AVL Trees – Height-Balanced Search Trees

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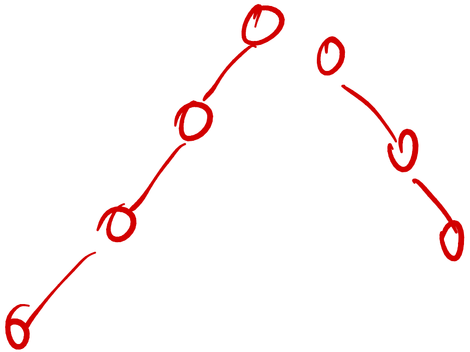
Programming and Data Structures with Python

Lecture 21, 01 Nov 2022

Operations on search trees

- `find()`, `insert()` and `delete()` all walk down a single path
- Worst-case: height of the tree
- An unbalanced tree with n nodes may have height $O(n)$
- Balanced trees have height $O(\log n)$
- How can we maintain balance as tree grows and shrinks



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Defining balance

- Left and right subtrees should be "equal"
  - Two possible measures: `size` and `height`
- `self.left.size()` and `self.right.size()` are equal?
  - Only possible for complete binary trees
- `self.left.size()` and `self.right.size()` differ by at most 1?
  - Plausible, but difficult to maintain

Height balanced trees

- `self.height()` — number of nodes on longest path from root to leaf
  - 0 for empty tree
  - 1 for tree with only a root node
  - $1 + \max$ of heights of left and right subtrees, in general

- Height balance
  - `self.left.height()` and `self.right.height()` differ by at most 1
  - AVL trees — Adelson-Velskii, Landis
- Does height balance guarantee $O(\log n)$ height?

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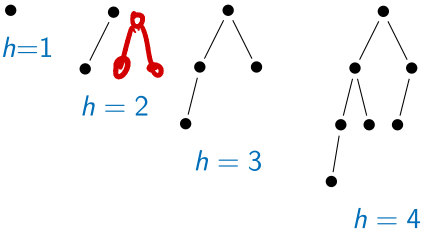
- Does height balance guarantee $O(\log n)$ height?

- Minimum size height-balanced trees



- General strategy to build a small balanced tree of height h
  - Smallest balanced tree of height $h-1$ as left subtree
  - Smallest balanced tree of height $h-2$ as right subtree

Height balanced trees

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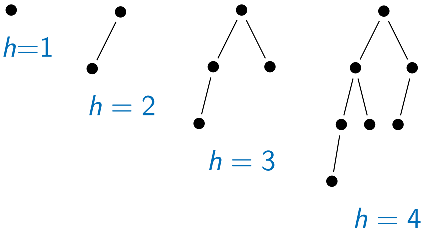

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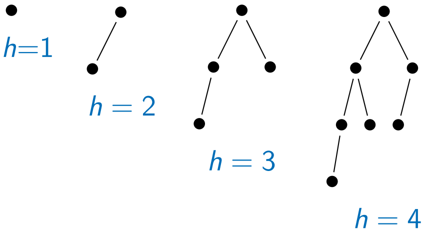

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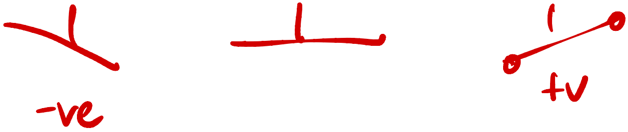
- $S(h)$, size of smallest height-balanced tree of height h
- Recurrence
  - $S(0) = 0$, $S(1) = 1$
  - $S(h) = 1 + S(h-1) + S(h-2)$
- Compare to Fibonacci sequence
  - $F(0) = 0$, $F(1) = 1$
  - $F(n) = F(n-1) + F(n-2)$
- $S(h)$ grows exponentially with h
- For size n, h is $O(\log n)$

Correcting imbalance

- Slope of a node : `self.left.height() - self.right.height()`



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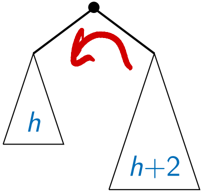
Left rotation



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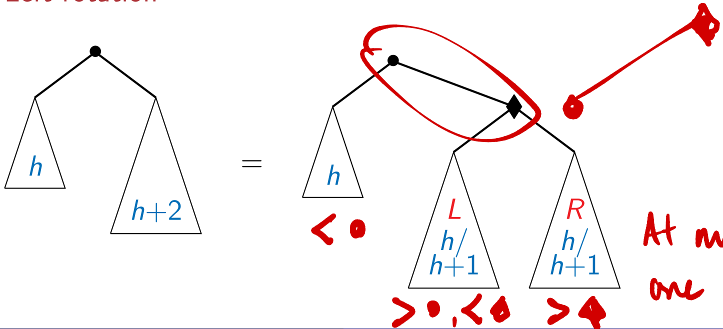

