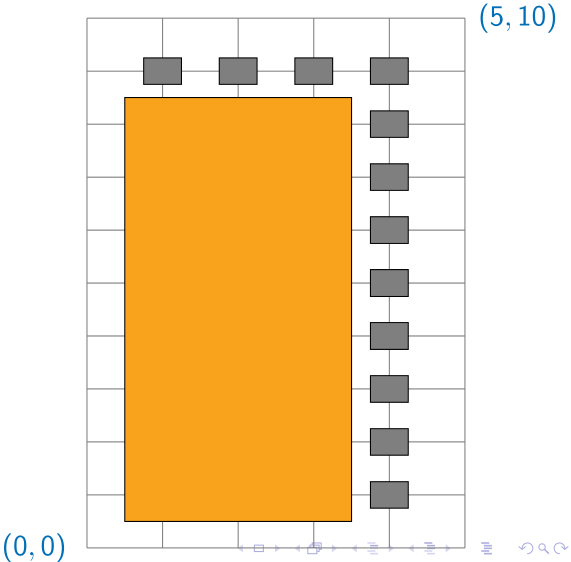
- Memo table has $O(m+n)$ entries



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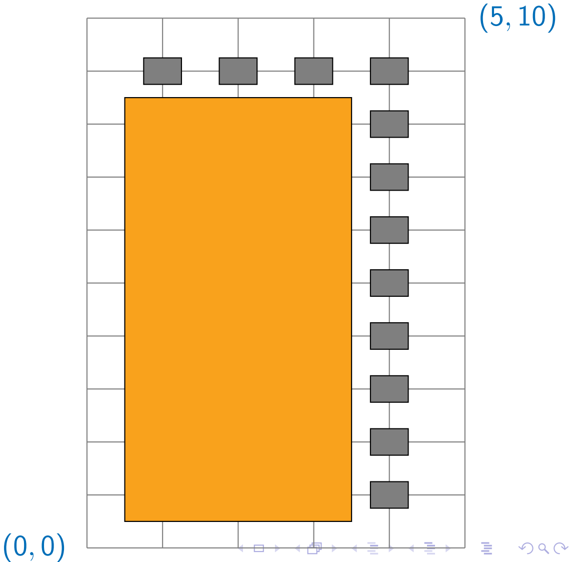

Memoization vs dynamic programming

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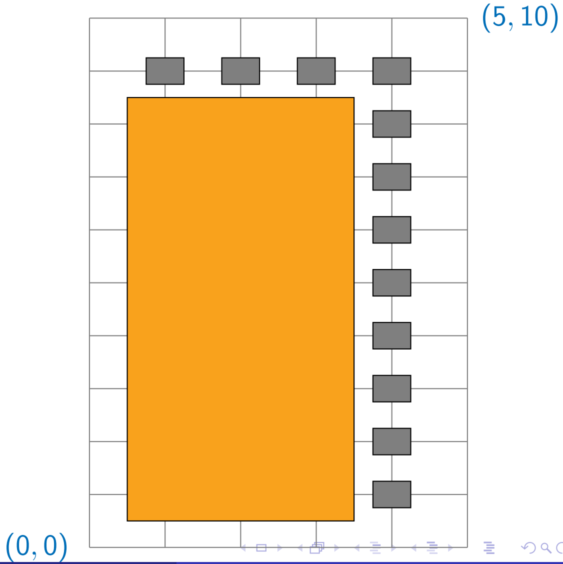
- Memo table has $O(m+n)$ entries
- Dynamic programming blindly fills all mn cells of the table
- Tradeoff between recursion and iteration
  - "Wasteful" dynamic programming still better in general



Longest common subword

- Given two strings, find the (length of the) longest common subword
  - `"secret"`, `"secretary"` — `"secret"`, length 6
  - `"bisect"`, `"trisect"` — `"isect"`, length 5
  - `"bisect"`, `"secret"` — `"sec"`, length 3
  - `"director"`, `"secretary"` — `"ec"`, `"re"`, length 2

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- Formally
  - $u = a_0 a_1 \ldots a_{m-1}$
  - $v = b_0 b_1 \ldots b_{n-1}$

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  - $u = a_0 a_1 \dots a_{m-1}$
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  - Common subword of length k — for some positions i and j, $a_i a_{i+1} a_{i+k-1} = b_j b_{j+1} b_{j+k-1}$

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  - Find the largest such k — length of the longest common subword