At most one is h

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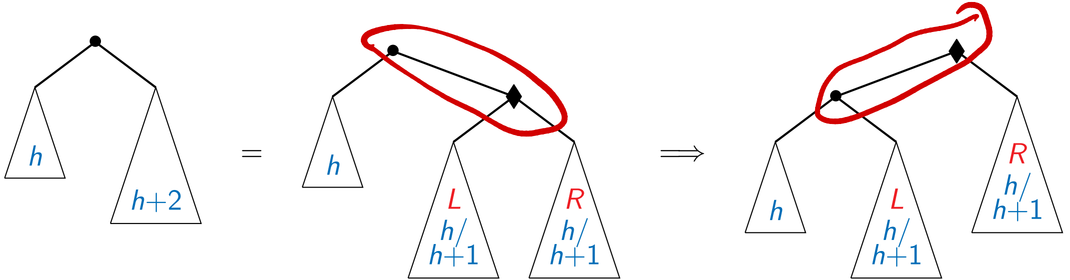
Left rotation — converts slope -2 to $\{0, 1, 2\}$



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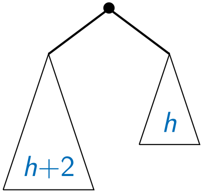
Right rotation



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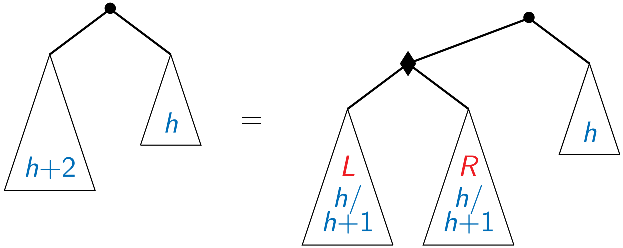

Correcting imbalance

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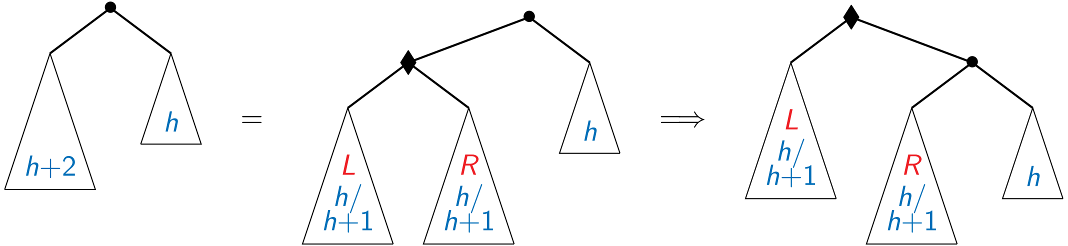
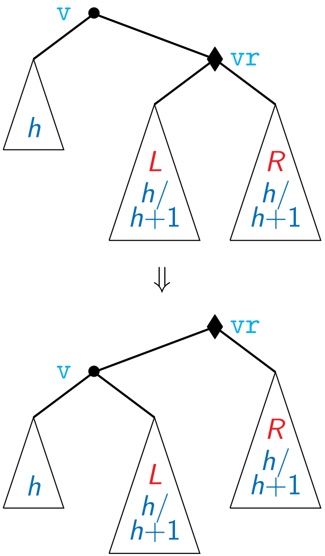
- `t.insert(v)`, `t.delete(v)` can alter slope to -2 or $+2$

Right rotation — converts slope $+2$ to $\{-2, -1, 0\}$

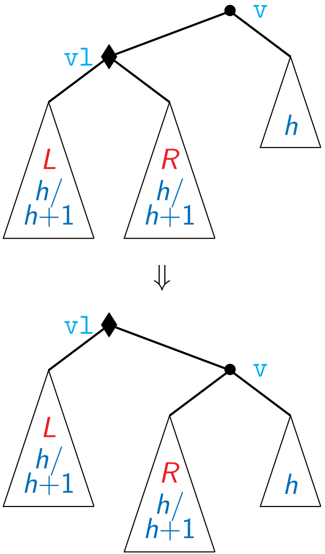


Implementing rotations




```
class Tree:
  ...

  def leftrotate(self):
    v = self.value
    vr = self.right.value
    tl = self.left
    trl = self.right.left
    trr = self.right.right

    newleft = Tree(v)
    newleft.left = tl
    newleft.right = trl

    self.value = vr
    self.left = newleft
    self.right = trr

    return
```

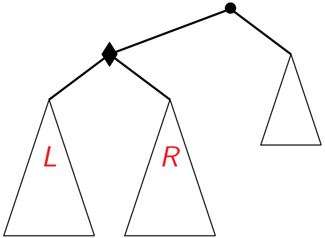
Implementing rotations




```
class Tree:
  ...

  def rightrotate(self):
    v = self.value
    vl = self.left.value
    tll = self.left.left
    tlr = self.left.right
    tr = self.right

    newright = Tree(v)
    newright.left = tlr
    newright.right = tr

    self.value = vl
    self.left = tll
    self.right = newright

    return
```

Rebalancing, root has slope $+2$

- Rebalance bottom-up, assume subtrees are balanced



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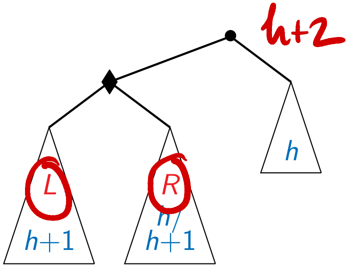
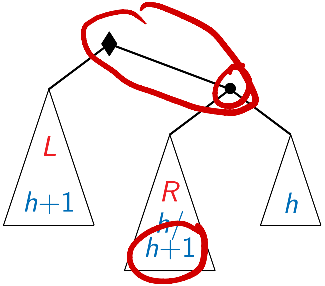
- Case 1: Slope at ◆ is in $\{0, 1\}$