Brute force

- $u = a_0 a_1 \dots a_{m-1}$
- $v = b_0 b_1 \dots b_{n-1}$
- Find the largest k such that for some positions i and j, $a_i a_{i+1} a_{i+k-1} = b_j b_{j+1} b_{j+k-1}$

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- Try every pair of starting positions i in u, j in v
  - Match $(a_i, b_j), (a_{i+1}, b_{j+1}), \dots$ as far as possible
  - Keep track of longest match

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  - Keep track of longest match
- Assuming $m > n$, this is $O(mn^2)$
  - mn pairs of starting positions
  - From each starting position, scan could be $O(n)$

Inductive structure

- $u = a_0 a_1 \ldots a_{m-1}$
- $v = b_0 b_1 \ldots b_{n-1}$
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- $LCW(i,j)$ — length of longest common subword in $a_i a_{i+1} \ldots a_{m-1}$, $b_j b_{j+1} \ldots b_{n-1}$
  - If $a_i \neq b_j$, $LCW(i,j) = 0$
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  - Base case: $LCW(m,n) = 0$

Valid indices are $0 \leftrightarrow m-1$
$0 \leftrightarrow n-1$

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Subproblem dependency

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							
1	i							
2	s							
3	e							
4	c							
5	t							
6	•							

Subproblem dependency

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							✓
1	i				i,j			✓
2	s					i+1,j+1		✓
3	e							✓
4	c							✓
5	t							✓
6	•	✓	✓	✓	✓	✓	✓	✓

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- Start at bottom right and fill row by row or column by column

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							**0**
1	i							**0**
2	s							**0**
3	e							**0**
4	c							**0**
5	t							**0**
6	•						0	**0**

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		0	1	2	3	4	5	6
		s	e	c	r	e	$\boxed{\texttt{t}}$	•
0	~~b~~						**0**	**0**
1	~~i~~						**0**	**0**
2	~~s~~						**0**	**0**
3	~~e~~						**0**	**0**
4	~~c~~						**0**	**0**
5	t						**1**	**0**
6	•						**0**	**0**

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		0	1	2	3	4	5	6
		s	e	c	r	~~e~~	t	•
0	b					0	0	0
1	i					0	0	0
2	s					0	0	0
3	~~e~~					1	0	0
4	c					0	0	0
5	t					0	1	0
6	•					0	0	0

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b				0	0	0	0
1	i				0	0	0	0
2	s				0	0	0	0
3	e				0	1	0	0
4	c				0	0	0	0
5	t				0	0	1	0
6	•				0	0	0	0

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b			**0**	**0**	**0**	**0**	**0**
1	i			**0**	**0**	**0**	**0**	**0**
2	s			**0**	**0**	**0**	**0**	**0**
3	e			**0**	**0**	**1**	**0**	**0**
4	c			**1**	**0**	**0**	**0**	**0**
5	t			**0**	**0**	**0**	**1**	**0**
6	•			**0**	**0**	**0**	**0**	**0**

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b		0	0	0	0	0	0
1	i		0	0	0	0	0	0
2	s		0	0	0	0	0	0
3	e		2	1+1 0	0	1	0	0
4	c		0	$\boxed{1}$	0	0	0	0
5	t		0	0	0	0	1	0
6	•		0	0	0	0	0	0

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		s	e	c	r	e	t	•
0	b	0	0	0	0	0	0	0
1	i	0	0	0	0	0	0	0
2	s	3	0	0	0	0	0	0
3	e	0	2	0	0	1	0	0
4	c	0	0	1	0	0	0	0
5	t	0	0	0	0	0	1	0
6	•	0	0	0	0	0	0	0

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Reading off the solution

- Find entry (i,j) with largest LCW value

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	0	0	0	0	0	0	0
1	i	0	0	0	0	0	0	0
2	s	3	0	0	0	0	0	0
3	e	0	2	0	0	1	0	0
4	c	0	0	1	0	0	0	0
5	t	0	0	0	0	0	1	0
6	•	0	0	0	0	0	0	0

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Reading off the solution

- Find entry (i, j) with largest LCW value
- Read off the actual subword diagonally

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	0	0	0	0	0	0	0
1	i	0	0	0	0	0	0	0
2	s	3	0	0	0	0	0	0
3	e	0	2	0	0	1	0	0
4	c	0	0	1	0	0	0	0
5	t	0	0	0	0	0	1	0
6	•	0	0	0	0	0	0	0