Rebalancing, root has slope $+2$

- Rebalance bottom-up, assume subtrees are balanced
- Case 1: Slope at ♦ is in $\{0, 1\}$
  - Rotate right at •
  - All nodes are balanced



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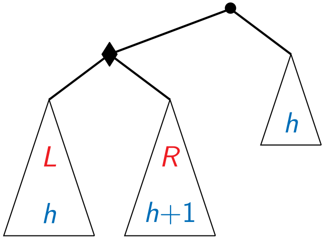

Rebalancing, root has slope $+2$

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- Case 1: Slope at ♦ is in $\{0, 1\}$
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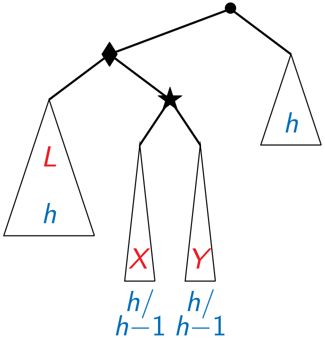
  - Expand *R*



Rebalancing, root has slope $+2$

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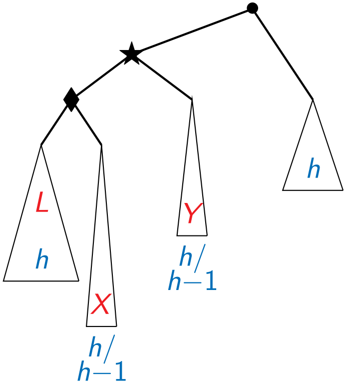

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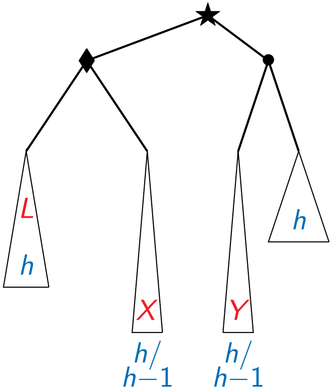

Rebalancing, root has slope $+2$

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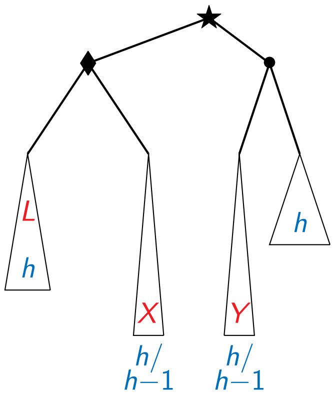
  - Rotate left at •
- Rebalance with root slope -2 is symmetric



- Use the rebalancing strategy to define a function `rebalance()`
- Rebalance each time the tree is modified
- Automatically rebalances bottom up

```
class Tree:
    ...
    def insert(self,v):
        if self.isempty():
            self.value = v
            self.left = Tree()
            self.right = Tree()

        if self.value == v:
            return

        if v < self.value:
            self.left.insert(v)
            self.left.rebalance()
            return

        if v > self.value:
            self.right.insert(v)
            self.right.rebalance()
            return
```

Update insert() and delete()

- Use the rebalancing strategy to define a function `rebalance()`
- Rebalance each time the tree is modified
- Automatically rebalances bottom up

```
class Tree:
    ...
    def delete(self,v):
        ...
        if v < self.value:
            self.left.delete(v)
            self.left.rebalance()
            return
        if v > self.value:
            self.right.delete(v)
            self.right.rebalance()
            return
        if v == self.value:
            if self.isleaf():
                self.makeempty()
            elif self.left.isempty():
                self.copyright()
            elif self.right.isempty():
                self.copyleft()
            else:
                self.value = self.left.maxval()
                self.left.delete(self.left.maxval())
            return
```

Computing slope

- To compute the slope we need heights of subtrees
- But, computing height is $O(n)$

```
class Tree:
    ...
    def height(self):
        if self.isempty():
            return(0)
        else:
            return(1 +
                   max(self.left.height(),
                       self.right.height()))
```

Computing slope

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- But, computing height is $O(n)$
- Instead, maintain a field `self.height`

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Computing slope

- To compute the slope we need heights of subtrees
- But, computing height is $O(n)$
- Instead, maintain a field `self.height`
- After each modification, update `self.height` based on `self.left.height`, `self.right.height`

```
class Tree:
    ...
    def insert(self,v):
        ...
        if v < self.value:
            self.left.insert(v)
            self.left.rebalance()
            self.height = 1 +
                          max(self.left.height,
                              self.right.height)
            return

        if v > self.value:
            self.right.insert(v)
            self.right.rebalance()
            self.height = 1 +
                          max(self.left.height,
                              self.right.height)
            return
```

Summary

- Using rotations, we can maintain height balance
- Height balanced trees have height $O(\log n)$
- `find()`, `insert()` and `delete()` all walk down a single path, take time $O(\log n)$

Memoization

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Inductive definitions, recursive programs, subproblems

- Factorial
  - $fact(0) = 1$
  - $fact(n) = n \times fact(n-1)$
- Insertion sort
  - $isort([\,]) = [\,]$ $\cancel{p}$
  - $isort([x_0, x_1, \ldots, x_n]) =$
    $insert(isort(\underline{[x_0, x_1, \ldots, x_{n-1}]}), x_n)$

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```
def fact(n):
  if n <= 0:
    return(1)
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    return(n * fact(n-1))
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- $fact(n-1)$ is a subproblem of $fact(n)$
  - So are $fact(n-2)$, $fact(n-3)$, ..., $fact(0)$
- $isort([x0, x_1, \ldots, x_{n-1}])$ is a subproblem of $isort([x_0, x_1, \ldots, x_n])$
  - So is $isort([x_i, \ldots, x_j])$ for any $0 \leq i < j \leq n$
- Solution to original problem can be derived by combining solutions to subproblems

Evaluating subproblems

- Fibonacci numbers
  - $fib(0) = 0$
  - $fib(1) = 1$
  - $fib(n) = fib(n-1) + fib(n-2)$

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def fib(n):
  if n <= 1:
    value = n
  else:
    value = fib(n-1) + fib(n-2)
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Evaluating subproblems

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Evaluating `fib(5)`

`fib(5)`

0	1	2	3	4	5
0	1	1	2	3	5

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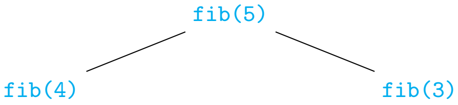

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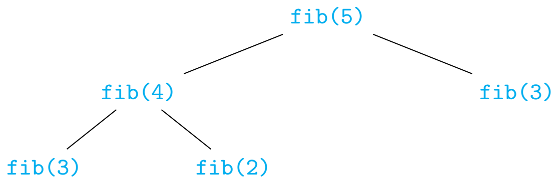

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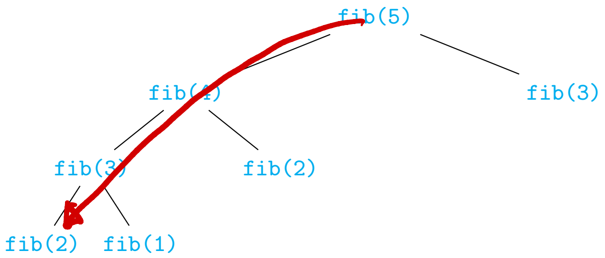

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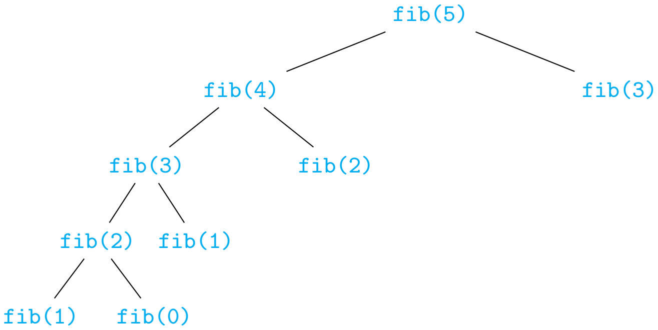

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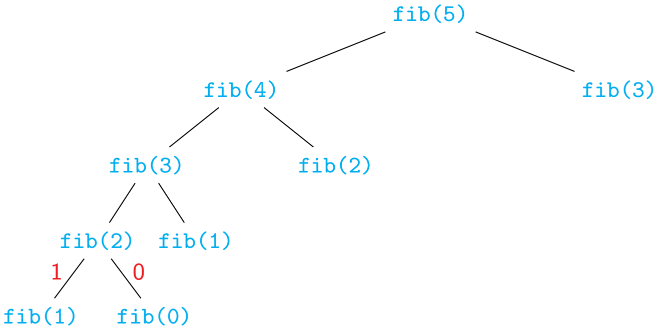

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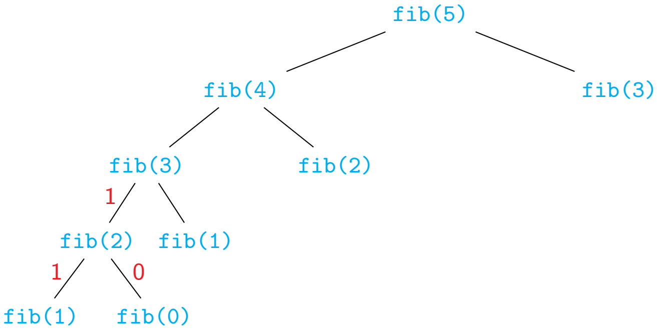

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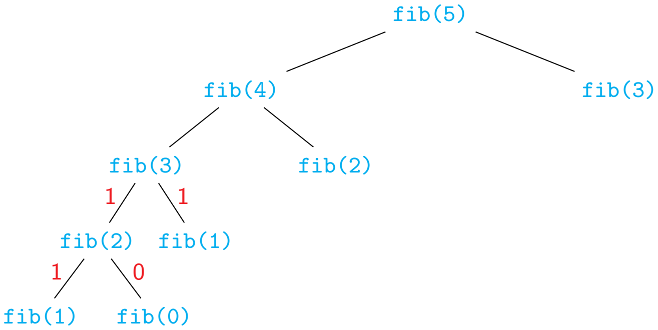

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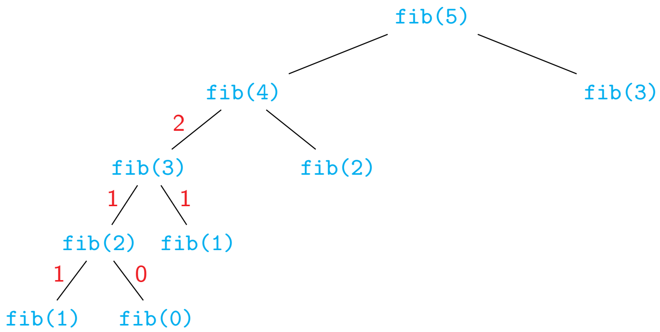