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```
def LCW(u,v):
  import numpy as np
  (m,n) = (len(u),len(v))
  lcw = np.zeros((m+1,n+1))

  maxlcw = 0

  for j in range(n-1,-1,-1):
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      if u[i] == v[j]:
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- Recall that brute force was $O(mn^2)$

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  - Fill a table of size $O(mn)$
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Longest common subsequence

- Subsequence — can drop some letters in between
- Given two strings, find the (length of the) longest common subsequence
  - `"secret"`, `"secretary"` — `"secret"`, length 6
  - `"bisect"`, `"trisect"` — `"isect"`, length 5
  - `"bisect"`, `"secret"` — `"sect"`, length 4
  - `"director"`, `"secretary"` — `"ectr"`, `"retr"`, length 4

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- LCS is the longest path connecting non-zero LCW entries, moving right/down

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	0	0	0	0	0	0	0
1	i	0	0	0	0	0	0	0
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Applications

- Analyzing genes
  - DNA is a long string over A, T, G, C
  - Two species are similar if their DNA has long common subsequences

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1	i	**0**	**0**	**0**	**0**	**0**	**0**	**0**
2	s	**3**	**0**	**0**	**0**	**0**	**0**	**0**
3	e	**0**	**2**	**0**	**0**	**1**	**0**	**0**
4	c	**0**	**0**	**1**	**0**	**0**	**0**	**0**
5	t	**0**	**0**	**0**	**0**	**0**	**1**	**0**
6	•	**0**	**0**	**0**	**0**	**0**	**0**	**0**

Applications

- Analyzing genes
  - DNA is a long string over `A`, `T`, `G`, `C`
  - Two species are similar if their DNA has long common subsequences
- `diff` command in Unix/Linux
  - Compares text files
  - Find the longest matching subsequence of lines
  - Each line of text is a "character"

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1	i	**0**	**0**	**0**	**0**	**0**	**0**	**0**
2	s	**3**	**0**	**0**	**0**	**0**	**0**	**0**
3	e	**0**	**2**	**0**	**0**	**1**	**0**	**0**
4	c	**0**	**0**	**1**	**0**	**0**	**0**	**0**
5	t	**0**	**0**	**0**	**0**	**0**	**1**	**0**
6	•	**0**	**0**	**0**	**0**	**0**	**0**	**0**

Inductive structure

- $u = a_0 a_1 \ldots a_{m-1}$
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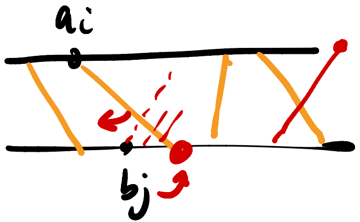
  - Can assume (a_i, b_j) is part of LCS



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  - Which one should we drop?
  - Solve $LCS(i, j+1)$ and $LCS(i+1, j)$ and take the maximum
- Base cases as with LCW
  - $LCS(i, n) = 0$ for all $0 \leq i \leq m$
  - $LCS(m, j) = 0$ for all $0 \leq j \leq n$

$LCS(i,j)$

$a_i \neq b_j$

$= \max\big(LCS(i+1,j),\ LCS(i,j+1)\big)$

Subproblem dependency

- Subproblems are $LCS(i, j)$, for $0 \leq i \leq m$, $0 \leq j \leq n$

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							
1	i							
2	s							
3	e							
4	c							
5	t							
6	•							

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0	b							
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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							**0**
1	i							**0**
2	s							**0**
3	e							**0**
4	c							**0**
5	t							**0**
6	•							**0**

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		s	e	c	r	e	t	•
0	b						**0**	**0**
1	i						**0**	**0**
2	s						**0**	**0**
3	e						**0**	**0**
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5	t						**1**	**0**
6	•						**0**	**0**

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		s	e	c	r	e	t	•
0	b					2	01	0
1	i					2	01	0
2	s					2	01	0
3	e					2	01	0
4	c					1	01	0
5	t					1	1	0
6	•					0	0	0

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Reading off the solution

- Trace back the path by which each entry was filled

		0	1	2	3	4	5	6
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1	i	4	3	2	1	1	0	0
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Reading off the solution

- Trace back the path by which each entry was filled
- Each diagonal step is an element of LCS

		0	1	2	3	4	5	6
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0	b	4	3	2	1	1	0	0
1	i	4	3	2	1	1	0	0
2	s	4	3	2	1	1	0	0
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