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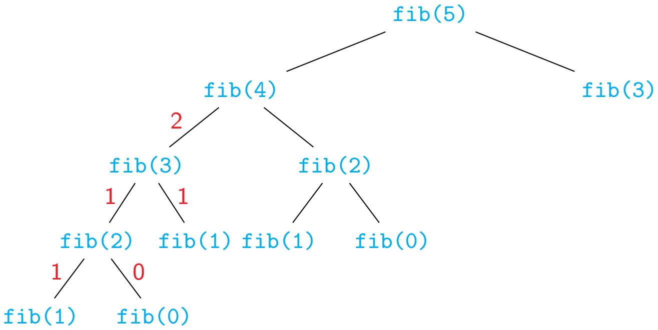

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  return(value)
```
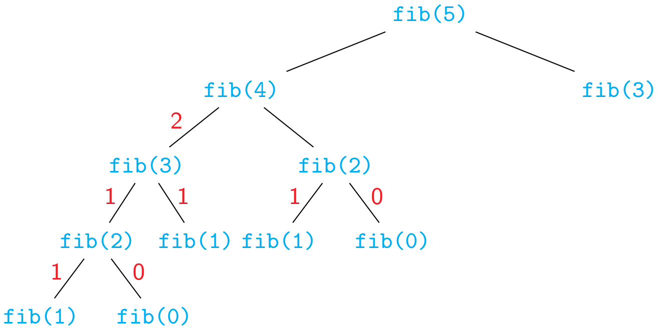

Evaluating `fib(5)`



```
def fib(n):
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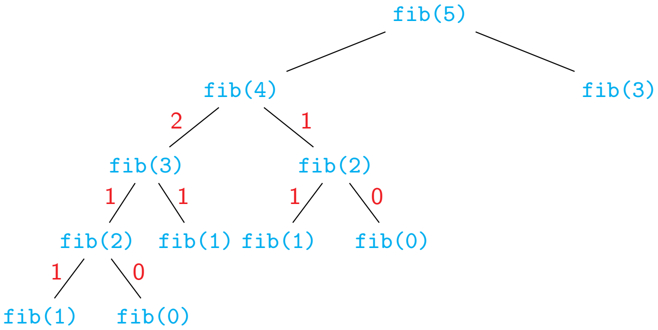
Evaluating fib(5)



Evaluating subproblems

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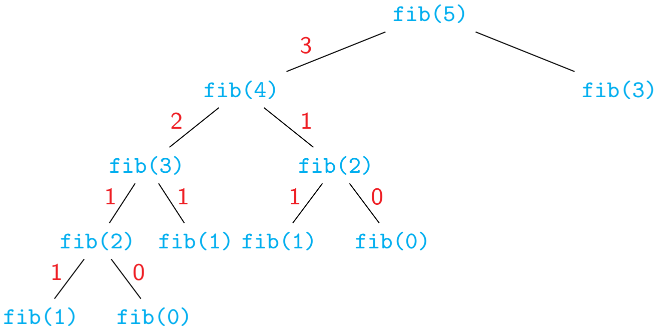

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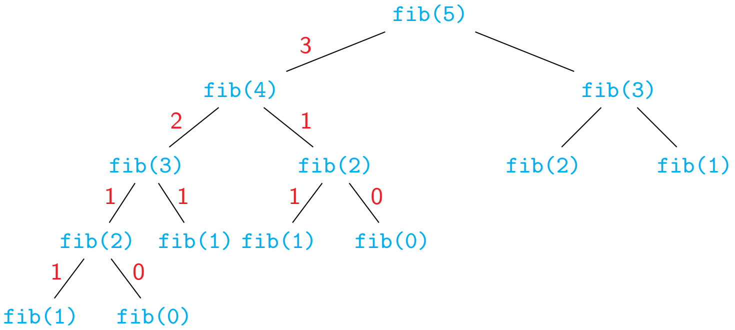

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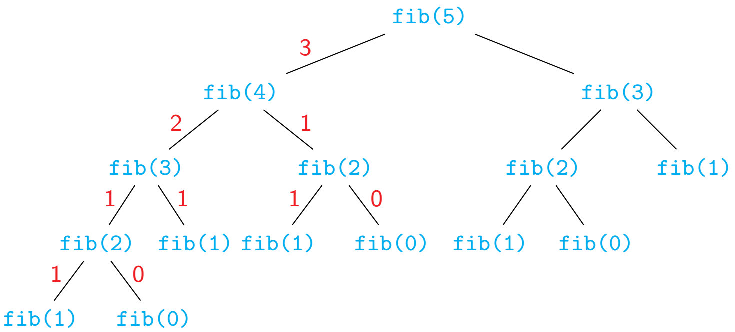

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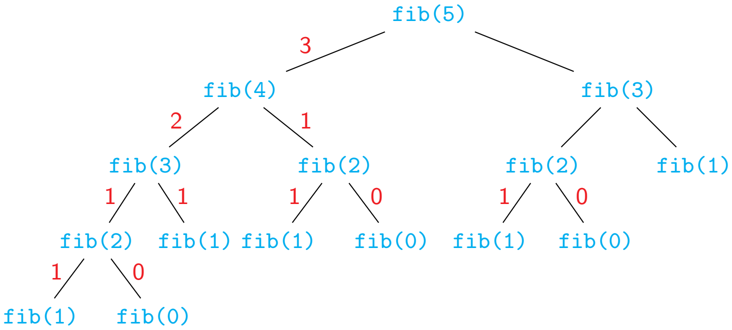

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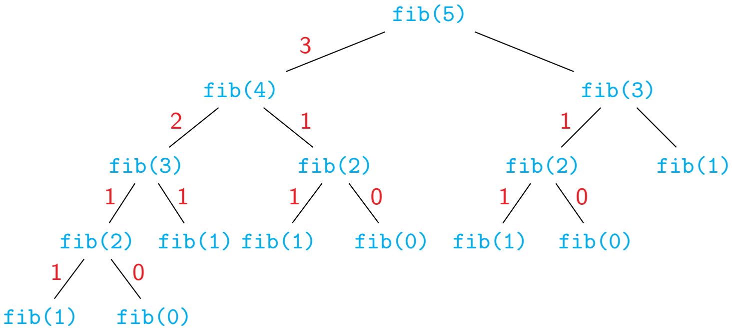

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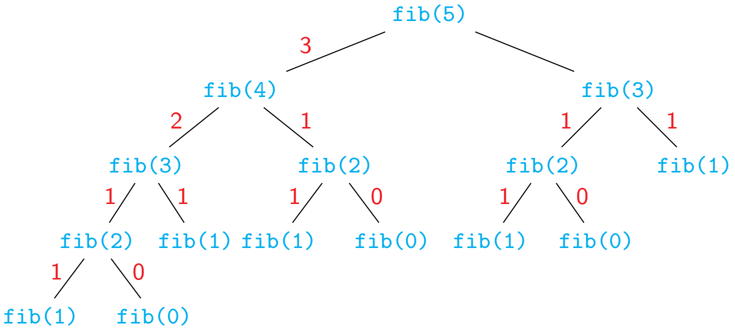

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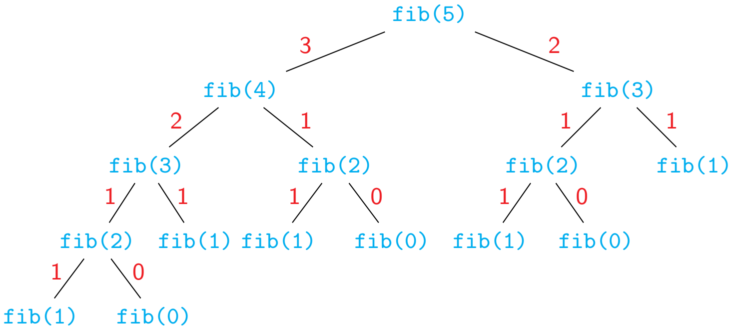

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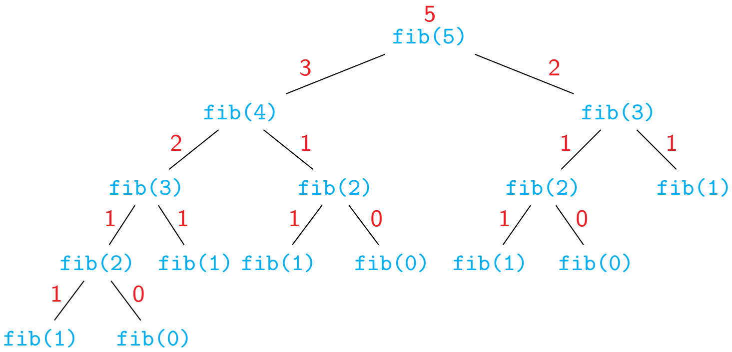

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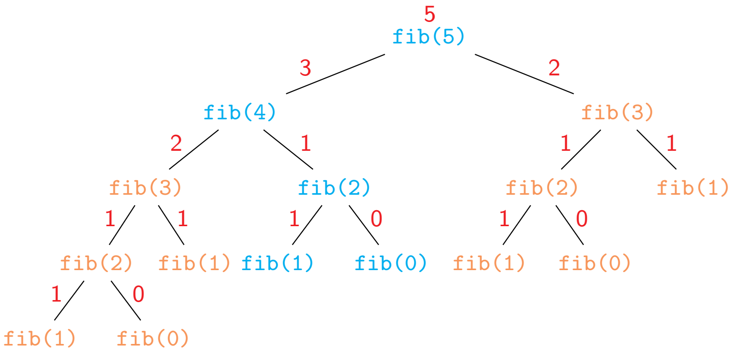
- Wasteful recomputation
- Computation tree grows exponentially

Evaluating `fib(5)`



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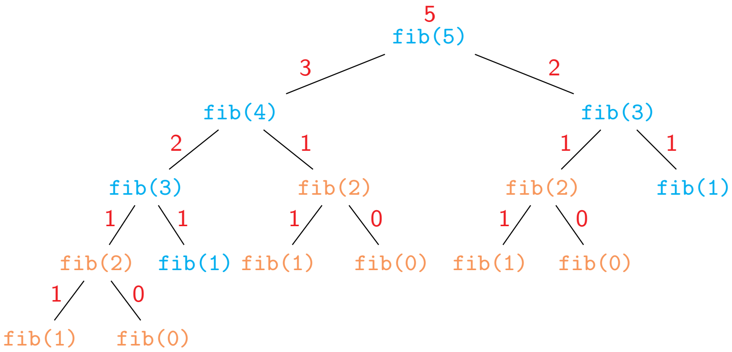

Evaluating subproblems

- Build a table of values already computed
  - Memory table

Evaluating subproblems

- Build a table of values already computed
  - Memory table
- Memoization
  - Check if the value to be computed was already seen before

Evaluating subproblems

- Build a table of values already computed
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Evaluating subproblems

- Build a table of values already computed
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- Look up the table before making a recursive call

Evaluating subproblems

- Build a table of values already computed
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- Computation tree becomes linear

Evaluating subproblems

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`fib(5)`

k						
fib(k)						

Evaluating subproblems

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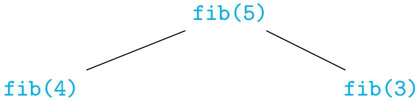

k						
fib(k)						

Evaluating subproblems

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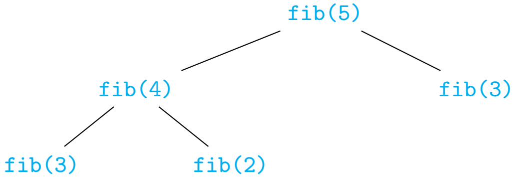

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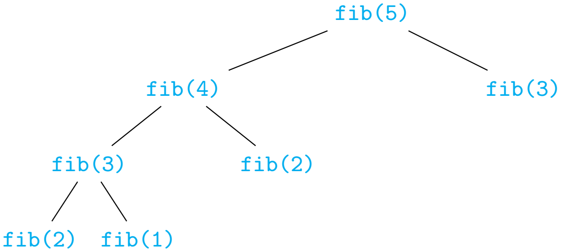

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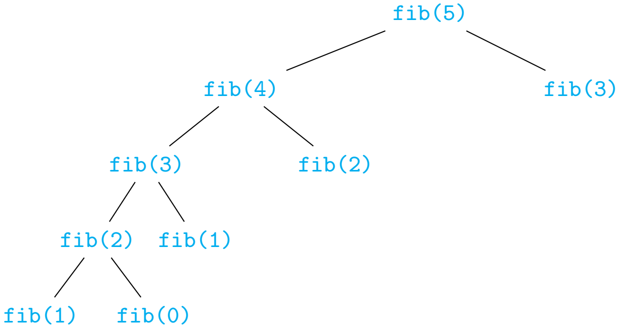

k						
fib(k)						

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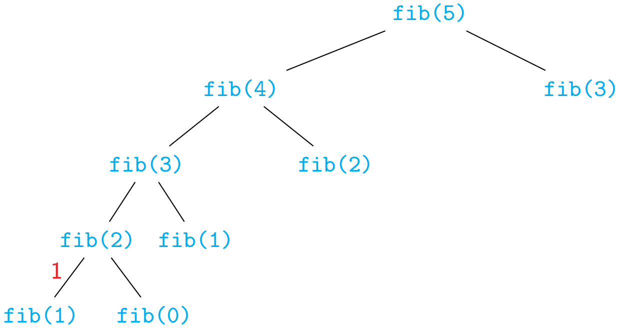

k	1					
fib(k)	1					

Evaluating subproblems

- Build a table of values already computed
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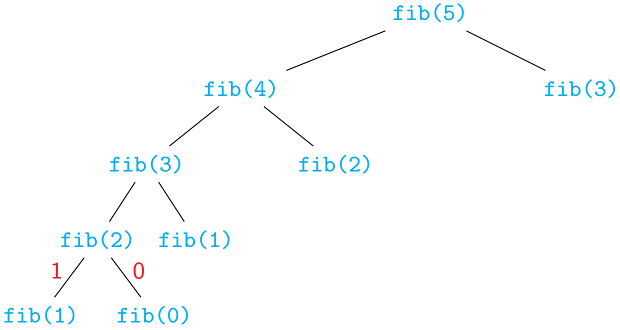

k	1	0				
fib(k)	1	0				

Evaluating subproblems

- Build a table of values already computed
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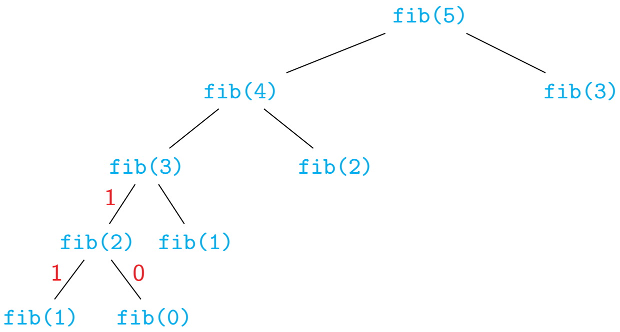

k	1	0	2			
fib(k)	1	0	1			

Evaluating subproblems

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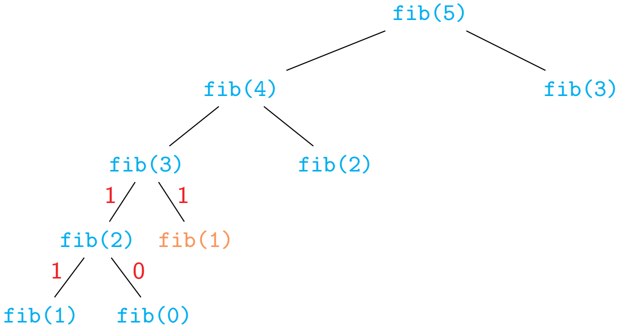

k	1	0	2			
fib(k)	1	0	1			

Evaluating subproblems

- Build a table of values already computed
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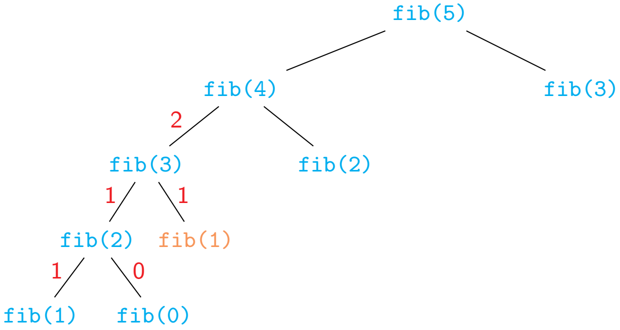

k	1	0	2	3		
fib(k)	1	0	1	2		

- Build a table of values already computed
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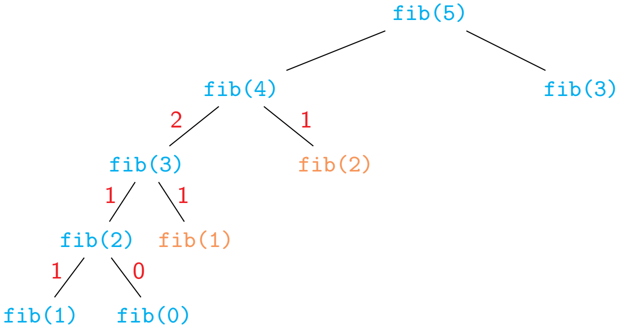
- Computation tree becomes linear



k	1	0	2	3		
fib(k)	1	0	1	2		

Evaluating subproblems

- Build a table of values already computed
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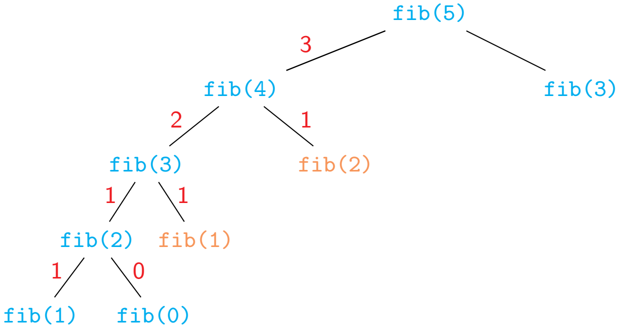

k	1	0	2	3	4	
fib(k)	1	0	1	2	3	

Evaluating subproblems

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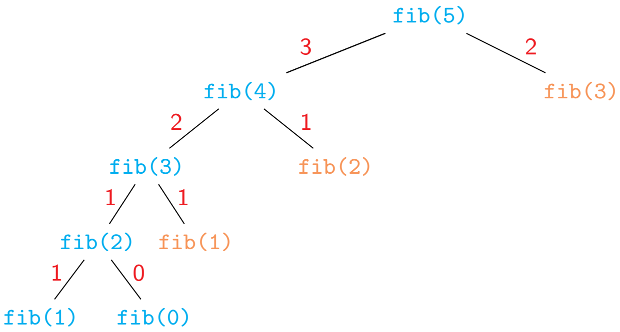

k	1	0	2	3	4	
fib(k)	1	0	1	2	3	

Evaluating subproblems

- Build a table of values already computed
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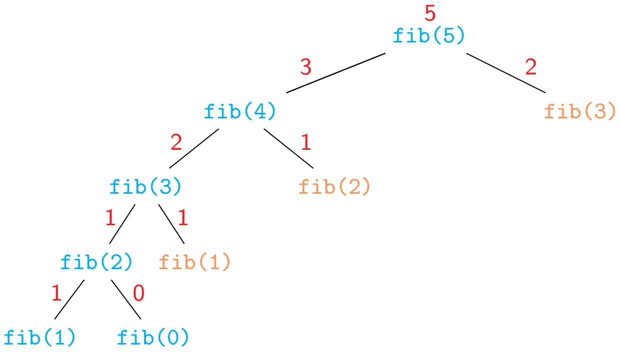

k	1	0	2	3	4	5
fib(k)	1	0	1	2	3	5

Memoizing recursive implmentations

```
def fib(n):
  if n <= 1:
    value = n
  else:
    value = fib(n-1) + fib(n-2)
  return(value)
```

Memoizing recursive implmentations

```
def fib(n):
  if n in fibtable.keys():
    return(fibtable[n])
  if n <= 1:
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    value = fib(n-1) + fib(n-2)
  fibtable[n] = value
  return(value)
```

Memoizing recursive implmentations

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  fibtable[n] = value
  return(value)
```

In general

```
def f(x,y,z):
  if (x,y,z) in ftable.keys():
    return(ftable[(x,y,z)])
  recursively compute value
    from subproblems
  ftable[(x,y,z)] = value
  return(value)
```

0 1 1 2 3 5 8 13 21 34 55